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On the calculation of Eigenvlaues and Eigenvectors of matrix Polynomials and Orthogonality relations between their Eigenvectors

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ABSTRACT

In the paper, the computation of the eigenvalues and eigenvectors of polynomial eigenvalue problem via standard eigenvalue problems is presented. We also establish orthogonality relations between the eigenvectors of matrix polynomials. A numerical example is given to illustrate the applicability of the obtained theoretical results.

Keywords: polynomial eigenvalue problem; orthogonality relations; matrix polynomial.

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1. INTRODUCTION

Consider the polynomial eigenvalue problem

(1.1) $$P(\lambda)x = \left(\lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0\right)x = 0$$

which arises in the analysis and numerical solution of high order systems of ordinary differential equations [4,5] of the form

(1.2) $$M_k \frac{d^k}{dt^k} v + M_{k-1} \frac{d^{k-1}}{dt^{k-1}} v + \cdots + M_1 \frac{d}{dt} v + M_0 v = 0,$$

where $\{M_k, M_{k-1}, \cdots, M_0\}$ are constant $n \times n$ matrices and M_k is nonsingular.

The polynomial

(1.3) $$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0$$

is very often referred to as a lambda matrix, or matrix polynomial of degree k [4,6]. The polynomial eigenvalue problem is the problem of determining all the eigenvalues λ_i and the corresponding eigenvectors x_i of the matrix polynomial P. Note that the standard eigenvalue problem $Ax = \lambda x$ is a special case of (1.1).

The paper is organized as follows: In section 2, we study the polynomial eigenvalue problem and we show how to compute the eigenvalues and eigenvectors of the polynomial eigenvalue problem (1.1) via the standard eigenvalue problem $Ax = \lambda x$. In section 3, we introduce orthogonality relations between the eigenvectors of the polynomial eigenvalue problem (1.1) of degree k. The orthogonality relations between the eigenvectors of matrix polynomial of degree 2 were considered, see for example [1,2,3] which is special case of (1.1). These orthogonality relations play an important role in control theory especially for partial eigenvalue assignment problem [2,6]. A numerical example is given in section 4 to show the applicability of the obtained theoretical results.3

2. Computing The Eigenvalues and Eigenvectors of Polynomial Eigenvalue Problem

Let us start with the following preliminary definitions, which are needed throughout the rest of the paper..

Definition 1 A scalar $\lambda \in C$ such that $\det(P(\lambda)) = 0$ is called an eigenvalue of the matrix polynomial P. The set of eigenvalues is the called the spectrum of P.

Definition 2 The nonzero vectors x and y are, respectively, called the right and left eigenvectors, corresponding to the eigenvalue λ of the matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0$ if

(2.1) $$\left(\lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0\right)x = 0$$

and

(2.2) $$y^H \left(\lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0\right) = 0$$

where y^H is the conjugate transpose of the vector y.

Definition 3 The triplet (λ, x, y) is called the eigenpair of P.

Definition 4 The pairs (λ, x) and (λ, y) are called, respectively, right and left the eigenpairs of P.

Definition 5 The matrix polynomial P is called singular if for any $\lambda \in C$ the matrix $P(\lambda)$ is singular. Otherwise the matrix polynomial P is called regular. In this paper we restrict ourselves to regular matrix polynomial P.

In the following theorem, we show how to compute the eigenvalues and eigenvectors of the polynomial eigenvalue problem (1.1)

Theorem 1

A scalar $\lambda \in C$ is an eigenvalue of the matrix polynomial

$$P(\lambda)=\lambda^k M_k+\lambda^{k-1}M_{k-1}+\cdots+\lambda M_1+M_0,$$

with the corresponding right eigenvector x and the left eigenvector y if and only if λ is an eigenvalue of the $kn\times kn$ matrix

(2.3)
$$A=\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix}$$

with the corresponding right eigenvector $\hat{x}$ and left eigenvector $\hat{y}$ such that.

(2.4)
$$\hat{x}=\begin{pmatrix} x \\ \lambda x \\ \lambda^2 x \\ \vdots \\ \lambda^{k-1}x \end{pmatrix} \text{ and } \hat{y}=\begin{pmatrix} (\lambda^{k-1}M_k^H+\lambda^{k-2}M_{k-1}^H+\cdots+M_1^H)y \\ (\lambda^{k-2}M_k^H+\lambda^{k-3}M_{k-1}^H+\cdots+M_2^H)y \\ (\lambda^{k-3}M_k^H+\lambda^{k-4}M_{k-1}^H+\cdots+M_3^H)y \\ \vdots \\ (M_k^H)y \end{pmatrix}$$

Proof.

Suppose the pair (λ, x) is a right eigenpairs of the matrix polynomial P, and then we have

(2.5)
$$(\lambda^k M_k+\lambda^{k-1}M_{k-1}+\cdots+\lambda M_1+M_0)x=0$$

From (2.5) it follows that

$$A\hat{x}=\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix}\begin{pmatrix} x \\ \lambda x \\ \lambda^2 x \\ \vdots \\ \lambda^{k-1}x \end{pmatrix}=\begin{pmatrix} \lambda x \\ \lambda^2 x \\ \lambda^3 x \\ \vdots \\ \lambda^k x \end{pmatrix}=\lambda\begin{pmatrix} x \\ \lambda x \\ \lambda^2 x \\ \vdots \\ \lambda^{k-1}x \end{pmatrix}=\lambda\hat{x}.$$

Notes: $-(\lambda^{k-1}M_{k-1}+\cdots+\lambda M_1+M_0)x=\lambda^k M_k x$

Similarly, suppose the pair (λ, y) is a left eigenpairs of the matrix polynomial P, and then we have

(2.6)
$$y^H(\lambda^k M_k+\lambda^{k-1}M_{k-1}+\cdots+\lambda M_1+M_0)=0.$$

Hence,

$$\hat{y}^H A=\left(y^H(\lambda^{k-1}M_k+\lambda^{k-2}M_{k-1}+\cdots+M_1),\ y^H(\lambda^{k-2}M_k+\lambda^{k-3}M_{k-1}+\cdots+M_2),\ \cdots,\ y^H(\lambda M_k+M_{k-1}),\ y^H M_k\right)$$

$$\times\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix}$$

and we have

$$\hat{y}^H A = \left(-y^H M_0,\quad y^H\left(\lambda^{k-1}M_k + \lambda^{k-2}M_{k-1} + \cdots + \lambda\, M_2\right),\quad \cdots\quad y^H\left(\lambda^2 M_k + \lambda\, M_{k-1}\right),\quad y^H \lambda M_k\right).$$

Notes:- $y^H\left(\lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda\, M_1\right) = -y^H M_0$.

Hence

$$\hat{y}^H A = \lambda\left(y^H\left(\lambda^{k-1}M_k + \lambda^{k-2}M_{k-1} + \cdots + M_1\right),\quad y^H\left(\lambda^{k-2}M_k + \lambda^{k-3}M_{k-1} + \cdots + M_2\right),\quad \cdots\quad y^H\left(\lambda\, M_k + M_{k-1}\right),\quad y^H M_k\right)$$
$$= \lambda y^H$$

which proves that $(\lambda, \hat{x}, \hat{y})$ is an eigenpair of the matrix A.

Next, suppose that λ is an eigenvalue of A and $\hat{x}$ is the associated right eigenvector. Then

(2.7)
$$A\hat{x} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix}\hat{x} = \lambda\hat{x},$$

where $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_k \end{pmatrix}$ with $\hat{x}_i$ are of $n \times 1$ column vectors, $i = 1,\cdots,k$.

The equation (2.7) can be written as

(2.8)
$$\begin{aligned} \hat{x}_2 &= \lambda\hat{x}_1 \\ \hat{x}_3 &= \lambda\hat{x}_2 \\ &\vdots \\ \hat{x}_k &= \lambda\hat{x}_{k-1} \end{aligned}$$

and

(2.9)
$$-M_k^{-1}M_0\hat{x}_1 - M_k^{-1}M_1\hat{x}_2 - \cdots - M_k^{-1}M_{k-1}\hat{x}_k = \lambda\hat{x}_k$$

Substituting (2.8) into (2.9) and multiplying by M_k on the left, we get

$$-M_0\hat{x}_1 - \lambda M_1\hat{x}_1 - \lambda^2 M_2\hat{x}_1 - \cdots - \lambda^{k-1}M_{k-1}\hat{x}_1 = \lambda^k M_k\hat{x}_1$$

i.e.

$$\left(\lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda\, M_1 + M_0\right)\hat{x}_1 = 0.$$

This shows that λ is the eigenvalue of $P(\lambda)$ with right eigenvector $\hat{x}_1$. If we consider the right eigenvector x of $P(\lambda)$ is determined by $x = \hat{x}_1$

Similarly, if $\hat{y}$ is the left eigenvector of A associated with the eigenvalue λ, then

(2.10) $$\hat{y}^H A = \hat{y}^H \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix} = \lambda \hat{y}^H$$

where $\hat{y}^H = \left(\hat{y}_1^H, \quad \hat{y}_2^H \quad \cdots \quad \hat{y}_{k-1}^H \quad \hat{y}_k^H\right)$

The equation (2.10) can be written as

(2.11) $$\begin{aligned} -\hat{y}_k^H M_k^{-1} M_0 &= \lambda \hat{y}_1^H \\ -\hat{y}_1^H - \hat{y}_k^H M_k^{-1} M_1 &= \lambda \hat{y}_2^H \\ &\vdots \\ -\hat{y}_{k-2}^H - \hat{y}_k^H M_k^{-1} M_{k-2} &= \lambda \hat{y}_{k-1}^H \end{aligned}$$

(2.12) $$-\hat{y}_{k-1}^H - \hat{y}_k^H M_k^{-1} M_{k-1} = \lambda \hat{y}_k^H$$

Substituting the equations (2.11) into (2.12) after multiplication by λ on the left, we obtain.

$-\left(\hat{y}_k^H M_k^{-1}\right)M_0 - \lambda\left(\hat{y}_k^H M_k^{-1}\right)M_1 - \lambda^2\left(\hat{y}_k^H M_k^{-1}\right)M_2 - \cdots - \lambda^{k-1}\left(\hat{y}_k^H M_k^{-1}\right)M_{k-1} = \lambda^k\left(\hat{y}_k^H M_k^{-1}\right)M_k$ hence

$$\hat{y}_k^H M_k^{-1}\left(\lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + M_0\right) = 0$$

which shows that λ is the eigenvalue of $P(\lambda)$ with the left eigenvector $\left(\hat{y}_k^H M_k^{-1}\right)$. If we consider the left eigenvector y of $P(\lambda)$ is determined by $y^H = \hat{y}_k^H M_k^{-1}$.

3. Orthogonality Relations between the Eigenvectors of Matrix Polynomial.

In this section, we first state recent result on the orthogonality relation between the eigenvectors of a given $n \times n$ matrix [6].

Theorem 2 [6] (Orthogonality of the Eigenvectors of a Matrix A)

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of a matrix $A \in C^{n \times n}$ and let $\hat{X}$ and $\hat{Y}$ be respectively the right and the left eigenvector matrices of A. Assume that $\{\lambda_1, \cdots, \lambda_m\} \cap \{\lambda_{m+1}, \cdots, \lambda_n\} = \Phi$ and $m < n$. Partition $\hat{X} = \left(\hat{X}_1, \hat{X}_2\right)$ and $\hat{Y} = \left(\hat{Y}_1, \hat{Y}_2\right)$, where $\hat{X}_1 = (\hat{x}_1, \cdots, \hat{x}_m)$; $\hat{X}_2 = (\hat{x}_{m+1}, \cdots, \hat{x}_n)$, $\hat{Y}_1 = (\hat{y}_1, \cdots, \hat{y}_m)$ and $\hat{Y}_2 = (\hat{y}_{m+1}, \cdots, \hat{y}_n)$.

Then

(3.1) $$\hat{Y}_1^H \hat{X}_2 = 0$$

and

(3.2) $$\hat{Y}_1^H A \hat{X}_2 = 0$$

If, in addition, A is real symmetric, then

(3.3) $$\hat{X}_1^T \hat{X}_2 = 0 \text{ and } \hat{X}_1^T A \hat{X}_2 = 0.$$

The following theorem establishes the orthogonality relations between the eigenvectors for the matrix polynomial (1.1) using its connection with the standard eigenvalues problem given in Theorem 2.

Theorem 3 (Orthogonality of the Eigenvectors of the Matrix Polynomial)

Let $\lambda_1, \lambda_2, \cdots, \lambda_{kn}$ be the eigenvalues of the $kn \times kn$ matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda M_1 + M_0$ and let X and Y be respectively the right and left eigenvector matrices. Assume that $\{\lambda_1,\cdots,\lambda_m\}\cap\{\lambda_{m+1},\cdots,\lambda_{kn}\}=\Phi$ and $m<kn$. Partition $X=(X_1,X_2)$, $Y=(Y_1,Y_2)$ and $\Lambda=diag(\Lambda_1,\Lambda_2)$

where $X_1=(x_1,\cdots,x_m)$; $X_2=(x_{m+1},\cdots,x_{kn})$, $Y_1=(y_1,\cdots,y_m)$ and $Y_2=(y_{m+1},\cdots,y_{kn})$,

with $\Lambda_1=diag(\lambda_1,\cdots,\lambda_m)$ and $\Lambda_2=diag(\lambda_{m+1},\cdots,\lambda_{kn})$

Then

(3.4)
$$\sum_{i=1}^{k-1}\left[\sum_{j=1}^{i}\left[\Lambda_1^{j}Y_1^{H}M_{k-i+j}\right]\right]X_2\Lambda_2^{k-i}-Y_1^{H}M_0X_2=0$$

and

(3.5)
$$\sum_{i=1}^{k}\left[\sum_{j=1}^{i}\left[\Lambda_1^{j-1}Y_1^{H}M_{k-i+j}\right]\right]X_2\Lambda_2^{k-i}=0$$

Proof.

From Theorem 1, the matrix

$$A=\begin{pmatrix} 0 & I & 0 & \cdots & 0\\ 0 & 0 & I & & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & & I\\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & -M_k^{-1}M_2 & \cdots & -M_k^{-1}M_{k-1}\end{pmatrix}$$

has the right eigenvector matrix $\hat{X}$ and the left eigenvector matrix $\hat{Y}$ given by

$$\hat{X}=\begin{pmatrix} X\\ X\Lambda\\ \vdots\\ X\Lambda^{k-1}\end{pmatrix}$$

and

$\hat{Y}^H=\left(\left(\Lambda^{k-1}Y^HM_K+\Lambda^{k-2}Y^HM_{K-1}+\cdots+Y^HM_1\right),\ \left(\Lambda^{k-2}Y^HM_K+\Lambda^{k-3}Y^HM_{K-1}+\cdots+Y^HM_2\right),\ \cdots,\ Y^HM_k\right)$ where $\hat{X}=(\hat{x}_1,\cdots,\hat{x}_{kn})$ and $\hat{Y}=(\hat{y}_1,\cdots,\hat{y}_{kn})$

From equation (3.1) of Theorem 2, we have

$$0=\hat{Y}_1^H\hat{X}_2=\left(\left(\Lambda^{k-1}Y_1^HM_K+\Lambda^{k-2}Y_1^HM_{K-1}+\cdots+Y^HM_1\right)\ \left(\Lambda^{k-2}Y_1^HM_K+\Lambda^{k-3}Y_1^HM_{K-1}+\cdots+Y_1^HM_2\right),\ \cdots,\ Y_1^HM_k\right)\begin{pmatrix} X_2\\ X_2\Lambda_2\\ \vdots\\ X_2\Lambda_2^{k-1}\end{pmatrix}.$$

Then

$$Y_1^HM_kX_2\Lambda_2^{k-1}+\left(\Lambda_1Y_1^HM_k+Y_1^HM_{k-1}\right)X_2\Lambda_2^{k-2}+\cdots+$$
$$\left(\Lambda_1^{k-2}Y_1^HM_k+\Lambda_1^{k-3}Y_1^HM_{k-1}+\cdots+Y_1^HM_2\right)X_2\Lambda_2+\left(\Lambda_1^{k-1}Y_1^HM_k+\Lambda_1^{k-2}Y_1^HM_{k-1}+\cdots+Y_1^HM_1\right)X_2=0$$

This relation can be summarized as follows

$$\sum_{i=1}^{k}\left[\sum_{j=1}^{i}\left[\Lambda_1^{j-1}Y_1^H M_{k-i+j}\right]\right]X_2\Lambda_2^{k-i}=0$$

which proves relation (3.5).

Similarly, from equation (3.2) we obtain (3.4) as follows

$$0=\hat{Y}_1^H A\hat{X}_2=\left(\left(\Lambda^{k-1}Y_1^H M_K+\Lambda^{k-2}Y_1^H M_{K-1}+\cdots+Y_1^H M_1\right),\ \left(\Lambda^{k-2}Y_1^H M_K+\Lambda^{k-3}Y_1^H M_{K-1}+\cdots+Y_1^H M_2\right),\ \cdots,\ Y_1^H M_k\right)$$
$$\begin{pmatrix} 0 & I & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & I \\ -M_k^{-1}M_0 & -M_k^{-1}M_1 & \cdots & -M_k^{-1}M_{k-1} \end{pmatrix}\begin{pmatrix} X_2 \\ X_2\Lambda_2 \\ \vdots \\ X_2\Lambda_2^{k-1} \end{pmatrix}.$$

Then

$$\begin{aligned}&-Y_1^H M_0X_2+\left(\Lambda_1^{k-1}Y_1^H M_k+\Lambda_1^{k-2}Y_1^H M_{k-1}+\cdots+\Lambda_1Y_1^H M_2+Y_1^H M_1-Y_1^H M_kM_k^{-1}M_1\right)X_2\Lambda_2\\&+\left(\Lambda_1^{k-2}Y_1^H M_k+\Lambda_1^{k-3}Y_1^H M_{k-1}+\cdots+\Lambda_1Y_1^H M_3+Y_1^H M_2-Y_1^H M_kM_k^{-1}M_2\right)X_2\Lambda_2^2+\cdots\\&+\left(\Lambda_1^2Y_1^H M_k+\Lambda_1Y_1^H M_{k-1}+Y_1^H M_{k-2}-Y_1^H M_kM_k^{-1}M_{k-2}\right)X_2\Lambda_2^{k-2}\\&+\left(\Lambda_1Y_1^H M_k+Y_1^H M_{k-1}-Y_1^H M_kM_k^{-1}M_{k-1}\right)X_2\Lambda_2^{k-1}=0\end{aligned}$$

i.e.

$$\begin{aligned}&\Lambda_1Y_1^H M_k\Lambda_2^{k-1}+\left(\Lambda_1^2Y_1^H M_k+\Lambda_1Y_1^H M_{k-1}\right)X_2\Lambda_2^{k-2}\\&+\cdots+\left(\Lambda_1^{k-1}Y_1^H M_k+\Lambda_1^{k-2}Y_1^H M_{k-1}+\cdots+\Lambda_1Y_1^H M_2\right)X_2\Lambda_2-Y_1^H M_0X_2=0\end{aligned}$$

This relation can be also summarized as follows

$$\sum_{i=1}^{k-1}\left[\sum_{j=1}^{i}\left[\Lambda_1^{j}Y_1^H M_{k-i+j}\right]\right]X_2\Lambda_2^{k-i}-Y_1^H M_0X_2=0$$

This proves the relation (3.4). The theorem is then proved.

We give some illustrative examples to show orthogonality relations between eigenvectors for matrix polynomial of different degrees

Example 1 matrix polynomial of degree 2 (Quadratic polynomial)

Put $k=2$ in the above relations, we get the quadratic polynomial

(3.6) $$P(\lambda)x=\left(\lambda^2M_2+\lambda M_1+M_0\right)x=0$$

Orthogonality relations of this quadratic polynomial are

$$\Lambda_1Y_1^H M_2\Lambda_2^2-Y_1^H M_0X_2=0$$

$$Y_1^H M_2X_2\Lambda_2+\left(\Lambda_1Y_1^H M_2+Y_1^H M_1\right)X_2=0$$

Example 2 matrix polynomial of degree 3 (Cubic polynomial)

Put $k=3$ in the above relations, we get the cubic polynomial

(3.7) $$P(\lambda)x=\left(\lambda^3M_3+\lambda^2M_2+\lambda M_1+M_0\right)x=0$$

Orthogonality relations of this cubic polynomial are

(3.8) $$\Lambda_1Y_1^H M_3X_2\Lambda_2^2+\left(\Lambda_1^2Y_1^H M_3+\Lambda_1Y_1^H M_2\right)X_2\Lambda_2-Y_1^H M_0X_2=0$$

(3.9) $$Y_1^H M_3X_2\Lambda_2^2+\left(\Lambda_1Y_1^H M_3+Y_1^H M_2\right)X_2\Lambda_2+\left(\Lambda_1^2Y_1^H M_3+\Lambda_1Y_1^H M_2+Y_1^H M_1\right)X_2=0.$$

4. Numerical Example

In this section, a numerical example to compute eigenvalues and (right and left) eigenvectors problem of matrix polynomial of degree 3 (cubic polynomial) is presented. We, also show orthogonality relations between eigenvectors of this matrix polynomial.

We generate the randomly matrices M_3, M_2, M_1 and M_0 (size 4) (using MATLAB 5.3) as follows

$$M_3 = \begin{bmatrix} 0.9501 & 0.8913 & 0.8214 & 0.9218 \\ 0.2311 & 0.7621 & 0.4447 & 0.7382 \\ 0.6068 & 0.4565 & 0.6154 & 0.1763 \\ 0.4860 & 0.0185 & 0.7919 & 0.4057 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.3046 & 0.3028 & 0.3784 & 0.4966 \\ 0.1897 & 0.5417 & 0.8600 & 0.8998 \\ 0.1934 & 0.1509 & 0.8537 & 0.8216 \\ 0.6822 & 0.6979 & 0.5936 & 0.6449 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0.4451 & 0.8462 & 0.8381 & 0.8318 \\ 0.9318 & 0.5252 & 0.0196 & 0.5028 \\ 0.4660 & 0.2026 & 0.6813 & 0.7095 \\ 0.4186 & 0.6721 & 0.3795 & 0.4289 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0.9355 & 0.0579 & 0.1389 & 0.2722 \\ 0.9169 & 0.3529 & 0.2028 & 0.1988 \\ 0.4103 & 0.8132 & 0.1987 & 0.0153 \\ 0.8936 & 0.0099 & 0.6038 & 0.7468 \end{bmatrix}.$$

The polynomial eigenvalue problem $P(\lambda)x = \left(M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0\right)x$ can be reduced to standard eigenvalue problem $A\hat{x} = \lambda\hat{x}$ such that $\hat{x} = \left(x \quad \lambda x \quad \lambda^2 x\right)^T$ is the right eigenvector and the matrix A has the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.8224 & 1.0864 & 0.6558 & 0.3535 & 1.6784 & -0.1404 & -1.2791 & -0.0836 & 0.2994 & 1.1211 & 1.9587 & 1.8037 \\ -0.0123 & -1.7574 & 0.2548 & 0.8474 & -1.0760 & 0.4793 & -0.0852 & -0.6637 & 0.5012 & 0.1647 & -1.4879 & -1.3291 \\ -1.2599 & -1.6610 & -1.1429 & -0.7765 & -1.7163 & -0.2579 & 0.1029 & -0.7091 & -0.8854 & -1.3455 & -2.5928 & -2.4290 \\ -0.7278 & 1.9967 & -0.0545 & -0.7871 & 0.3569 & -1.0069 & 0.3998 & 0.4574 & -0.3347 & -0.4443 & 1.3198 & 1.0518 \end{pmatrix}$$

Then the matrix A and the cubic polynomial $P(\lambda) = M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0$ have eigenvalues as shown in Table 1.

Table 1

	Eigenvalues of $P(\lambda)=M_3\lambda^3+M_2\lambda^2+M_1\lambda+M_0$
$\lambda_1 =$	-2.1822
$\lambda_2 =$	-1.6155
$\lambda_3 =$	1.1687 - 0.8481i
$\lambda_4 =$	1.1687 + 0.8481i
$\lambda_5 =$	-0.9321
$\lambda_6 =$	0.0375 + 0.9101i
$\lambda_7 =$	0.0375 - 0.9101i
$\lambda_8 =$	-0.2044 + 0.5375i
$\lambda_9 =$	-0.2044 - 0.5375i
$\lambda_{10} =$	0.6132
$\lambda_{11} =$	0.1275
$\lambda_{12} =$	0.9085

Now, we show how to compute the right eigenvectors $x_i \quad i=1,2,\cdots,12$ and left eigenvectors $y_i^H \quad i=1,2,\cdots,12$ of the matrix polynomial $P(\lambda)=M_3\lambda^3+M_2\lambda^2+M_1\lambda+M_0$.

For example, we consider the eigenvalue $\lambda_1=-2.1822$ of the matrix *A* has corresponding the right eigenvector $\hat{x}_1=(-0.1159 \quad 0.0961 \quad 0.0837 \quad -0.0740 \quad 0.2530 \quad -0.2098 \quad -0.1826 \quad 0.1615 \quad -0.5521 \quad 0.4579 \quad 0.3985 \quad -0.3524)^T$ and the corresponding left eigenvector

$\hat{y}_1^H=(-0.4246 \quad -0.1873 \quad -0.2301 \quad -0.2242 \quad -0.2223 \quad -0.1272 \quad 0.1395 \quad -0.0730 \quad -0.1057 \quad -0.2915 \quad -0.5436 \quad -0.4465)$.

Since from Theorem 2, $\hat{x}_1=\left(x_1 \quad \lambda x_1 \quad \lambda^2 x_1\right)^T$ is the right eigenvector of matrix *A,* then

$x_1=(-0.1159 \quad 0.0961 \quad 0.0837 \quad -0.0740)^T$ is the right eigenvector of matrix polynomial $P(\lambda)$.

Also since from Theorem 2, $\hat{y}_1^H=\left(y_1^H\left(\lambda^2 M_3+\lambda M_2+M_1\right) \quad y_1^H\left(\lambda M_3+M_2\right) \quad y_1^H\left(M_3\right)\right)$ is l the eft eigenvector of matrix *A,* then

$y_1^H=(-0.1057 \quad -0.2915 \quad -0.5436 \quad -0.4465)\times M_3^{-1}$ is the left eigenvector of matrix polynomial $P(\lambda)$.

Similarly, we can compute the reminder right eigenvectors $x_2,x_3,\cdots,x_{12}$ and left eigenvectors $y_2^H,y_3^H,\cdots,y_{12}^H$ of the matrix polynomial $P(\lambda)=M_3\lambda^3+M_2\lambda^2+M_1\lambda+M_0$ as shown in Table 2 and Table 3 respectively.

Table 2

	Right Eigenvectors of $P(\lambda)=M_3\lambda^3+M_2\lambda^2+M_1\lambda+M_0$
$x_1=$	$(-0.1159 \quad 0.0961 \quad 0.0837 \quad -0.0740)^T$
$x_2=$	$(-0.1432 \quad 0.1928 \quad 0.0729 \quad -0.1816)^T$
$x_3=$	$(0.0565+0.0080i \quad -0.1843-0.1308i \quad -0.1038-0.0506i \quad 0.1989+0.1652i)^T$
$x_4=$	$(0.0565-0.0080i \quad -0.1843+0.1308i \quad -0.1038+0.0506i \quad 0.1989-0.1652i)^T$
$x_5=$	$(0.3550 \quad 0.2606 \quad 0.0484 \quad -0.4299)^T$
$x_6=$	$(0.1604+0.2012i \quad 0.0982+0.1697i \quad 0.1937+0.4862i \quad -0.0087-0.1359i)^T$
$x_7=$	$(0.1604-0.2012i \quad 0.0982-0.1697i \quad 0.1937-0.4862i \quad -0.0087+0.1359i)^T$
$x_8=$	$(0.1375-0.0457i \quad -0.0809+0.2227i \quad 0.2093-0.3994i \quad -0.2761+0.5811i)^T$
$x_9=$	$(0.1375+0.0457i \quad -0.0809-0.2227i \quad 0.2093+0.3994i \quad -0.2761-0.5811i)^T$
$x_{10}=$	$(0.3760 \quad -0.1562 \quad 0.3097 \quad -0.6303)^T$
$x_{11}=$	$(0.1285 \quad 0.2187 \quad -0.6789 \quad 0.6771)^T$
$x_{12}=$	$(0.3658 \quad 0.0923 \quad 0.0391 \quad -0.5051)^T$

Table 3

	Left Eigenvectors of $P(\lambda)=M_3\lambda^3+M_2\lambda^2+M_1\lambda+M_0$
y_1^H	$(0.5084 \quad -0.8702 \quad -0.1538 \quad -0.6056)$
y_2^H	$(-0.1904 \quad 0.5160 \quad -0.0037 \quad 0.5484)$
y_3^H	$(0.2502+0.6450i \quad -0.1664-0.3686i \quad 0.5635-0.4570i \quad -0.7573+0.0997i)$
y_4^H	$(0.2502-0.6450i \quad -0.1664+0.3686i \quad 0.5635+0.4570i \quad -0.7573-0.0997i)$
y_5^H	$(-0.2608+0.1464i \quad 0.1821+0.1516i \quad -0.1153-0.0228i \quad -0.0627-0.0130i)$
y_6^H	$(-0.2608-0.1464i \quad 0.1821-0.1516i \quad -0.1153+0.0228i \quad -0.0627+0.0130i)$
y_7^H	$(-0.0559 \quad -0.1674 \quad -0.3047 \quad -0.1433)$
y_8^H	$(0.9966 \quad -0.5645 \quad -0.1242 \quad -0.4244)$
y_9^H	$(-0.3706+0.2971i \quad 0.3144+0.3934i \quad -0.1192-0.3685i \quad 0.0493-0.4635i)$
y_{10}^H	$(-0.3706-0.2971i \quad 0.3144-0.3934i \quad -0.1192+0.3685i \quad 0.0493+0.4635i)$
y_{11}^H	$(0.9049 \quad -0.4036 \quad -0.0697 \quad -0.4484)$
y_{12}^H	$(0.6497 \quad -0.5797 \quad 0.1805 \quad -0.1378)$

Now, we satisfy the orthogonality relations (3.8) and (3.9) from example 2. We take the first m $(m=4)$ eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ from Table 1 and the associated left eigenvectors y_1^H,y_2^H,y_3^H,y_4^H from Table 3 such that

$$\Lambda_1=diag(\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4)=diag(-2.1822 \quad -1.6155 \quad 1.1687-0.8481i \quad 1.1687+0.8481i)$$

$$Y_1^H=\begin{pmatrix} 0.5084 & -0.8702 & -0.1538 & -0.6056 \\ -0.1904 & 0.5160 & -0.0037 & 0.5484 \\ 0.2502+0.6450i & -0.1664-0.3686i & 0.5635-0.4570i & -0.7573+0.0997i \\ 0.2502-0.6450i & -0.1664+0.3686i & 0.5635+0.4570i & -0.7573-0.0997i \end{pmatrix}.$$

$\Lambda_2=diag(\lambda_5 \quad \lambda_6 \quad \cdots \quad \lambda_{12})$ from Table 1 and the associated right eigenvectors $X_2=(x_5 \quad x_6 \quad \cdots \quad x_{12})$ from Table 2. Then, we can easily verify that:

$$\Lambda_1 Y_1^H M_3 X_2 \Lambda_2^2 + \left(\Lambda_1^2 Y_1^H M_3 + \Lambda_1 Y_1^H M_2\right) X_2 \Lambda_2 - Y_1^H M_0 X_2 = 0$$

$$Y_1^H M_3 X_2 \Lambda_2^2 + \left(\Lambda_1 Y_1^H M_3 + Y_1^H M_2\right) X_2 \Lambda_2 + \left(\Lambda_1^2 Y_1^H M_3 + \Lambda_1 Y_1^H M_2 + Y_1^H M_1\right) X_2 = 0$$

5. Conclusion

In this paper, we showed how to compute eigenvalues and eigenvectors of polynomial eigenvalue problem. We derived orthogonalityrelationsbetween eigenvectors for the matrix polynomial.
$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda\, M_1 + M_0$. These orthogonality relations play important role in control theory, for example for solving the partial eigenvalue assignment problem. The study of the partial eigenvalue assignment problem of higher order control system using the orthognality relation presented in this work is under preparation by the authors.

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