ISSN 2347-1921

A Note on the sums of Reciprocal k–Fibonacci Numbers of Subscript $2^n a$

Sergio Falcon
Department of Mathematics
University of Las Palmas de Gran Canaria
Las Palmas. Spain

ABSTRACT

In this article we find the finite sum of reciprocal k–Fibonacci numbers of subscript 2^n a, then we find the infinite sum of these numbers. Special cases of these sums for the classical Fibonacci sequence and the Pells equence are indicated.

Finally we propose a new way to find the infinite sum of the reciprocal k–Fibonacci numbers with odd subscripts and, consequently, the sum of all reciprocal k–Fibonacci numbers, but without finding the answer to this problem (Erdös).

Keywords

k-Fibonacci numbers; k-Lucas numbers; Binet Identity.

Mathematics Subject Classification:

MSC 2000: 11B39; 11C20; 15A36.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics
Vol 9, No 1
editor@cirjam.org
www.cirjam.com, www.cirworld.com

1 INTRODUCTION

Classical Fibonacci numbers have been very used in as different Sciences as the Biology, Demographyor Economy [6]. Recently they have been applied even in the high-energy physics [8,9]. But there exist generalizations of these numbers given by researches as Horadam [7] and recently by Weoursef [3–5].

1.1 k–Fibonacci Numbers

For any positive real number k, the k–Fibonacci sequence, say $F_k = \{F_{k,n}\}_{n\in\mathbb{N}}$ is defined by the recurrence relation

$$F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \qquad (1)$$

With the initial conditions $F_{k,o} = 0$ and $F_{k,1} = 1$.

For k = 1, classical Fibonacci sequence is obtained and for k = 2, Pell sequence appears.

In similar form, the k–Lucas numbers are defined as $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$ (see [1,2]).

The well–known Binet formula in the Fibonacci numbers theory [3,7,10] allows us to express the k–Fibonacci numbers and the k–Lucas numbers by mean of the rootsσ_1 and σ_2 of the characteristic equation, associated to the recurrence

relation $r^2 = k\,r + 1$. If $\sigma_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2}$, then

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \qquad (2)$$

$$L_{k,n} = \sigma_1^n + \sigma_2^n \qquad (3)$$

As consequence $L_{k,n} = F_{k,n-1} + F_{k,n+1}$

Among other properties, σ_1 and σ_2 verify:

$$\sigma_1 + \sigma_2 = k, \quad \sigma_1 - \sigma_2 = \sqrt{k^2+4}, \quad \sigma_1 \cdot \sigma_2 = -1, \quad \sigma^2 = k\sigma + 1 \to \sigma^n = F_{k,n}\sigma + F_{k,n-1}$$

$$\lim_{n\to\infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma_1^r, \quad \sigma_2 < \sigma_1 \to \lim_{n\to\infty} \frac{\sigma_2^n}{\sigma_1^n} = 0$$

Even subscript:

$$F_{k,2m} = F_{k,m} \cdot L_{k,m} \qquad (4)$$

2 FINITE SUM OF RECIPROCAL k-FIBONACCI NUMBERS OF SUBSCRIPT 2^na

In this section we study both the finite and infinite sum of reciprocal k–Fibonacci numbers of subscript 2^na. As consequence, a new form of calculating the infinite sum of the k–Fibonacci numbers is given.

2.1 Theorem 1

The finite sum of reciprocal k–Fibonacci numbers of subscript 2^na is given by the formula

$$\sum_{j=1}^{n} \frac{1}{F_{k,2^j a}} = \frac{1}{F_{k,2^n a}} \frac{L_{k,2^n a} - L_{k,(2^n-2)a}}{L_{k,2a} - 2} \qquad (5)$$

Proof : by induction.

For n = 1, the Left Hand Side (LHS) of (5) is $(LHS)=\sum_{j=1}^{1}\frac{1}{F_{k,2^j a}}=\frac{1}{F_{k,2a}}$ while the Right Hand Side (RHS) is

$$(RHS)=\frac{1}{F_{k,2a}}\frac{L_{k,2a}-L_{k,0}}{L_{k,2a}-2}=\frac{1}{F_{k,2a}}$$

Let us suppose Formula (5) is true for n > 1. Then

$$\sum_{j=1}^{n+1}\frac{1}{F_{k,2^j a}}=\sum_{j=1}^{n}\frac{1}{F_{k,2^j a}}+\frac{1}{F_{k,2^{n+1}a}}=\frac{1}{F_{k,2^n a}}\frac{L_{k,2^n a}-L_{k,(2^n-2)a}}{L_{k,2a}-2}+\frac{1}{F_{k,2^{n+1}a}}$$

$$=\frac{(L_{k,2^n a})^2-L_{k,(2^n-2)a}L_{k,2^n a}+L_{k,2a}-2}{F_{k,2^{n+1}a}(L_{k,2a}-2)}$$

Because from Equation (4) $F_{k,2^{n+1}a}=F_{k,2\cdot 2^n a}=F_{k,2^n a}\cdot L_{k,2^n a}$

The numerator of this last relation, from Binet Identity, is

$$A=\left(\sigma_1^{2^n a}+\sigma_2^{2^n a}\right)^2-\left(\sigma_1^{(2^n-2)a}+\sigma_2^{(2^n-2)a}\right)\left(\sigma_1^{2^n a}+\sigma_2^{2^n a}\right)+\left(\sigma_1^{2a}+\sigma_2^{2a}\right)-2$$

$$=\left(\sigma_1^{2^{n+1}a}+\sigma_2^{2^{n+1}a}+2\right)-\left(\sigma_1^{(2^{n+1}-2)a}+\sigma_2^{(2^{n+1}-2)a}+\sigma_1^{-2a}+\sigma_2^{-2a}\right)+\left(\sigma_1^{2a}+\sigma_2^{2a}-2\right)$$

$$=L_{k,2^{n+1}a}+2-L_{k,(2^{n+1}-2)a}-\frac{\sigma_2^{2a}+\sigma_1^{2a}}{(\sigma_1\sigma_2)^{2a}}+\sigma_1^{2a}+\sigma_2^{2a}-2=L_{k,2^{n+1}a}-L_{k,(2^{n+1}-2)a}$$

because $\sigma_1\cdot\sigma_2=-1$. Then $\sum_{j=1}^{n+1}\frac{1}{F_{k,2^j a}}=\frac{1}{F_{k,2^{n+1}a}}\frac{L_{k,2^{n+1}a}-L_{k,(2^{n+1}-2)a}}{L_{k,2a}-2}$

2.1.1 Particular Case

If a =1:

$$\sum_{j=1}^{n}\frac{1}{F_{k,2^j}}=\frac{1}{F_{k,2^n}}\frac{L_{k,2^n}-L_{k,2^n-2}}{L_{k,2}-2}=\frac{1}{F_{k,2^n}}\frac{kL_{k,2^n-1}}{k^2}=\frac{1}{k}\frac{F_{k,2^n-2}+F_{k,2^n}}{F_{k,2^n}}$$

$$=\frac{1}{k}\frac{2F_{k,2^n}-kF_{k,2^n-1}}{F_{k,2^n}}=\frac{2}{k}-\frac{F_{k,2^n-1}}{F_{k,2^n}}$$

2.2 Theorem 2

$$\sum_{j=1}^{\infty}\frac{1}{F_{k,2^j a}}=\frac{\sqrt{k^2+4}}{\sigma_1^{2a}-1} \qquad (6)$$

Proof: From Equation (5)

$$\sum_{j=1}^{\infty}\frac{1}{F_{k,2^j a}}=\frac{1}{L_{k,2a}-2}\lim_{n\to\infty}\left(\frac{L_{k,2^n a}-L_{k,(2^n-2)a}}{F_{k,2^n a}}\right)=\frac{\sigma_1-\sigma_2}{L_{k,2a}-2}\lim_{n\to\infty}\left(\frac{\sigma_1^{2^n a}+\sigma_2^{2^n a}-\frac{\sigma_1^{2^n a}}{\sigma_1^{2^n}}-\frac{\sigma_2^{2^n a}}{\sigma_1^{2^n}}}{F_{k,2^n a}}\right)$$

$$=\frac{\sigma_1-\sigma_2}{\sigma_1^{2^n}+\sigma_2^{2^n}-2}\left(1-\frac{1}{\sigma_1^2\cdot a}\right)=\frac{(\sigma_1-\sigma_2)(\sigma_1^{2a}-1)}{\sigma_1^{4a}+1-2\sigma_1^{2a}}=\frac{(\sigma_1-\sigma_2)(\sigma_1^{2a}-1)}{(\sigma_1^{2a}-1)^2}$$

$$=\frac{\sigma_1-\sigma_2}{\sigma_1^{2a}-1}=\frac{\sqrt{k^2+4}}{\sigma_1^{2a}-1}$$

If a = 1, and taking into account $\sigma^2=k\sigma+1$, Formula (6) becomes $\sum_{j=1}^{\infty}\frac{1}{F_{k,2^j}}=\frac{\sqrt{k^2+4}}{k\sigma_1}$

Moreover, if a = k = 1, for the classical Fibonacci numbers it is $\sum_{j=1}^{\infty}\frac{1}{F_{2^j}}=\frac{\sqrt{5}}{\Phi}$, being Φ the Golden Ratio, $\Phi=\frac{1+\sqrt{5}}{2}$

Finally, the sum of reciprocal Fibonacci numbers with odd subscriptsis

$$S_o=\sum_{p=0}^{\infty}S_{2p+1}=\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{F_{k,2^n(2p+1)}}=\sum_{m=1}^{\infty}\frac{1}{F_{k,2m}} \qquad (7)$$

The exact answer has not been found so far.

If the sum of Equation (6) begins in j = 0, the last sum is the sum of reciprocal

k–Fibonacci numbers $S=\sum_{j=1}^{\infty}\frac{1}{F_{k,j}}$

Formula (6) can also be written more compactly as

$$\sum_{j=1}^{\infty}\frac{1}{F_{k,2^j a}}=\frac{\sqrt{k^2+4}}{\sigma_1^a\cdot L_{k,a}} \qquad (8)$$

because, as $\sigma_1\sigma_2=-1$and a is odd, then $\sigma_1^{2a}-1=\sigma_1^{2a}+(\sigma_1\sigma_2)^a=\sigma_1^a\left(\sigma_1^a+\sigma_2^a\right)=\sigma_1^a L_{k,a}$

Consequently, $S_o=\sqrt{k^2+4}\sum_{p=0}^{\infty}\frac{1}{\sigma_1^{2p+1}L_{k,2p+1}}$

3 REFERENCES

[1] Sergio Falcon, On the k–Lucas Numbers, Int. J. Contemp. Math. Sciences, Vol.6, 2011, no. 21, 1039 – 1050

[2] Sergio Falcon, Onthe k–Lucas triangle and its relationship with the k–Lucas numbers, J. Math. Comput. Sci, 6 (2012), no. 3, 425 – 434

[3] S. Falcon, A. Plaza, Onthe Fibonacci k–numbers. Chaos, Solitons & Fractals(2006), doi:10.1016/j.chaos.2006.09.022.

[4] S. Falcon, A. Plaza, The k–Fibonacci sequence and the Pascal 2–triangle, Chaos,Solit. & Fract. 33(1) (2007) 38–49.

[5] S. Falcon, A. Plaza, On k–Fibonacci numbers of arithmetic indexes. Applied Mathematics and Computation, 208 (2009), 180–185

[6] Hogat V.E. Fibonacci and Lucas numbers, Palo Alto, C: Houghton-Mifflin, 1969.

[7] Horadam AF. A generalized Fibonacci sequence. Math Mag 1961;68: 455-9.

[8] El Naschie MS. On the cohomology and instantons number in E-infinity Cantorian space time. Chaos, Solitons & Fractals 2005;26: 13-7.

[9] El Naschie MS. On dimensions on Cantor sets related systems. Chaos, Solitons & Fractals 1993; 3: 675-85.

[10] Spinadel VW. The family of metallic means. Vis Math 1999;1(3). Available from: http://members.tripod.com/vismath/.