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# Approximation of homomorphisms and derivations of additive functional equation of n-Apollonius type in induced fuzzy Lie C*-algebras

[1]Ali Ebadian, [2] Rasoul Aghalary, [3] Mohammad Ali Abolfathi

[1] Department of Mathematics, Urmia University
P.O.Box 165, Urmia, Iran.
a.ebadian@urmia.ac.ir

[2] Department of Mathematics, Urmia University
P.O.Box 165, Urmia, Iran.
r.aghalary@urmia.ac.ir

[3] Department of Mathematics, Urmia University
P.O.Box 165, Urmia, Iran.
m.abolfathi@urmia.ac.ir

## ABSTRACT

Using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms and derivations of additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i) = -\frac{1}{n}\sum_{1\le i<j\le n} f(x_i+x_j) = nf(z-\frac{1}{n^2}\sum_{i=1}^{n} x_i)$$

for a fixed positive $n$ with $n \geq 2$ on induced fuzzy C*-algebras and induced fuzzy Lie C*-algebras.

**Keywords:** Hyers-Ulam -Rasias stability; fuzzy Banach $*-$algebra; induced fuzzy C*-algebra; induced fuzzy Lie C*-algebra; fixed point method.

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generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [6] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [12]. Some mathematics have define fuzzy normed on a vector form various point of view [15, 20, 25]. In particular, Bang and Samanta [2] following Cheng and Mordeson [4], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type [13]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

A classical equation in the theory of functional equations is the following: ”when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?”. If the problem accepts a solution, we say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [23] in 1940. In the next year Hyers [8] gave a first affirmative answer to the question of Ulam in context of Banach spaces. In 1978, Rassias [22] proved a generalization of the Hayers’ theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Gˇavrut¸a [7] provided a further generalization of Rassias’ theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ in by a general control function $\varphi(x,y)$. Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem [1, 9, 10, 11].

In the following, we will given some notations that are needed in this paper.

Let $X$ is a Banach algebra, then an involution on $x$ is a mapping $x \to x^*$ from $X$ into $X$ which satisfies

(1) $(x^*)^* = x$ for all $x \in X$;

(2) $(\alpha x+\beta y)^* = \overline{\alpha}x^* + \overline{\beta}y^*$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$;

(3) $(xy)^* = y^* x^*$ for all $x, y \in X$;

If, in addition, $\|xx^*\| = \|x\|^2$ for all $x \in X$, then $X$ is a C*-algebra.

**Definition 1.1.** Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on $X$ if for all $x, y \in X$ and all $t, s \in \mathbb{R}$

$(N1)$ $N(x,t) = 0$ for $t \le 0$;

$(N2)$ $N(x,t) = 1$ for all $t > 0$ if and only if $x = 0$;

$(N3)$ $N(cx,t) = N(x,\frac{t}{|c|})$ for each $c \ne 0$;

$(N4)$ $N(x+y,s+t) \ge \min\{N(x,t), N(y,s)\}$;

$(N5)$ $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

$(N6)$ for $x \ne 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair $(X,N)$ is called a fuzzy normed linear space.

On may regard $N(x,t)$ as the truth value of the statement ” the norm of $x$ is less than or equal to the real number $t$ ".

**Example 1.2.** Let $(X,\|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t)=\begin{cases}\dfrac{\alpha t}{\beta t+\|x\|}, & t>0, \quad x\in X,\\ 0, & t\le 0, \quad x\in X\end{cases}$$

is a fuzzy norm on $X$.

**Example 1.2.** Let $(X,\|.\|)$ be a normed linear space. Then

$$N(x,t)=\begin{cases}0, & t<0,\\ \dfrac{t}{\|x\|}, & 0<t\le\|x\|, \quad x\in X,\\ 0, & t>\|x\|, \quad x\in X\end{cases}$$

is a fuzzy norm on $X$.

Let $(X,N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In that case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty}x_n=x.$

A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\varepsilon>0$ and each $t>0$ there exists an $n_0\in N$ such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon.$

It is well-known that every convergent sequence in fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy normed is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f:X\to Y$ between fuzzy normed vector space $X,Y$ is continuous at point $x_0\in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$ If $f:X\to Y$ is continuous at each $x\in X$, then $f$ is said to be continuous on $X$ [3].

**Definition 1.4.** Let $X$ be a $*$-algebra and $(X,N)$ be a fuzzy normed space.

(1) The fuzzy normed space $(X,N)$ is called a fuzzy normed $*$-algebra if

$$N(xy,st)\ge N(x,t).N(y,s),$$
$$N(x^*,t)=N(x,t)$$

for all $x,y\in X$ and all positive real numbers $t$.

(2) A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

**Example 1.5.** Let $(X,\|.\|)$ be a normed $*$-algebra. Let

$$N(x,t)=\begin{cases}\dfrac{t}{t+\|x\|}, & t>0, \quad x\in X,\\ 0, & t\le 0, \quad x\in X.\end{cases}$$

Then $(X,N)$ is a fuzzy normed $*$-algebra.

**Definition 1.6.** Let $(X,\|.\|)$ be a normed C*-algebra and $N$ be a fuzzy norm on $X$..

(1)The fuzzy normed $*-$algebra $(X,N)$ is called an induced fuzzy normed $*-$algebra.

(2)The fuzzy Banach $*-$algebra $(X,N)$ is called an induced fuzzy C*-algebra.

Let $(X,N)$ and $(Y,N)$ be induced fuzzy normed $*-$algebras. Then a $\mathbb{C}$-linear mapping $f:(X,N)\to(Y,N)$ is called a fuzzy $*$-homomorphism if $f(xy)=f(x)f(y)$, $f(x^*)=f(x)^*$ and a $\mathbb{C}$-linear mapping $f:(X,N)\to(X,N)$ is called a fuzzy $*$-derivation if $f(xy)=f(x)y+xf(y)$, $f(x^*)=f(x)^*$ for all $x,y\in X$.

Let $X$ be a nonempty set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on

$X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$ for $x,y\in X$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Let be $(X,d)$ a generalized metric space. An operator $T:X\to X$ satisfies a Lipschitz condition with Lipschitz constant $L$, if there exists a constant $L\ge 0$ such that $d(Tx,Ty)\le Ld(x,y)$ for all $x,y\in X$ . If the Lipschitz constant $L$ is less than $1$, then the operator $T$ is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Diaz and Margolis.

**Theorem 1.7.** (see.[16, 21]) Let $(X,d)$ be a complete generalized metric space and $J:X\to X$ be a strictly contractive mapping with Lipshitz constant $L<1$. Then, for each given $x\in X$, either

$$d(J^n x, J^{n+1}x)=\infty \qquad \text{for all } n\ge 0$$

or there exists a natural number $n_0$ such that

(1) $d(J^n x, J^{n+1}x)<\infty$ for all $n\ge n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;

(3) $y^*$ is the unique fixed point of $J$ in the set $Y=\{y\in X: d(J^{n_0},y)<\infty\}$;

(4) $d(y,y^*)\le \frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

In 1996, Isac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors [5, 19].

In this paper we consider a mapping $f:X\to Y$ satisfying the following of additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i)=-\frac{1}{n}\sum_{1\le i<j\le n} f(x_i+x_j)=nf\Big(z-\frac{1}{n^2}\sum_{i=1}^{n}x_i\Big) \tag{1.1}$$

for all $z,x_1,x_2,...,x_n\in X$, which $n$ is fixed positive integer with $n\ge 2$ and establish the homomorphisms and derivations of functional equation (1.1) on induced fuzzy C*-algebras and induced fuzzy Lie C*-algebras. Throughout this article, assume that $(X,N)$ is a fuzzy Banach $*-$algebra and that $(Y,N)$ is an induced fuzzy C*-algebra.

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## 2. Approximate fuzzy -homomorphisms in fuzzy Banach C*-algebras

In this section, we prove the Hyers- Ulam stability of homomorphisms on fuzzy Banach $*-$algebra s related to additive functional equation of n-Apollonius type.

**Theorem 2.1.** Let $\varphi : X^{n+1} \to [0,\infty)$ be a function such that there exists an $L < \frac{n^2-1}{n^2}$ with

$$\varphi\left(\frac{n^2-1}{n^2}z, \frac{n^2-1}{n^2}x_1, \frac{n^2-1}{n^2}x_2, \ldots, \frac{n^2-1}{n^2}x_n\right) \le \frac{n^2-1}{n^2}L\varphi(z, x_1, x_2, \ldots, x_n) \qquad (2.1)$$

for all $z, x_1, x_2, \ldots, x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0)=0$ such that

$$N\left(\sum_{i=1}^{n}\mu f(z-x_i) + \frac{1}{n}\sum_{1\le i<j\le n} f(\mu x_i + \mu x_j) - nf(\mu z - \frac{1}{n^2}\sum_{i=1}^{n}\mu x_i), t\right) \ge \frac{t}{t+\varphi(z, x_1, x_2, \ldots, x_n)} \qquad (2.2)$$

$$N\left(f(xy) - f(x)f(y), t\right) \ge \frac{t}{t+\varphi(x, y, 0, \ldots, 0)} \qquad (2.3)$$

$$N\left(f(x^*) - f(x)^*, t\right) \ge \frac{t}{t+\varphi(x, 0, 0, \ldots, 0)} \qquad (2.4)$$

for all $x, y, z, x_1, x_2, \ldots, x_n \in X$, all $\mu \in \mathbb{T}^1 := \{u \in C : |u| = 1\}$ and all $t > 0$.

Then $H(x) = N - \lim_{k\to\infty}(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)$ exists for each $x \in X$, and defines a unique fuzzy $*$-homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \ge \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, 0, \ldots, \underbrace{x}_{jth}, 0, 0, \ldots 0)} \qquad (2.5)$$

for all $x \in X$ and all $t > 0$.

**Proof.** Consider the set $\Omega := \{g : X \to Y, g(0) = 0\}$ and introduce the generalized metric

$$d(g,h) = \inf\{\eta \in R^+ : N(g(x) - h(x), \eta t) \ge \frac{t}{t+\varphi(x, 0, 0, \ldots, \underbrace{x}_{jth}, 0, 0, \ldots.0)}$$

where $\inf \phi = +\infty$. The proof of the fact $(\Omega, d)$ is a complete generalized metric space can befound in [5].

Now we consider the mapping $J : \Omega \to \Omega$ defined by $Jg(x) = \frac{n^2}{n^2-1}g(\frac{n^2-1}{n^2}x)$ for all $g \in \Omega$ and $x \in X$. Let $\varepsilon > 0$ and $f, g \in \Omega$ be given such that $d(g, f) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t+\varphi(x, 0, 0, \ldots, \underbrace{x}_{jth}, 0, 0, \ldots 0)}$$

for all $x \in X$ and all $t > 0$. Hence

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$$N(Jg(x)-Jh(x),L\varepsilon t)=N(\frac{n^2}{n^2-1}g(\frac{n^2-1}{n^2}x)-\frac{n^2}{n^2-1}h(\frac{n^2-1}{n^2}x),L\varepsilon t)$$

$$=N(g(\frac{n^2-1}{n^2}x)-h(\frac{n^2-1}{n^2}x),\frac{n^2-1}{n^2}L\varepsilon t)$$

$$\geq\frac{\frac{n^2-1}{n^2}Lt}{\frac{n^2-1}{n^2}Lt+\varphi(\frac{n^2-1}{n^2}x,0,0,...,\underbrace{\frac{n^2-1}{n^2}x}_{jth},0,0,...,0)}$$

$$\geq\frac{\frac{n^2-1}{n^2}Lt}{\frac{n^2-1}{n^2}Lt+\frac{n^2-1}{n^2}L\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...,0)}$$

$$=\frac{t}{t+\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...,0)}.$$

So $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)=L\varepsilon$, for all $g,h\in\Omega$. Letting $\mu=1$ and $z=x_j=x$ for each $1\leq k\leq n$ with $k\neq j, x_k=0$ in (2.2), we have

$$N(\frac{n^2-1}{n}f(x)-nf(\frac{n^2-1}{n^2}x),t)\geq\frac{t}{t+\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...,0)} \qquad (2.6)$$

for all $x\in X$ and all $t>0$. It follows from (2.6) that $d(f,Jf)=\frac{n}{n^2-1}$. By Theorem 1.7 there exists a mapping $H:X\to Y$ such that the following holds:

(1) $H$ is a fixed point of $J$, that is,

$$H(\frac{n^2-1}{n^2}x)=\frac{n^2-1}{n^2}H(x) \qquad (2.7)$$

for all $x\in X$. The mapping $H$ is a unique fixed point of $J$ in the set $\Delta=\{h\in\Omega:d(g,h)<\infty\}$. This implies that $H$ is a unique mapping satisfying (2.7) such that there exists $\eta\in(0,\infty)$ satisfying

$$N(f(x)-H(x),\eta t)\geq\frac{t}{t+\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...0)},$$

for all $x\in X$ and all $t>0$.

(2) $d(J^k f,H)\to 0$ as $k\to\infty$. This implies the equality

$$N-\lim_{k\to\infty}\frac{n^{2k}}{(n^2-1)^k}f((\frac{n^2-1}{n^2})^k x)=H(x)$$

exists for each $x\in X$,

(3) $d(f,H)\le \frac{1}{1-L}d(f,Jf)$, which implies inequality

$$d(f,H)\le \frac{1}{\frac{n^2-1}{n}-\frac{n^2-1}{n}L}$$

and so

$$N(f(x)-H(x),t)\ge \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t+n\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...0)}.$$

It follows from (2.1) and (2.2)that

$$N\left(\sum_{i=1}^{n}\mu H(z-x_i)+\frac{1}{n}\sum_{1\le i<j\le n}H(\mu x_i+\mu x_j)-nH(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i),t\right)$$

$$=N-\lim_{k\to\infty}\left((\frac{n^2}{n^2-1})^k\sum_{i=1}^{n}\mu f((\frac{n^2-1}{n^2})^k(z-x_i))\right.$$

$$\left.+\frac{1}{n}\sum_{1\le i<j\le n}f((\frac{n^2-1}{n^2})^k(\mu x_i+\mu x_j))-n(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k(z-\frac{1}{n^2}\sum_{i=1}^{n}x_i)),t\right)$$

$$\ge \lim_{k\to\infty}\frac{(\frac{n^2-1}{n^2})^k t}{(\frac{n^2-1}{n^2})^k t+\varphi((\frac{n^2-1}{n^2})^k z,(\frac{n^2-1}{n^2})^k x_1,...,(\frac{n^2-1}{n^2})^k x_n)}$$

$$\ge \lim_{k\to\infty}\frac{(\frac{n^2-1}{n^2})^k t}{t+(\frac{n^2}{n^2-1})^k\varphi((\frac{n^2-1}{n^2})^k z,(\frac{n^2-1}{n^2})^k x_1,...,(\frac{n^2-1}{n^2})^k x_n)}$$

$$\ge \lim_{k\to\infty}\frac{t}{t+L^k\varphi(z,x_1,...,x_n)}\to 1$$

for all $z,x_1,x_2,...,x_n\in X$, $t\ge 0$ and $\mu\in\mathbb{T}^1$. Thus

$$\sum_{i=1}^{n}\mu H(z-x_i)=-\frac{1}{n}\sum_{1\le i<j\le n}H(\mu x_i+\mu x_j)+nH(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i) \qquad (2.8)$$

for all $z,x_1,x_2,...,x_n\in X$, all $t\ge 0$ and all $\mu\in\mathbb{T}^1$. By [18] $H:X\to Y$ is Cauchy additive, that is, $H(x+y)=H(x)+H(y)$ for all $x,y\in X$. By a Similar method to the proof of [16], one can show that the mapping is $\mathbb{C}$-linear.

By (2.3) we have

$$N((\frac{n^2}{n^2-1})^{2k}f((\frac{n^2-1}{n^2})^{2k}xy)-(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k y),t)$$

$$\geq \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + \varphi((\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, 0, \ldots, 0)}$$

$$\geq \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, 0, \ldots, 0)}$$

for all $x, y \in X$ and all $t > 0$. Since

$$\lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, 0, \ldots, 0)} = 1$$

for all $x, y \in X$ and $t > 0$, hence

$$H(xy) = H(x)H(y)$$

By (2.4), we have

$$N((\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x^*) - (\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)^*, t)$$

$$\geq \frac{(\frac{n^2-1}{n^2})^k t}{(\frac{n^2-1}{n^2})^k t + \varphi((\frac{n^2-1}{n^2})^k x, 0, 0, \ldots, 0)}$$

$$\geq \frac{t}{t + L^k \varphi(x, y, 0, \ldots, 0)}$$

for all $x, y \in X$ and $t > 0$. Since

$$\lim_{k\to\infty} \frac{t}{t + L^k \varphi(x, 0, 0, \ldots, 0)} = 1$$

for all $x, y \in X$ and all $t > 0$, hence

$$H(x^*) = H(x)^*.$$

**Corollary 2.2.** Let $X$ be a normed vector space with norm $\|.\|, \delta \geq 0$ and $p$ be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying

$$N\left(\sum_{i=1}^{n} \mu f(z - x_i) + \frac{1}{n} \sum_{1\leq i<j\leq n} f(\mu x_i + \mu x_j) - nf(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i), t\right)$$

$$\geq \frac{t}{t + \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)} \tag{2.9}$$

$$N\left(f(xy)-f(x)f(y),t\right) \geq \frac{t}{t+\delta(\|x\|^p+\|y\|^p)} \qquad (2.10)$$

$$N\left(f(x^*)-f(x)^*,t\right) \geq \frac{t}{t+\delta\|x\|^p} \qquad (2.11)$$

for all $z, x_1, x_2, ..., x_n \in X$, all $t \geq 0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique fuzzy $*-$homomorphism $H : X \to Y$ such that

$$N(f(x)-H(x),t) \geq \frac{(n^2-1)^{1-p}-n^{2(1-p)}t}{(n^2-1)^{1-p}-n^{2(1-p)}t+2n\delta(n^2-1)^{-p}\|x\|^p}$$

for all $x \in X$ and all $t > 0$,

**Proof.** The proof follows from Theorem 2.1 by taking

$$\phi(z, x_1, x_2, ..., x_n) := \delta(\|z\|^p + \sum_{i=1}^{n}\|x_i\|^p)$$

for all $z, x_1, x_2, ..., x_n \in X$, all $t \geq 0$ and all $\mu \in \mathbb{T}^1$. It follows from (2.9) that $f(0)=0$, we can choose $L = (\frac{n^2}{n^2-1})^{1-p}$ to get the desired result.

**Theorem 2.3.** Let $\varphi : X^{n+1} \to [0,\infty)$ be a function such that there exists an $L < 1$ such that

$$\varphi\left(\frac{n^2}{n^2-1}z, \frac{n^2}{n^2-1}x_1, \frac{n^2}{n^2-1}x_2, ..., \frac{n^2}{n^2-1}x_n\right) \leq \frac{n^2}{n^2-1}L\varphi(z, x_1, x_2, ..., x_n) \qquad (2.12)$$

for all $z, x_1, x_2, ..., x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0)=0$ and (2.2), (2.3), (2.4). Then $H(x) = N - \lim_{k\to\infty}(\frac{n^2-1}{n^2})^k f((\frac{n^2}{n^2-1})^k x)$ exists for each $x \in X$, and defines a unique fuzzy $*$- homomorphism $H : X \to Y$ such that

$$N(f(x)-H(x),t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t+n\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...0)} \qquad (2.13)$$

for all $x \in X$ and all $t > 0$.

**Proof.** Let $(\Omega, d)$ be the generalized metric space in the proof of Theorem 2.1. Consider the mapping $J : \Omega \to \Omega$ defined by $Jg(x) = \frac{n^2-1}{n^2}g(\frac{n^2}{n^2-1}x)$ for all $g \in \Omega$ and $x \in X$. We can conclude that $J$ is a strictly contractive self mapping of with the Lipschitz constant $L$. Replacing $x$ by $\frac{n^2}{n^2-1}x$ in (2.6), we obtain

$$N(\frac{n^2-1}{n^2}f(\frac{n^2}{n^2-1}x)-f(x),t) \geq \frac{nt}{nt+\varphi(\frac{n^2}{n^2-1}x,0,0,...,\underbrace{\frac{n^2}{n^2-1}x}_{jth},0,0,...,0)}$$

$$\geq \frac{nt}{nt + \frac{n^2}{n^2-1} L\varphi(x, 0, 0, ..., \underbrace{x}_{jth}, 0, 0, ..., 0)} \quad (2.14)$$

It follows that $d(f, Jf) \leq \frac{nL}{n^2-1}$.

By Theorem 2.1, there exists a mapping $H : X \to Y$ satisfying

(1) $H$ is a fixed point of $J$ , that is,

$$H(\frac{n^2}{n^2-1}x) = \frac{n^2}{n^2-1}H(x) \quad (2.15)$$

for all $x \in X$. The mapping $H$ is a unique fixed point of $J$ in the set $\Delta = \{h \in \Omega : d(g,h) < \infty\}$. This implies that $H$ is a unique mapping satisfying (2.15) such that there exists $\eta \in (0,\infty)$ satisfying

$$N(f(x) - H(x), \eta t) \geq \frac{t}{t + \varphi(x, 0, 0, ..., \underbrace{x}_{jth}, 0, 0, ...0)}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^k f, H) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} (\frac{n^2-1}{n^2})^k f((\frac{n^2}{n^2-1})^k x) = H(x)$$

exists for each $x \in X$ ,

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies inequality

$$d(f, H) \leq \frac{nL}{(n^2-1) - (n^2-1)L}$$

The rest the proof is similar to the Theorem 2.1.

**Corollary 2.4.** Let $X$ be a normed vector space with norm $\|.\|, \delta \geq 0$ and $p$ be a real number with $p < 1$. Let $f : X \to Y$ be a mapping satisfying (2.9), (2.10) and (2.11). Then there exists a unique fuzzy $*-$homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{(n^2-1)^{p-1} - n^{2(p-1)}t}{(n^2-1)^{p-1} - n^{2(p-1)}t + 2n^{2p-1}\delta(n^2-1)^{-1}\|x\|^p}$$

for all $x \in X$ and all $t > 0$,

**Proof.** The proof follows from Theorem 2.3 by taking

$$\varphi(z, x_1, x_2, ..., x_n) := \delta(\|z\|^p + \sum_{i=1}^{n}\|x_i\|^p)$$

for all $z, x_1, x_2, ..., x_n \in X$, all $t \geq 0$ and all $\mu \in \mathbb{T}^1$. It follows from (2.9) that $f(0) = 0$, we can choose $L = (\frac{n^2}{n^2-1})^{p-1}$ to get the desired result.

From now on, assume that $X$ is a unital C*-algebra with unite e and a unitary group $U(X) := \{u \in X : u^* u = uu^* = e\}$ and that $Y$ is a unital C*-algebra.

**Theorem 2.5.** Let $\varphi : X^{n+1} \to [0,\infty)$ be a function such that there exists an $L < \frac{n^2-1}{n^2}$ satisfying (2.1) and let $f : X \to Y$ be a mapping satisfying $f(0)=0$, (2.2) and such that

$$N\left(f(uv)-f(u)f(v),t\right) \geq \frac{t}{t+\varphi(u,v,0,\ldots,0)} \qquad (2.16)$$

$$N\left(f(u^*)-f(u)^*,t\right) \geq \frac{t}{t+\varphi(u,0,0,\ldots,0)} \qquad (2.17)$$

for all $u,v \in U(X)$ and all $t>0$. Then there exists a unique fuzzy $*$- homomorphism $H : X \to Y$ satisfying (2.5).

**proof**. By Theorem 2.1 there is a $\mathbb{C}$-linear mapping ($N-\lim_{k\to\infty}(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x) = \mathrm{H}(x)$) satisfying (2.5).

By (2.1) and (2.16)

$$N\left((\frac{n^2}{n^2-1})^{2k} f((\frac{n^2-1}{n^2})^{2k} uv) - (\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k u)(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k y),t\right)$$

$$\geq \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + \varphi((\frac{n^2-1}{n^2})^k u, (\frac{n^2-1}{n^2})^k v, 0,\ldots,0)}$$

$$\geq \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(u,v,0,\ldots,0)}$$

for all $x,y \in X$ and $t>0$. Since

$$\lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(u,v,0,\ldots,0)} = 1$$

for all $u,v \in U(X)$ and $t>0$, hence

$$H(uv) = H(u)H(v). \qquad (2.18)$$

Since $H$ is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, that is, $x = \sum_{i=1}^{n} \alpha_i u_i$ for $\alpha_i \in \mathbb{C}$ and $u_i \in U(X)$. It follows from (2.18) that

$$H(xv) = H(\sum_{i=1}^{n} \alpha_i u_i v) = \sum_{i=1}^{n} H(\alpha_i u_i v) = \sum_{i=1}^{n} H(\alpha_i u_i)H(v) = H(\sum_{i=1}^{n} \alpha_i u_i)H(v) = \mathrm{H(x)H}(v)$$

for all $v \in U(X)$. Similarly, one can obtain that $H(xy) = H(x)H(y)$ for all $x,y \in X$.

By (2.1) and (2.17)

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$$N\left(\left(\frac{n^2}{n^2-1}\right)^k f\left(\left(\frac{n^2-1}{n^2}\right)^k u^*\right)-\left(\frac{n^2}{n^2-1}\right)^k f\left(\left(\frac{n^2-1}{n^2}\right)^k u\right)^*, t\right)$$

$$\geq \frac{\left(\frac{n^2-1}{n^2}\right)^k t}{\left(\frac{n^2-1}{n^2}\right)^k t+\varphi\left(\left(\frac{n^2-1}{n^2}\right)^k u, 0, 0, \ldots, 0\right)}$$

$$\geq \frac{t}{t+L^k\varphi(u,0,0,\ldots,0)}$$

for all $u \in U(X)$ and $t > 0$. Since

$$\lim_{k\to\infty}\frac{t}{t+L^k\varphi(u,0,\ldots,0)}=1$$

for all $u \in U(X)$ and all $t > 0$, hence

$$H(u^*) = H(u)^*. \qquad (2.19)$$

Since $H$ is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, that is, $x=\sum_{i=1}^{n}\alpha_i u_i$ for $\alpha_i \in \mathbb{C}$ and $u_i \in U(X)$. It follows from (2.19) that

$$H(x^*) = H\left(\sum_{i=1}^{n}\overline{\alpha_i} u_i^*\right)=\sum_{i=1}^{n}\overline{\alpha_i} H(u_i^*)=\sum_{i=1}^{n}\overline{\alpha_i} H(u_i)^* = H\left(\sum_{i=1}^{n}\alpha_i u_i\right)^* = H(x)^*$$

for all $x \in X$. Therefore the mapping $H: X \to Y$ is a $*$- homomorphism. Similarly, we have the following. We will omit the proof.

**Theorem 2.6.** Let $\varphi: X^{n+1} \to [0,\infty)$ be a function such that there exists an $L<1$ satisfying (2.1) and let $f: X \to Y$ be a mapping satisfying $f(0)=0$, (2.2) ,(2.16) and (2.17). Then there exists a unique fuzzy $*$-homomorphism $H: X \to Y$ satisfying (2.13).

## 3. Approximate fuzzy $*$-derivations in fuzzy Banach C*-algebras

In this section, we prove the Hyers- Ulam stability of derivations on fuzzy Banach $*$−algebras related to additive functional equation of n-Apollonius type.

**Theorem 3.1.** Let $\varphi: X^{n+1} \to [0,\infty)$ be a function such that there exists an $L<\frac{n^2-1}{n^2}$ with

$$\varphi\left(\frac{n^2-1}{n^2}z, \frac{n^2-1}{n^2}x_1, \frac{n^2-1}{n^2}x_2, \ldots, \frac{n^2-1}{n^2}x_n\right) \leq \frac{n^2-1}{n^2}L\varphi(z, x_1, x_2, \ldots, x_n) \qquad (3.1)$$

for all $z, x_1, x_2, \ldots, x_n \in X$. Let $f: X \to X$ be a mapping satisfying $f(0)=0$ such that

$$N\left(\sum_{i=1}^{n}\mu f(z-x_i)+\frac{1}{n}\sum_{1\leq i<j\leq n} f(\mu x_i+\mu x_j)-nf\left(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i\right), t\right) \geq \frac{t}{t+\varphi(z, x_1, x_2, \ldots, x_n)} \qquad (3.2)$$

$$N\left(f(xy)-f(x)y-xf(y), t\right) \geq \frac{t}{t+\phi(x, y, 0, \ldots, 0)} \qquad (3.3)$$

$$N\left(f(x^*)-f(x)^*,t\right) \geq \frac{t}{t+\varphi(x,0,0,...,0)} \qquad (3.4)$$

for all $x, y, z, x_1, x_2,..., x_n \in X$, all $\mu \in \mathrm{T}^1 := \{u \in C : |u| = 1\}$ and all $t > 0$.

Then $D(x) = N - \lim_{k\to\infty} \frac{n^{2k}}{(n^2-1)k} f\left(\left(\frac{n^2-1}{n^2}\right)^k x\right)$ exists for each $x \in X$, and defines a unique fuzzy $*$-derivation $D : X \to Y$ such that

$$N(f(x)-D(x),t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x,0,0,..., \underbrace{x}_{jth}, 0,0,...0)} \qquad (3.5)$$

for all $x \in X$ and all $t > 0$.

**Proof**. The proof is similar to the proof of Theorem 2.1.

**Theorem 3.2.** Let $\varphi : X^{n+1} \to [0,\infty)$ be a function such that there exists an $L < 1$ such that

$$\varphi\left(\frac{n^2}{n^2-1}z, \frac{n^2}{n^2-1}x_1, \frac{n^2}{n^2-1}x_2, ..., \frac{n^2}{n^2-1}x_n\right) \leq \frac{n^2}{n^2-1} L\varphi(z, x_1, x_2, ..., x_n) \qquad (3.6)$$

for all $z, x_1, x_2,..., x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2), (3.3), (3.4). Then $D(x) = N - \lim_{k\to\infty} \frac{n^{2k}}{(n^2-1)k} f\left(\left(\frac{n^2-1}{n^2}\right)^k x\right)$ exists for each $x \in X$, and defines a unique fuzzy $*$-derivation $D : X \to Y$ such that

$$N(f(x)-D(x),t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + nL\varphi(x,0,0,..., \underbrace{x}_{jth}, 0,0,...0)} \qquad (3.7)$$

for all $x \in X$ and all $t > 0$.

## 4. Approximate of homomorphisms and derivations in induced fuzzy Lie C*-algebras

A induced fuzzy C*-algebra $\Im$ endowed with the Lie product

$$[x, y] = \frac{xy - yx}{2}$$

on $\Im$, is called a induced fuzzy Lie C*-algebra.

**Definition 4.1.** Let $(X, N)$ and $(Y, N)$ be induced fuzzy Lie C*-algebras. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a Lie C*-algebra homomorphism if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in X$.

Throughout this section, we assume that $(X, N)$ and $(Y, N)$ are induced fuzzy Lie C*-algebras. We prove the generalized Hyers-Ulam stability of homomorphisms in induced fuzzy Lie C*-algebras for the functional equation (1.1).

**Theorem 4.2.** Let $\varphi : X^{n+1} \to [0,\infty)$ be a function such that there exists an $L < \frac{n^2-1}{n^2}$ with condition (2.1).

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that (2.2) and (2.4) hold, and

$$N\left(f([x, y]) - [f(x), f(y)], t\right) \geq \frac{t}{t + \varphi(x, y, 0,...,0)} \qquad (4.1)$$

for all $x, y \in X$ and all $t > 0.$

Then exists a unique homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, 0, \ldots, \underbrace{x}_{jth}, 0, 0, \ldots 0)} \tag{4.2}$$

for all $x \in X$ and all $t > 0.$

**Proof.** By the same reasoning as in the proof of Theorem 2.1, we can fined the mapping $H(x) = N - \lim_{k\to\infty} (\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)$ exists for each $x \in X$. It follows from (2.1) and (4.1) that

$$N(H([x, y]) - [H(X), H(y)], t) = N - \lim_{k\to\infty} \left( (\frac{n^2}{n^2-1})^{2k} f([(\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y]) \right.$$

$$\left. -(\frac{n^2}{n^2-1})^k [f((\frac{n^2-1}{n^2})^k x), f((\frac{n^2-1}{n^2})^k y)], t \right)$$

$$\geq \lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + \varphi((\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, 0, 0, \ldots, 0)}$$

$$\geq \lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, 0, \ldots, 0)}$$

for all $x, y \in X$ and $t > 0$. Since

$$\lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, 0, \ldots, 0)} = 1$$

for all $x, y \in X$ and $t > 0$, hence

$$H([x, y]) = [H(x), H(y)].$$

for all $x, y \in X$ and $t > 0$. Thus $H$ is a Lie homomorphism satisfying (4.2), as desired.

**Corollary 4.3.** Let $X$ be a normed vector space with norm $\|.\|$, $\delta \geq 0$ and $p$ be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying

$$N\left( \sum_{i=1}^{n} \mu f(z - x_i) + \frac{1}{n} \sum_{1\leq i<j\leq n} f(\mu x_i + \mu x_j) - nf(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i), t \right)$$

$$\geq \frac{t}{t + \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)} \tag{4.3}$$

$$N\left(f(xy)-f(x)f(y),t\right)\geq\frac{t}{t+\delta(\|x\|^{p}+\|y\|^{p})}$$

$$N\left(f(x^{*})-f(x)^{*},t\right)\geq\frac{t}{t+\delta\|x\|^{p}}$$

for all $z,x_1,x_2,...,x_n\in X$, all $t\geq 0$ and all $\mu\in\mathbb{T}^1$. Then there exists a unique Lie homomorphism $H:X\to Y$ such that

$$N(f(x)-H(x),t)\geq\frac{(n^2-1)^{1-p}-n^{2(1-p)}t}{(n^2-1)^{1-p}-n^{2(1-p)}t+2n\delta(n^2-1)^{-p}\|x\|^{p}}$$

for all $x\in X$ and all $t>0$,

**Proof.** The proof follows from Theorem 4.2.

**Definition 4.4.** Let $(X,N)$ and $(Y,N)$ be induced fuzzy Lie C*-algebras. A $\mathbb{C}$-linear mapping $D:X\to Y$ is called a Lie C*-algebra derivation if $D([x,y])=[D(x),y]+[x,D(y)]$ for all $x,y\in X$.

**Theorem 4.5.** Let $\varphi:X^{n+1}\to[0,\infty)$ be a function such that there exists an $L<\frac{n^2-1}{n^2}$ with condition (3.1).

Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ such that (2.2) and (2.4) hold, and

$$N\left(f([x,y])-[f(x),y]+[x,f(y)],t\right)\geq\frac{t}{t+\varphi(x,y,0,...,0)}\tag{4.4}$$

for all $x,y\in X$ and all $t>0$.

Then exists a unique Lie derivation $D:X\to X$ such that

$$N(f(x)-D(x),t)\geq\frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t+n\varphi(x,0,0,...,\underbrace{x}_{jth},0,0,...0)}\tag{4.5}$$

for all $x\in X$ and all $t>0$.

**Proof.** By the same reasoning as in the proof of Theorem 3.1, we can fined the mapping $D(x)=N-\lim_{k\to\infty}\frac{n^{2k}}{(n^2-1)k}f((\frac{n^2-1}{n^2})^k x)$ exists for each $x\in X$. It follows from (3.1) and (4.4) that

$$N\left(f([x,y])-[f(x),y]+[x,f(y)],t\right)=N-\lim_{k\to\infty}((\frac{n^2}{n^2-1})^{2k}f([(\frac{n^2-1}{n^2})^k x,(\frac{n^2-1}{n^2})^k y])$$

$$-(\frac{n^2}{n^2-1})^{2k}[f((\frac{n^2-1}{n^2})^k x),(\frac{n^2-1}{n^2})^k y])-(\frac{n^2}{n^2-1})^{2k}[(\frac{n^2-1}{n^2})^k x,f((\frac{n^2-1}{n^2})^k y)],t)$$

$$\geq\lim_{k\to\infty}\frac{(\frac{n^2-1}{n^2})^{2k}t}{(\frac{n^2-1}{n^2})^{2k}t+\varphi((\frac{n^2-1}{n^2})^k x,(\frac{n^2-1}{n^2})^k y,0,0,...,0)}$$

$$\geq \lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^{k} L^k \varphi(x,y,0,...,0)}$$

for all $x, y \in X$ and $t > 0$. Since

$$\lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{2k} t}{(\frac{n^2-1}{n^2})^{2k} t + (\frac{n^2-1}{n^2})^{k} L^k \varphi(x,y,0,...,0)} = 1$$

for all $x, y \in X$ and $t > 0$, hence

$$D([x,y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in X$. Thus $D$ is a Lie derivation satisfying (4.5), as desired.

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