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Functional and Operator Variants of Jensen's, Chord's, and Mercer's Inequality

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ABSTRACT

The paper examines functional and operator variants of Jensen's, chord's, and Mercer's inequality with a convex function on the interval of real numbers. In this research we relied on general functional and operator forms of Jensen's inequality. Among others, corresponding means are also observed. The inequalities for operators are observed without operator convexity.

Keywords and phrases

convex combination; convex function; Jensen's inequality; functional convex combination; operator convex combination

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1. INTRODUCTION

Through this paper $I \subseteq \mathbb{R}$ will be a non-degenerate interval, $[a,b] \subset \mathbb{R}$ will be a non-degenerate closed segment, and $<a,b> \subset \mathbb{R}$ will be a non-degenerate open segment. If $x_i \in I$ are numbers, and $p_i \in [0,1]$ are coefficients such that $\sum_{i=1}^{n} p_i = 1$, then the sum $\sum_{i=1}^{n} p_i x_i$ belongs to I, and it is called the convex combination on I. For a continuous function $f: I \to \mathbb{R}$ the convex combination $\sum_{i=1}^{n} p_i f(x_i)$ belongs to $f(I)$. A convex hull of a set X will be denoted by $\text{co}X$.

If $f: I \to \mathbb{R}$ is a function, then the chord line joining the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f with $a < b$ will be denoted by $f_{[a,b]}^{\text{cho}}$, that is,

$$f_{[a,b]}^{\text{cho}}(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b). \tag{1.1}$$

Theorem A. 1 (The Jensen-chord-Mercer inequality) *Let* $\sum_{i=1}^{n} p_i x_i$ *be a convex combination on* $[a,b]$*.*

Then every convex function $f:[a,b] \to \mathbb{R}$ *verifies the series of inequalities*

$$\begin{aligned} f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) &\le \sum_{i=1}^{n} p_i f(a+b-x_i) \\ &\le f(a)+f(b)-\sum_{i=1}^{n} p_i f_{[a,b]}^{\text{cho}}(x_i). \\ &\le f(a)+f(b)-\sum_{i=1}^{n} p_i f(x_i) \end{aligned} \tag{1.2}$$

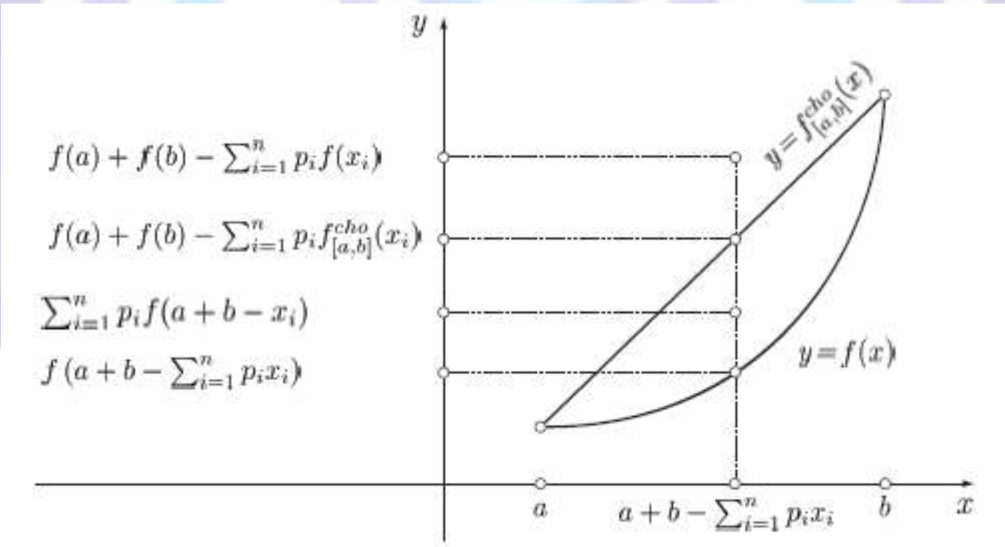

Figure 1. Graphical presentation of the inequality in (1.2)

The inequality in (1.2) is possible because the expression

$$a+b-\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i(a+b) - \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i(a+b-x_i)$$

is the convex combination on $[a,b]$. Namely, since $x_i \in [a,b]$, then $a+b-x_i \in [a,b]$. Our aim is to expand the series of inequalities in (1.2) with more points. The third inequality in (1.2) is Mercer's inequality which was derived in [4] by using Jensen's inequality. It can be derived using only the chord line. If $f:[a,b] \to \mathbb{R}$ is a convex function, and

$\sum_{i=1}^{n} p_i x_i$ is a convex combination on $[a,b]$, then using the chord as the affine function $h(x) = f_{[a,b]}^{\text{cho}}(x) = \alpha x + \beta$, it follows

$$f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) \le h\left(a+b-\sum_{i=1}^{n} p_i x_i\right) = h(a)+h(b)-\sum_{i=1}^{n} p_i h(x_i)$$
$$\le f(a)+f(b)-\sum_{i=1}^{n} p_i f(x_i).$$

Lemma 1.1. 2 *Let $x_{ij}, y_k \in \mathbb{R}$ be numbers, and $p_i \in \mathbb{R}$ be coefficients such that $\sum_{i=1}^{n} p_i = 1$.*

Then every affine function $h:\mathbb{R}\to\mathbb{R}$ verifies the equality

$$h\left(\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij}\right) = \sum_{k=1}^{m+1} h(y_k) - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i h(x_{ij}). \quad (1.3)$$

The equality in (1.3) can be adapted to a convex function $f: I \to \mathbb{R}$ as the inequality

$$f\left(\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij}\right) \le \sum_{k=1}^{m+1} f(y_k) - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i f(x_{ij}) \quad (1.4)$$

with coefficients $p_i \in [0,1]$, numbers $x_{ij}, y_k \in I$ so that $y_k \in I \setminus \langle a,b \rangle$ where $[a,b] = \text{co}\{x_{ij}\}$, and with the requirement that the number $\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij}$ belongs to $[a,b]$. This inequality can be easily verified using the chord line $h(x) = f_{[a,b]}^{\text{cho}}(x) = \alpha x + \beta$, and applying the equality in (1.3).

The inequality in (1.4) with a convex continuous function f was obtained as the main result in [7, Theorem 2.1] using the majorization assumptions

$$(x_{i1},\ldots,x_{im+1}) \prec (y_1,\ldots,y_{m+1}) \ \text{for every}\ i = 1,\ldots,n$$

where $x_{im+1} = \sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m} x_{ij}$, instead of requires that $y_k \in I \setminus \langle a,b \rangle$ and $\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij} \in [a,b]$.

Using the Jensen inequality and the chord line, the inequality in (1.4) can be refined to the series of inequalities as it follows:

Theorem 1.2. 3 *Let $x_{ij}, y_k \in I$ be numbers so that $y_k \in I \setminus \langle a,b \rangle$ where $[a,b] = \text{co}\{x_{ij}\}$. Let the numbers $x_i = \sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m} x_{ij}$ belong to $[a,b]$. Let $p_i \in [0,1]$ be coefficients such that $\sum_{i=1}^{n} p_i = 1$.*

Then every convex function $f: I \to \mathbb{R}$ verifies the series of inequalities

$$f\left(\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij}\right) \le \sum_{i=1}^{n} p_i f\left(\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m} x_{ij}\right)$$
$$\le \sum_{k=1}^{m+1} f_{[a,b]}^{\text{cho}}(y_k) - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i f_{[a,b]}^{\text{cho}}(x_{ij}). \quad (1.5)$$
$$\le \sum_{k=1}^{m+1} f(y_k) - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i f(x_{ij})$$

Proof. The proof can be done by applying Jensen's inequality to the convex combination

$$\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij} = \sum_{k=1}^{m+1}\sum_{i=1}^{n} p_i y_k - \sum_{j=1}^{m}\sum_{i=1}^{n} p_i x_{ij} = \sum_{i=1}^{n} p_i \left(\sum_{k=1}^{m+1} y_k - \sum_{j=1}^{m} x_{ij} \right)$$

which belongs to $[a,b]$, and then using the chord line $h(x) = f^{\text{cho}}_{[a,b]}(x)$. □

2. FUNCTIONAL VARIANTS OF INEQUALITIES

Let X be a non-empty set, and $\mathcal{X}$ be a real vector space of functions $g : X \to \mathbb{R}$. A linear functional $\mathrm{P} : \mathcal{X} \to \mathbb{R}$ is positive (non-negative) or monotone if $\mathrm{P}(g) \geq 0$ for every non-negative function $g \in \mathcal{X}$. If a space $\mathcal{X}$ contains a unit function $\mathbf{1}$, by definition $\mathbf{1}(x) = 1$ for every $x \in X$, and $\mathrm{P}(\mathbf{1}) = 1$, we say that the functional P is unital or normalized.

Let $\mathrm{P} : \mathcal{X} \to \mathbb{R}$ be a positive linear functional, $w \in \mathcal{X}$ be a non-negative function such that $\mathrm{P}(w) = 1$, and $g : X \to I$ be a function where I is a closed interval. If $wg \in \mathcal{X}$, then $\mathrm{P}(wg) \in I$. If the interval I is not closed, then it can happen that $\mathrm{P}(wg) \notin I$. The following example shows such an undesirable situation:

Example 2.1. 4 *Let* $X = I = \langle 0,1]$ *and*

$$\mathcal{X} = \left\{ g : I \to \mathbb{R} \mid \lim_{x \to 0+} g(x) \text{ is finite} \right\}.$$

If $\mathrm{P} : \mathcal{X} \to \mathbb{R}$ *is defined by*

$$\mathrm{P}(g) = \lim_{x \to 0+} g(x),$$

then P *is a positive linear functional. The functional* P *is also unital because* $\mathbf{1} \in \mathcal{X}$ *and* $\mathrm{P}(\mathbf{1}) = 1$*. If we take* $w = \mathbf{1}$*, and* $g(x) = x$ *for* $x \in I$*, then* $g \in \mathcal{X}$ *and its image is in* I*, but* $\mathrm{P}(wg) = \mathrm{P}(g) = 0 \notin I$*.*

Let $\mathrm{P}_i : \mathcal{X} \to \mathbb{R}$ be positive linear functionals, $w_i \in \mathcal{X}$ be non-negative functions such that $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = 1$, and $g_i : X \to I$ be functions where I is a closed interval. If the functions $w_i g_i$ belong to $\mathcal{X}$, then the functional sum $\sum_{i=1}^{n} \mathrm{P}_i(w_i g_i)$ belongs to I, and the sum itself can be called a functional convex combination on I.

The backbone of this section is the next functional form of Jensen's inequality (the part of [6, Theorem 4.3]):

Theorem B. 5 *Let* $\mathrm{P}_i : \mathcal{X} \to \mathbb{R}$ *be positive linear functionals,* $w_i \in \mathcal{X}$ *be non-negative functions such that* $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = 1$*, and* $g_i : X \to I$ *be functions where* I *is a closed interval.*

Then every convex continuous function $f : I \to \mathbb{R}$ *verifies the inequality*

$$f\left(\sum_{i=1}^{n} \mathrm{P}_i(w_i g_i) \right) \leq \sum_{i=1}^{n} \mathrm{P}_i(w_i f(g_i)) \tag{2.1}$$

provided that the functions $w_i g_i$ *and* $w_i f(g_i)$ *belong to* $\mathcal{X}$*.*

Corollary 2.2. 6 *Let* $\mathrm{P}_i : \mathcal{X} \to \mathbb{R}$ *be positive linear functionals,* $w_i \in \mathcal{X}$ *be non-negative functions such that* $\sum_{i=1}^{n} \mathrm{P}_i(w_i) = 1$*, and* $g_i : X \to [a,b]$ *be functions.*

Then every convex continuous function $f : [a,b] \to \mathbb{R}$ *verifies the series of inequalities*

$$f\left(a+b-\sum_{i=1}^{n}\mathrm{P}_i(w_i g_i)\right) \le \sum_{i=1}^{n}\mathrm{P}_i\left(w_i f(a+b-g_i)\right)$$
$$\le f(a)+f(b)-\sum_{i=1}^{n}\mathrm{P}_i\left(w_i f_{[a,b]}^{\mathrm{cho}}(g_i)\right) \quad (2.2)$$
$$\le f(a)+f(b)-\sum_{i=1}^{n}\mathrm{P}_i\left(w_i f(g_i)\right)$$

provided that the functions $w_i g_i$, $w_i f(g_i)$ *and* $w_i f(a+b-g_i)$ *belong to* $\mathcal{X}$.

Proof. First is the use of the inequality in (2.1) with the functional convex combination

$$a+b-\sum_{i=1}^{n}\mathrm{P}_i(w_i g_i)=\sum_{i=1}^{n}\mathrm{P}_i\left(w_i(a+b-g_i)\right).$$

After that we apply $f_{[a,b]}^{\mathrm{cho}}$ with its affinity, and thus obtain the inequality in (2.2). □

The special case of the inequality in (2.2) for $n=1$ was obtained in [1, Theorem 2.1] as the main result. The next is the generalization of Corollary (2.2) as well as the functional variant of Theorem 1.2.

Theorem 2.3. 7 *Let* $\mathrm{P}_i:\mathcal{X}\to\mathbb{R}$ *be positive linear functionals,* $w_i\in\mathcal{X}$ *be non-negative functions such that* $\sum_{i=1}^{n}\mathrm{P}_i(w_i)=1$, $g_{ij}:X\to I$ *be functions where* I *is a closed interval, and* $y_k\in I\backslash\langle a,b\rangle$ *be numbers where* $[a,b]=\mathrm{co}\{g_{ij}(X)\}$. *Let the images of the functions* $g_i=\sum_{k=1}^{m+1}y_k-\sum_{j=1}^{m}g_{ij}$ *are contained in* $[a,b]$.

Then every convex continuous function $f:I\to\mathbb{R}$ *verifies the series of inequalities*

$$f\left(\sum_{k=1}^{m+1}y_k-\sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{P}_i(w_i g_{ij})\right) \le \sum_{i=1}^{n}\mathrm{P}_i\left(w_i f\left(\sum_{k=1}^{m+1}y_k-\sum_{j=1}^{m}g_{ij}\right)\right)$$
$$\le \sum_{k=1}^{m+1}f_{[a,b]}^{cho}(y_k)-\sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{P}_i\left(w_i f_{[a,b]}^{cho}(g_{ij})\right) \quad (2.3)$$
$$\le \sum_{k=1}^{m+1}f(y_k)-\sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{P}_i\left(w_i f(g_{ij})\right)$$

provided that the functions $w_i g_{ij}$, $w_i f(g_{ij})$ *and* $w_i f(g_i)$ *belong to* $\mathcal{X}$.

Proof. Similarly as the proof of Corollary (2.2) because the expression

$$\sum_{k=1}^{m+1}y_k-\sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{P}_i(w_i g_{ij})=\sum_{i=1}^{n}\mathrm{P}_i\left(w_i\left(\sum_{k=1}^{m+1}y_k-\sum_{j=1}^{m}g_{ij}\right)\right)$$

is the functional convex combination on $[a,b]$.

Now we will use the third inequality in (2.2) to functional means. In the functional case quasi-arithmetic means are formed by the application of strictly monotone continuous functions to functional convex combinations. Thus, let I be a closed interval, $\varphi:I\to\mathbb{R}$ be a strictly monotone continuous function, and $\sum_{i=1}^{n}\mathrm{P}_i(w_i g_i)$ be a functional convex combination on I. The discrete functional φ-quasi-arithmetic mean of functions g_i with weighted functions w_i with respect to functionals P_i is the number

$$M_{\varphi}(g_i,w_i,\mathrm{P}_i)=\varphi^{-1}\left(\sum_{i=1}^{n}\mathrm{P}_i(w_i\varphi(g_i))\right) \quad (2.4)$$

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provided that the functions $w_i\varphi(g_i)$ belong to $\mathcal{X}$. This number belongs to I because the functional convex combination $\sum_{i=1}^{n} \mathrm{P}_i(w_i\varphi(g_i))$ belongs to $\varphi(I)$. Functional quasi-arithmetic means are invariant with respect to affine mappings, that is, they verify the equality

$$M_{\alpha\varphi+\beta}(g_i, w_i, \mathrm{P}_i) = M_\varphi(g_i, w_i, \mathrm{P}_i)$$

for every strictly monotone continuous function φ, and every pair of real numbers $\alpha \neq 0$ and β.

If $\varphi:[a,b] \to \mathbb{R}$ is a strictly monotone continuous function, and $g_i : X \to [a,b]$ are functions, then we can define the functional mean

$$N_\varphi(a,b,g_i,w_i,\mathrm{P}_i) = \varphi^{-1}\left(\varphi(a)+\varphi(b)-\sum_{i=1}^{n}\mathrm{P}_i(w_i\varphi(g_i))\right) \tag{2.5}$$

provided that the functions $w_i\varphi(g_i)$ belong to $\mathcal{X}$. The functional convex combination

$$\varphi(a)+\varphi(b)-\sum_{i=1}^{n}\mathrm{P}_i(w_i\varphi(g_i)) = \sum_{i=1}^{n}\mathrm{P}_i\left(w_i\left(\varphi(a)+\varphi(b)-\varphi(g_i)\right)\right)$$

belongs to $\varphi([a,b])$ because the images of the functions $\varphi(a)+\varphi(b)-\varphi(g_i)$ are contained in $\varphi([a,b])$. Therefore the number $N_\varphi = N_\varphi(a,b,g_i,w_i,\mathrm{P}_i)$ belongs to $[a,b]$. The functional means N_φ are also invariant with respect to affine mappings. If we take $y = \psi(x) = \alpha\varphi(x)+\beta$, then $x = \psi^{-1}(y) = \varphi^{-1}\left(\frac{y-\beta}{\alpha}\right)$. Thus, it follows

$$\begin{aligned} N_\psi &= \psi^{-1}\left(\psi(a)+\psi(b)-\sum_{i=1}^{n}\mathrm{P}_i(w_i\psi(g_i))\right) \\ &= \varphi^{-1}\left(\frac{[\alpha\varphi(a)+\beta]+[\alpha\varphi(b)+\beta]-\sum_{i=1}^{n}\mathrm{P}_i\left(w_i[\alpha\varphi(g_i)+\beta]\right)-\beta}{\alpha}\right) \\ &= \varphi^{-1}\left(\varphi(a)+\varphi(b)-\sum_{i=1}^{n}\mathrm{P}_i\left(w_i\varphi(g_i)\right)\right) = N_\varphi, \end{aligned}$$

which shows the invariant property of the observed functional means.

In applications of convexity we often use strictly monotone continuous functions $\varphi,\psi : I \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), that is, $f = \psi\circ\varphi^{-1}$ is convex (this terminology is taken from [8, Definition 1.19]). A similar notation is used for concavity.

Corollary 2.4. 8 *Let $\varphi,\psi : [a,b] \to \mathbb{R}$ be strictly monotone continuous functions. Let $\mathrm{P}_i : \mathcal{X} \to \mathbb{R}$ be positive linear functionals, $w_i \in \mathcal{X}$ be non-negative functions such that $\sum_{i=1}^{n}\mathrm{P}_i(w_i) = 1$, and $g_i : X \to [a,b]$ be functions.*

If ψ is either φ-convex and increasing or φ-concave and decreasing, then the inequality

$$N_\varphi\left(a,b,g_i,w_i,\mathrm{P}_i\right) \le N_\psi\left(a,b,g_i,w_i,\mathrm{P}_i\right) \tag{2.6}$$

holds provided that the functions $w_i\varphi(g_i)$ and $w_i\psi(g_i)$ belong to $\mathcal{X}$.

If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in (2.6).

Proof. Suppose that ψ is φ-convex and increasing. Put $f=\psi\circ\varphi^{-1}$. Using the third inequality in (2.2) with $J=\varphi(I)$ and convex function $f:J\to\mathbb{R}$, it follows

$$\psi\circ\varphi^{-1}\left(\varphi(a)+\varphi(b)-\sum_{i=1}^{n}\mathrm{P}_i\left(w_i\varphi(g_i)\right)\right)\le\psi(a)+\psi(b)-\sum_{i=1}^{n}\mathrm{P}_i(w_i\psi(g_i)).$$

After applying the increasing function ψ^{-1} on the above inequality, we have the inequality in (2.6).

If we use the functions $\varphi_{-1}(x)=x^{-1}$, $\varphi_0(x)=\ln x$, and $\varphi_1(x)=x$ in the inequality in (2.6), we have the following harmonic-geometric-arithmetic inequality for functional means N_φ:

Corollary 2.5. 9 *Let $\mathrm{P}_i:\mathcal{X}\to\mathbb{R}$ be positive linear functionals, $w_i\in\mathcal{X}$ be non-negative functions such that $\sum_{i=1}^{n}\mathrm{P}_i(w_i)=1$, and $g_i:X\to[a,b]\subset<0,\infty>$ be functions.*

Then the double inequality

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}-\sum_{i=1}^{n}\mathrm{P}_i\left(\frac{w_i}{g_i}\right)\right)^{-1} &\le \exp\left(\sum_{i=1}^{n}\mathrm{P}_i\left(w_i\ln\frac{ab}{g_i}\right)\right)\\ &\le a+b-\sum_{i=1}^{n}\mathrm{P}_i\left(w_ig_i\right)\end{aligned}\qquad(2.7)$$

holds if provided that the functions $\frac{w_i}{g_i}$, $w_i\ln g_i$ and w_ig_i belong to $\mathcal{X}$.

Proof. The left side of the inequality in (2.7) follows from the inequality in (2.6) with functions $\varphi(x)=x^{-1}$ and $\psi(x)=\ln x$, so in this case ψ is φ-convex ($\psi\circ\varphi^{-1}(x)=-\ln x$) and increasing.

The right side of the inequality in (2.7) follows from the inequality in (2.6) with functions $\varphi(x)=\ln x$ and $\psi(x)=x$, and as in the previous case ψ is φ-convex ($\psi\circ\varphi^{-1}(x)=\exp x$) and increasing. □

3. OPERATOR VARIANTS OF INEQUALITIES

Recall some notations and definitions. Let $\mathcal{H}$ be a Hilbert space,and $\mathcal{B}(\mathcal{H})$ be a C*-algebra of all bounded linear operators $A:\mathcal{H}\to\mathcal{H}$. The bounds of a self-adjoint operator $A\in\mathcal{B}(\mathcal{H})$ are defined by

$$a_A=\inf_{\|x\|=1}\langle Ax,x\rangle \quad and \quad b_A=\sup_{\|x\|=1}\langle Ax,x\rangle,$$

and if $\mathrm{Sp}(A)$ denotes its spectrum, then $\mathrm{Sp}(A)\subseteq[a_A,b_A]$. If 1_H denotes the identity operator on $\mathcal{H}$, then

$$a_A1_H\le A\le b_A1_H.$$

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Let $\Phi_i:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ be positive linear mappings, $W_i\in\mathcal{B}(\mathcal{H})$ be positive operators such that $\sum_{i=1}^{n}\Phi_i(W_i)=1_K$, and $A_i\in\mathcal{B}(\mathcal{H})$ be self-adjoint operators with spectra in I. Then the spectrum of the operator sum $\sum_{i=1}^{n}\Phi_i(W_iA_i)$ is contained in I, and the sum itself may be called an operator convex combination on I. For a continuous function $f:I\to\mathbb{R}$ the spectrum of the operator convex combination $\sum_{i=1}^{n}\Phi_i\left(W_if(A_i)\right)$ is contained in $f(I)$.

A continuous function $f: I \to \mathbb{R}$ is said to be operator increasing on I if $A \le B$ implies $f(A) \le f(B)$ for every pair of self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in I. A function f is said to be operator decreasing if the function $-f$ is operator increasing.

The most commonly used the operator form of Jensen's inequality for operator convex functions can be found in [2] where $W_i = 1_H$. Next is the operator form of Jensen's inequality for generally convex functions (the part of [6, Theorem 5.1] or [5, Theorem 1]):

Theorem C. 10 *Let* $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be positive linear mappings,* $W_i \in \mathcal{B}(\mathcal{H})$ *be positive operators such that* $\sum_{i=1}^{n} \Phi_i(W_i) = 1_K$, *and* $A_i \in \mathcal{B}(\mathcal{H})$ *be self-adjoint operators with spectra in* I. *Let* $a_M \le b_M$ *be bounds of the operator convex combination* $M = \sum_{i=1}^{n} \Phi_i(W_i A_i)$.

Then every convex continuous function $f: I \to \mathbb{R}$ *verifies the inequality*

$$f\left(\sum_{i=1}^{n} \Phi_i(W_i A_i)\right) \le \sum_{i=1}^{n} \Phi_i\left(W_i f(A_i)\right) \tag{3.1}$$

if provided that $[a_M, b_M] \cap \mathrm{Sp}(A_i) = \varnothing$ *or* $\{\mathrm{endpoint}\}$ *for all* A_i.

Corollary 3.1. 11 *Let* $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be positive linear mappings,* $W_i \in \mathcal{B}(\mathcal{H})$ *be positive operators such that* $\sum_{i=1}^{n} \Phi_i(W_i) = 1_K$, *and* $A_i \in \mathcal{B}(\mathcal{H})$ *be self-adjoint operators with spectra in* I. *Let* $a_M < b_M$ *be bounds of the operator convex combination* $M = \sum_{i=1}^{n} \Phi_i(W_i A_i)$. *Let the spectra of the operators* $B_i = a_M 1_H + b_M 1_H - A_i$ *are contained in* I.

Then every convex continuous function $f: I \to \mathbb{R}$ *verifies the inequalities*

$$f\left(a_M 1_K + b_M 1_K - \sum_{i=1}^{n} \Phi_i(W_i A_i)\right) \le \sum_{i=1}^{n} \Phi_i\left(W_i f(a_M 1_H + b_M 1_H - A_i)\right) \tag{3.2}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \Phi_i\left(W_i f(a_M 1_H + b_M 1_H - A_i)\right) &\ge f(a_M)1_K + f(b_M)1_K - \sum_{i=1}^{n} \Phi_i\left(W_i f_{[a_M, b_M]}^{\mathrm{cho}}(A_i)\right) \\ &\ge f(a_M)1_K + f(b_M)1_K - \sum_{i=1}^{n} \Phi_i(W_i f(A_i)) \end{aligned} \tag{3.3}$$

if provided that $[a_M, b_M] \cap \mathrm{Sp}(A_i) = \varnothing$ *or* $\{\mathrm{endpoint}\}$ *for all* A_i.

Proof. Let $N = a_M 1_K + b_M 1_K - M$. Then $[a_N, b_N] \subseteq [a_M, b_M]$, and therefore $[a_N, b_N] \cap \mathrm{Sp}(B_i) = \varnothing$ or $\{\mathrm{endpoint}\}$ for all B_i because the same is true for $[a_M, b_M]$ and $\mathrm{Sp}(B_i)$. We can apply the inequality in (3.1) on the operator convex combination $N = \sum_{i=1}^{n} \Phi_i(W_i B_i)$, that is,

$$a_M 1_K + b_M 1_K - \sum_{i=1}^{n} \Phi_i(W_i A_i) = \sum_{i=1}^{n} \Phi_i\left(W_i(a_M 1_H + b_M 1_H - A_i)\right),$$

and so get the inequality in (3.2).

The inequality in (3.3) is a consequence of inequalities

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$$f\left(a_M 1_H + b_M 1_H - A_i\right) \geq f^{\text{cho}}_{[a_M,b_M]}(a_M 1_H + b_M 1_H - A_i)$$
$$= f(a_M)1_H + f(b_M)1_H - f^{\text{cho}}_{[a_M,b_M]}(A_i)$$
$$\geq f(a_M)1_H + f(b_M)1_H - f(A_i)$$

for all $i = 1,\ldots,n$. □

The result related to the inequalities in (3.2) and (3.3) with operator convex functions was obtained in [3, Theorem 1] as the main result.

Theorem 3.2. 12 *Let* $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be positive linear mappings,* $W_i \in \mathcal{B}(\mathcal{H})$ *be positive operators such that* $\sum_{i=1}^n \Phi_i(W_i) = 1_K$, *and* $A_{ij} \in \mathcal{B}(\mathcal{H})$ *be self-adjoint operators with spectra in* I. *Let* $y_k \in I$ *be numbers so that the spectra of the operators* $B_i = \sum_{k=1}^{m+1} y_k 1_H - \sum_{j=1}^m A_{ij}$ *are contained in* I. *Let* $a_N < b_N$ *be bounds of the operator convex combination* $N = \sum_{k=1}^{m+1} y_k 1_K - \sum_{j=1}^m \sum_{i=1}^n \Phi_i(W_i A_{ij})$.

Then every convex continuous function $f : I \to \mathbb{R}$ *verifies the inequality*

$$f\left(\sum_{k=1}^{m+1} y_k 1_K - \sum_{j=1}^m \sum_{i=1}^n \Phi_i(W_i A_{ij})\right) \leq \sum_{i=1}^n \Phi_i\left(W_i f\left(\sum_{k=1}^{m+1} y_k 1_H - \sum_{j=1}^m A_{ij}\right)\right) \tag{3.4}$$

if provided that $[a_N, b_N] \cap \operatorname{Sp}(B_i) = \varnothing$ *or* $\{\text{endpoint}\}$ *for all* B_i, *as it verifies the inequality*

$$\sum_{i=1}^n \Phi_i\left(W_i f\left(\sum_{k=1}^{m+1} y_k 1_H - \sum_{j=1}^m A_{ij}\right)\right)$$
$$\geq \sum_{k=1}^{m+1} f^{\text{cho}}_{[a_N,b_N]}(y_k)1_K - \sum_{j=1}^m \sum_{i=1}^n \Phi_i\left(W_i f^{\text{cho}}_{[a_N,b_N]}(A_{ij})\right) \tag{3.5}$$
$$\geq \sum_{k=1}^{m+1} f(y_k)1_K - \sum_{j=1}^m \sum_{i=1}^n \Phi_i(W_i f(A_{ij}))$$

if additionally provided that all $y_k \in [a_N, b_N]$, *and* $[a_N, b_N] \cap \operatorname{Sp}(A_{ij}) = \varnothing$ *for all* A_{ij}.

Proof. Since

$$\sum_{k=1}^{m+1} y_k 1_K - \sum_{j=1}^m \sum_{i=1}^n \Phi_i(W_i A_{ij}) = \sum_{i=1}^n \Phi_i\left(W_i\left(\sum_{k=1}^{m+1} y_k 1_H - \sum_{j=1}^m A_{ij}\right)\right),$$

thus $N = \sum_{i=1}^n \Phi_i(W_i B_i)$. Considering the spectral conditions $[a_N, b_N] \cap \operatorname{Sp}(B_i) = \varnothing$ or $\{\text{endpoint}\}$ for all B_i, we are in a position to apply the inequality in (3.1) on the operator N, and so get the inequality in (3.4).

Considering all spectral conditions and using $f^{\text{cho}}_{[a_N,b_N]}$, we obtain the inequality in(3.5). □

We also want to use Theorem C to operator means. In the operator case quasi-arithmetic means are introduced by the application of strictly monotone continuous functions to operator convex combinations. Thus, let $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function, and $M = \sum_{i=1}^n \mathrm{F}_i(W_i A_i)$ be an operator convex combination on I. The discrete operator φ-quasi-arithmetic mean of self-adjoint operators $A_i \in \mathcal{B}(\mathcal{H})$ with weighted operators W_i with respect to positive linear mappings F_i is the operator

$$M_{\varphi}(A_i,W_i,\Phi_i)=\varphi^{-1}\left(\sum_{i=1}^{n}\Phi_i(W_i\varphi(A_i))\right). \tag{3.6}$$

The spectrum of this operator is contained in I because the spectrum of the operator convex combination $\sum_{i=1}^{n}\Phi_i(W_i\varphi(A_i))$ is contained in $\varphi(I)$. The operator quasi-arithmetic means $M_{\varphi}(A_i,W_i,\Phi_i)$ are invariant with respect to affine mappings, that is, they verify the equality

$$M_{\alpha\varphi+\beta}(A_i,W_i,\Phi_i)=M_{\varphi}(A_i,W_i,\Phi_i)$$

for every strictly monotone continuous function φ, and every pair of real numbers $\alpha\neq 0$ and β.

If the spectra of all A_i are contained in $[a,b]$, we can define the operator mean

$$N_{\varphi}(a1_K,b1_K,A_i,W_i,\Phi_i)=a1_K+b1_K-M_{\varphi}(A_i,W_i,\Phi_i). \tag{3.7}$$

Since the spectrum of the operator $M_{\varphi}(A_i,W_i,\Phi_i)$ is contained in $[a,b]$, the same is true for the spectrum of the operator $N_{\varphi}(a1_K,b1_K,A_i,W_i,\Phi_i)$.

Another operator mean can also be defined, using bounds $a_{\varphi}\le b_{\varphi}$ of the operator mean $M_{\varphi}(A_i,W_i,\Phi_i)$, with the expression

$$N_{\varphi}(a_{\varphi}1_K,b_{\varphi}1_K A_i,W_i,\Phi_i)=a_{\varphi}1_K+b_{\varphi}1_K-\mathcal{M}_{\varphi}(A_i,W_i,\Phi_i). \tag{3.8}$$

The operator means $N_{\varphi}(a1_K,b1_K,A_i,W_i,\Phi_i)$ and $N_{\varphi}(a_{\varphi}1_K,b_{\varphi}1_K A_i,W_i,\Phi_i)$ are also invariant with respect to affine mappings.

Corollary 3.3. 13 *Let $\varphi,\psi:[a,b]\to\mathbb{R}$ be strictly monotone continuous functions. Let $\Phi_i:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ be positive linear mappings, $W_i\in\mathcal{B}(\mathcal{H})$ be positive operators such that $\sum_{i=1}^{n}\Phi_i(W_i)=1_K$, and $A_i\in\mathcal{B}(\mathcal{H})$ be self-adjoint operators with spectra in $[a,b]$. Let $a_{\varphi}\le b_{\varphi}$ be bounds of the operator mean $M_{\varphi}(A_i,W_i,\Phi_i)$.*

If ψ is either φ-convex with operator increasing ψ^{-1} or φ-concave with operator decreasing ψ^{-1}, then the inequality

$$N_{\varphi}(a1_K,b1_K,A_i,W_i,\Phi_i)\ge N_{\psi}(a1_K,b1_K,A_i,W_i,\Phi_i) \tag{3.9}$$

holds provided that $[a_{\varphi},b_{\varphi}]\cap\mathrm{Sp}(A_i)=\varnothing$ or $\{\text{endpoint}\}$ for all A_i.

If ψ is either φ-concave with operator increasing ψ^{-1} or φ-convex with operator decreasing ψ^{-1}, then the reverse inequality is valid in (3.9).

Proof. Suppose ψ is φ-convex with operator increasing ψ^{-1}. Put $f=\psi\circ\varphi^{-1}$. Using the inequality in (3.1) with $J=\varphi(I)$ and convex continuous function $f:J\to\mathbb{R}$, it follows

$$\psi\circ\varphi^{-1}\left(\sum_{i=1}^{n}\Phi_i\left(W_i\varphi(A_i)\right)\right)\le\sum_{i=1}^{n}\Phi_i\left(W_i\psi(A_i)\right).$$

After applying the operator increasing function ψ^{-1} on the above inequality, multiplying by -1, and adding $a1_K+b1_K$, we have the inequality in (3.9).

The consequence of the inequality in **Error! Reference source not found.** is the following version of harmonic-geometric-arithmetic inequality for the operator means $N_{\varphi}(a1_K,b1_K,A_i,W_i,\Phi_i)$:

Corollary 3.4. 14 *Let* $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be positive linear mappings,* $W_i \in \mathcal{B}(\mathcal{H})$ *be strictly positive operators such that* $\sum_{i=1}^{n} \Phi_i(W_i) = 1_K$*, and* $A_i \in \mathcal{B}(\mathcal{H})$ *be strictly positive operators with spectra in* $[a,b] \subset <0,\infty>$*. Let* $a_0 \le b_0$ *be bounds of the operator mean* $M_{\ln}(A_i, W_i, \Phi_i)$.

Then the double inequality

$$\begin{aligned} (a+b)1_K - \left(\sum_{i=1}^{n} \Phi_i \left(W_i A_i^{-1} \right) \right)^{-1} &\ge (a+b)1_K - \exp\left(\sum_{i=1}^{n} \Phi_i \left(W_i \ln A_i \right) \right) \\ &\ge (a+b)1_K - \sum_{i=1}^{n} \Phi_i \left(W_i A_i \right) \end{aligned} \tag{3.10}$$

holds if provided that $[a_0, b_0] \cap \mathrm{Sp}(A_i) = \emptyset$ *or* $\{\text{endpoint}\}$ *for all* A_i.

Proof. The right side of the inequality in (3.10) follows from the inequality in (3.9) with functions $\varphi(x) = \ln x$ and $\psi(x) = x$, so ψ is φ-convex ($\psi \circ \varphi^{-1}(x) = \exp x$) and $\psi^{-1}(x) = x$ is operator increasing.

The left side of the inequality in (3.10) follows from the reverse of the inequality in (3.9) with functions $\varphi(x) = \ln x^{-1} = -\ln x$ and $\psi(x) = x^{-1}$, so in this case ψ is φ-convex ($\psi \circ \varphi^{-1}(x) = \exp x$) and $\psi^{-1}(x) = x^{-1}$ is operator decreasing. Invariant property of the observed means provides $N_{-\ln} = N_{\ln}$. □

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