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Pettis Integration via Statistical Convergence

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ABSTRACT

In this paper we extend the usual concept of Pettis integration to a statistical form. In order to achieve this, we prove some necessary statements such as Vitali theorem and use the statistically compactness. We obtain some properties of statistical Pettis integration which are well known for the Pettis integration.

Keywords

statistical convergence; statistical integration; statistical Pettis integration.

Academic Discipline And Sub-Disciplines

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INTRODUCTION

In recent years, statistical convergence has increasingly become an attractive area of research. The idea of statistical convergence was initially described by Zigmund [19]. The concept was further formalized by Steinhaus[14] and Fast [6]. A few years later, the concept was reintroduced by Schoenberg [16]. In this paper we follow the notions about the convergence of sequences introduced by Fridy [9],[8] as well as the approach of Schoenberg about integration.

The base line concept is the statistical Cauchy convergence of Fridy[7]. On the Banach space, we adopted the approach from the work of Connor et al.[4](1989).

2. Terminology

Let be A a subset of ordered natural set ℕ. It said to have *density* δ(A) if $\delta(A) = \lim_{n\to\infty} \frac{|A_n|}{n}$, where A = {k<n : k∈A} and with |A| denotes the cardinality of this one. It is clear that the finite sets have the density zero and δ(A′)= 1-δ(A) if A'=ℕ-A. If a property P(k)={k : k∈A} holds for all k∈A with δ(A)=1, we say that property P holds for almost all k, that is a.a.k. The vectorial sequence x is *statistically convergent* to the vector(element) p of a vectorial normed space X if for each ε >0

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : ||x_k - p|| \ge \varepsilon\}| = 0$$

in short form $||x_k - p|| < \varepsilon$ a.a.k.

We write st-limx_k =p. In same way, the sequence x is a *statistical Cauchy sequence* if for every ε>0, there exists a number N=N(ε) such that

$||x_k - x_N|| < \varepsilon$ a.a.k.

Now, we deals with generalization of pointwise statistical convergence of functions on normed space.

The sequence $\{f_k\}$ contains the functions with values in one vectorial normed space. For each x of the domain, we consider the functional sequence $\{f_k(x)\}$.

A sequence of functions $\{f_k(x)\}$ is said to be *pointwise statistically convergent to f* if for every ε>0

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : ||f_k(x) - f(x)|| \ge \varepsilon, \forall x \in S\}| = 0 \text{ ,}$$

i.e. for every x∈S, $||f_k(x)-f(x)|| < \varepsilon$ a.a.k.

We write st-limf_k(x)=f(x) or $f_k \xrightarrow{st} f$ on S.

This means that for every δ>0, there exists integer N such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : ||f_k(x) - f(x)|| \ge \varepsilon, \forall x \in S\}| < \delta$$

For all n>N = (N(ε,δ,x)) and for every ε>0.

If the inequality in (1) holds for all k except finite many k, then one obtain the usual limes, $\lim_{n\to\infty} f_n(x) = f(x)$ on S. It follows that this limes implies st-$\lim_{n\to\infty} f_k(x) = f(x)$. But the converse of this is not always true.

A family *H* of scalar integrable functions is *uniformly integrable* if

$$\lim_{\mu(E)\to 0} \int_E |h| \, d\mu = 0$$

uniformly for h∈ *H.*

If Σ_0 is a subalgebra of Σ, then E(h | Σ_0) denotes the conditional expectation of h with respect to Σ_0.

Further, we denote (S,∑,μ) the probability measure space, where S is any set and Σ sigma algebra of Borel.

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1. Vitali theorem

A function f : S →X, where X is a vectorial normed space is called simple function by μ, if there is a finite sequence of measurable sets {E_i}, such that $E_i \in S$, i=1,...,n $E_i \cap E_j = \varnothing$ for i≠j, S = $\bigcup_{i=1}^{n} E_i$ and f(s) = x_i for s ∈ E_i,

It is represented in a form f = $\sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is a characteristic function of E_i.

We denote T(μ,X) –the set of simple functions with domain S. T(μ,X) is a vectorial space with the addition of simple function and multiple with the real number(see [17]). The simple functions as it is known are the measurable functions.

The function f:S→X is called statistically *strongly measurable by μ* on set S(in short form st- measurable) if there exists a sequence of simple functions (f_n)∈ T(μ,X) that for every s ∈S and every ε>0 holds:

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : ||f_k(s) - f(s)|| \ge \varepsilon, \forall x \in S\}| = 0 .$$

for almost all s ∈S.

The function f : S→X is called *statistically strongly uniform measurable by* μ on S if every δ>0 and every ε>0 there exists a integer N(ε, δ) such that

$$\frac{1}{n} |\{k \le n : ||f_k(s) - f(s)|| \ge \varepsilon\}| < \delta$$

for k> N(ε, δ) almost for every s∈S. In this case it is said that the sequence $f_k(s)$ converges statistically strong almost everywhere uniformly by μ to the function f on S.

Proposition 1. A linear combination of st-measurable functions is a st-measurable function.

We modified some techniques developed in [12] in order to prove in [3] the following theorem.

Theorem 2. (Theorem Egorov) [3]. If a function f : S→X is st- strongly measurable by μ, then it is st- strong measurable uniformly almost everywhere on S.

Definition 3. The function f : S→X is *statistically weakly measurable* if the scalar function x*f is statistically strong measurable for every x* of dual space X*.

Definition 4. The integral of the simple function f : S→X, is called the element of vectorial normed space $\sum_{i=1}^{n} x_i \mu(E_i)$, symbolically

$$\int_S f(s) d\mu = \sum_{i=1}^{n} x_i \mu(E_i)$$

In case E is a measurable set and E ⊂ S, then the integral of simple function f on E is the integral of function f χ_E , we write

$$\int_E f(s) d\mu = \int_S (f.\chi_E)(s) d\mu$$

We define the map

$$||*||_1 : T(\mu,E) \to \mathbb{R} ; \; ||f||_1 = \int_S ||f(s)|| \, d\mu$$

It is easy to prove that $||f||_1$ is a seminorm.

Following the definition of Cauchy sequences introduced by Fridy [7] and their extension to the functional sequences (see for example [12]), the sequence (f_k) is called a *statistical Cauchy sequence* if for every ε>0 there exists an integer N(=N(ε,x)) with

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : ||f_k(x) - f_N(x)|| \ge \varepsilon \;\; \forall x \in S\}| = 0$$

In the set of st-Cauchy simple sequence, we define the equivalence relation:

(f_n)~(g_n) ⇔st – lim || f_n-g_n|| = 0.

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The following theorem extends in case of Banach space the results presented in [8].

Theorem 5. [3], [7], [12]. Let (f_k) be a sequence of functions on a set S with values to Banach space X. The following statements are equivalent:

a) the sequence (f_k) is pointwise statistically convergent on S;

b) the sequence (f_k) is a statistical Cauchy sequence on S.

Lemma 6. If the sequence of simple functions (f_n) is a st-Cauchy sequence on Banach space there exists the limes st

$$\lim_k \int_S f_k(s)d\mu .$$

Definition 7. The function f : S→X is called st- Bochner integrable if there exists a st- Cauchy sequence of simple functions (f_k) such that :

i) statistically converges a.e. by μ to the function f ;

ii) $st-\lim_k \int_s \| f_k(s) - f_N(s) \| d\mu = 0$ a.e.

st-lim $\int_S f_n(s)d\mu$ is called *st- Bochner integral* and denote with Bs - $\int_s f(x)d\mu$

This sequence (f_n) of simple functions is called determinant of function f. We have presented in our paper [3] a function that is statistical Bochner integrable but not Bochner integrable.

Theorem 8[3] . Let ($f_k(s)$) be the sequence of st-measurable functions statistically convergent almost everywhere to the function f(s). If for a.a.k and every s∈S $\|f_k(s)\| \leq \|f_{k+1}(s)\|$, then

$$st-\lim_r \int_S \| f_r \| d\mu = (Bs)\int_S \| f(s) \| d\mu$$

Lemma 9. [3] (Fatou) Let {$f_n(x)$} be the sequence of strong measurable functions from S into X, then for every A⊂S holds

$$\int_A st-\liminf \| f_n \| d\mu \leq st-\liminf \int_A \| f_n \| d\mu .$$

Lemma 10. [15] (Salat) A sequence (x_k) is statistically convergent to p if and only if there exists a set K={$k_1<k_2<\ldots$}⊂ℕ that δ(K)=1 and $\lim_{n\to\infty}(x_{k_n}) = p$.

The set K is directed and the sequence (x_{k_n}) is called the essential subsequence of (x_k). The above lemma can be formulated:

A sequence (x_k) is statistically convergent to p if and only if there exists an essential subsequence (x_{k_n}) which converges in usual meaning to limes p. We write $\lim_K x_k = p$. We can formulate an immediate corollary of Salat's lemma.

Preposition 11. The sequence {$f_k(x)$} where f_n : S→X, (X a vectorial normed space) is statistically convergent to f(x), if and only if, there exists an essential subsequence (f_{k_n}) of it that is convergent to f(x).

Corollary 12 . The sequence {$f_k(x)$}is statistically convergent almost everywhere to f(x) on S if there exists an essential subsequence (f_{k_n}) such that is convergent almost everywhere to f(x).

Theorem 13. (Vitali) Let (S, Σ, μ) be a positive measure space, μ finite and the sequence {f_n} where f_n : S→X is uniformly integrable . If

a) $\lim_K f_n = f$
b) $\|f(x)\| < \infty$

Then the following hold:

1. $f \in L_1(\mu)$
2. Bs-$\int_E \| f_n - f \| d\mu$

Proof. For proving statement 1, we use the Fatou's lemma $\int_S st-\liminf \| f_n \| d\mu \leq st-\liminf \int_S \| f_n \| d\mu$

Using uniform integrability we have $\int_E \| f \| d\mu < 1$ where E is a set such that μ(E)<δ.

By the Egorov theorem 2, the sequence f_n statistically uniformly converges on the set E^c and $\int_{E^c} \| f_n - f_N \| d\mu < 1$ for n>N and a.a.n.

Using triangle inequality we have $\int_{E^c} \| f_n \| d\mu \leq \int_{E^c} \| f_N \| d\mu + 1 = M$. This proves the statement 1. For the second statement we have

$$\int_S \| f - f_n \| \leq \int_E \| f \| d\mu + \int_E \| f_n \| d\mu + \int_{E^c} \| f - f_n \| d\mu.$$

Where E⊂S and μ(E)<δ. All the terms in the above inequality are bounded for a.a.n. This proves the statement 2.

1. Statistical Pettis integration

Definition 14. [2] A point p is called a statistically-sequential accumulation point of the set F if there is a sequence $x=(x_k)$ of points in F\{p} such that st-lim(x_k)=p. The set of all statistically –sequential accumulation points of F is called statistically-sequential closure of F. We say that a set is statistically-sequential closed if it contains all the points in its statistically-closure.

Definition 15 . A subset F of X is called statistically-sequential compact if whenever $x=(x_k)$ is a sequence of points in F there is a subsequence y= (y_{k_n}) of x with st-lim y_{k_n} =p∈F.

Preposition 16 . A subset F of X is sequentially compact if and only if it is statistically-sequential compact in it.

Proof. Let F be a subset of statistically-sequential compact set X. By definition, for every sequence x in F there is a subsequence (y_k) such that is statistically convergent to the point p∈F. But the sequence (y_k) has an essential subsequence (y_{k_n}) convergent to the same point p. This means that F is sequentially compact.

Let we extend the concept of Pettis integration by means of statistical convergence. Let (S, Σ, μ) be a measurable space with finite measure μ and X one Banach space.

Definition 17. Let E be a subset of the set S. The function f : S→X is called statistically Pettis integrable if

a) The function x*f is statistically Bohner integrable for every x*∈X*
b) There exists an element x_E of X such that

$$x^*(x_E)=st\text{-}\int_E x^*(f)d\mu \text{ for every } x^*\in X^*. \tag{1}$$

The element x_E is called indefinite statistical Pettis integral and we denote

$$x_E = st\text{-}P\text{-}\int_E f d\mu . \tag{2}$$

Proposition 18. Let X be a Banach space and there exists on it the sequence of simple functions $(f_n(s))$ statistically weakly convergent almost everywhere to the function f(s) such that

$$\int_E | x^* f_n - x^* f_m | d\mu \to 0 \text{ almost every } m,n\to\infty$$

for every x*∈X*. Then the function f is statistically Pettis integrable and

$$st-\lim \int_E f_n(s)d\mu = st-\int_E f(s)d\mu . \tag{3}$$

Proof. Since the real functions are μ-integrable then the integrability of the real function x*f for every x*∈X* is derives from the fact that statistical weak convergence of the sequence (f_n) implies statistical convergence of real sequence $x^*f_n \to x^*f$ and

$$\int_E x^* f_n d\mu \xrightarrow{st} \int_E x^* f d\mu . \tag{4}$$

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Considering the property of integration of simple functions we have

$$\int_E x^* f_n d\mu = x^* \int_E f_n d\mu \qquad (5)$$

From (3) and (4) and continuousness of the function x* follows the statistical convergence of the sequence $\left\{x^* \int_E f_n d\mu\right\}$. It follows that sequence $\left\{\int_E f_n d\mu\right\}$ is statistically Cauchy on the weakly topology of X. As the space X is a Banach space the sequence statistically converges to the point x_E ,

$$st - \int_E f_n d\mu \to x_E .$$

So we have

$$x^* \int_E f_n d\mu \xrightarrow{st} x^*(x_E) \qquad (6)$$

From the (4), (5) and (6) comes (1) and (2). This way, the statistical limit of the sequence of integrals $\left\{\int_E f_n d\mu\right\}$ of the simple functions defins the statistical integral of Pettis of the function f. This proves the equality (2).

A function $f : S \to X$ is weakly uniformly bounded if there is a constant μ such that $|x^* f| \le M \| x^* \|$ a.e (the exceptional set may vary with X^*).

In the following two theorems we adhere to the approach used in [11] and [13] for usual Pettis integration.

Theorem 19. Let $f : S \to X$ be a function and X a Banach space. If there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of X-valued st- Pettis integreable functions on S such that :

(a) The set $\{x^* f_n : x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable .

(b) $st - \lim x^* f_n = x^* f$ in measure, for each $x^* \epsilon X^*$

Then f is st- Pettis integrable and $st - \lim \int_E f_n d\mu = Ps - \int_E f d\mu$ weakly in X, for each E$\epsilon\Sigma$.

Proof: Fix E $\epsilon\Sigma$ and let C be the st-weakly closure of the set .$\{\int_E f_n d\mu : n \in \mathbb{N}\}$ Since Vitali's convergence (theorems 13) guaranties that $st - \lim \int_E x^* f_n d\mu = Bs - \int_E x^* f d\mu$ for each $x^* \epsilon X^*$, we see by the ([1] – Corollary 2.9) C is statistically bounded and $C \setminus \int_E f_n d\mu$: n $\in\mathbb{N}$ consists of at most one point. In order to prove our assertion it is sufficient to show that C is st-weakly compact, since this yields the existence of statistically weakly limit of $\{\int_E f_n d\mu : n \in \mathbb{N}\}$ in X. Clearly, the limit can only be equal to $\int_E f d\mu$ and so we be able to conclude that is st-Pettis integrable on E and hence on the whole of Σ. Suppose therefore that C is not weakly compact(so and statistically weakly compact- Proposition 16). Then, according to a theorem of James ([10] Theorem 1) there exists a bound sequence $\{x_n : n \in \mathbb{N}\} \subset C$ and ε>0, such that

$$x_n^*(x_k) = 0 \quad for\ k > n$$

and

$$x_n^*(x_k) > \varepsilon \quad for\ k \le n$$

Since $st - \lim x^* f_n = x^* f$ we find a set A $\subset \mathbb{N}$ that δ(n ∈ A : |x*f$_n$ – x*f|>σ)=0 for every σ>0. So for $n \notin A$ we have

$$\lim_{n \notin A} \int_E x^* f_n d\mu = Bs - \int_E x^* f d\mu .$$

We can choose a subsequence $\{g_m : m \in \mathbb{N} \setminus A\}$ of $\{f_n\}$ and a subsequence $\{y_m^*\}$ of $\{x_n^*\}$, such that

(i) $\int_E y_k^* g_m d\mu = 0$ k<m

(ii) $\int_E y_k^* g_m d\mu > \varepsilon$ k≥m

(iii) $st - \lim \int_E x^* f_m d\mu = \int_E x^* f d\mu$ for each x*∈X*.

Considerer now the set $\{y_m^* f : m \in \mathbb{N}\}$. It follows from (a) that this set is uniformly integrable and stastically bounded in $L_1(\mu)$. Hence it is relatively weakly compact. This yields the existence of a function $h \epsilon L_1(\mu)$ and a subsequence $\{z_j^* : j \in \mathbb{N}\}$ of $\{y_m^* : m \in \mathbb{N} \setminus A\}$ such that $st - \lim z_j^* f = h$ weakly in $L_1(\mu)$. Applying (iii) for all z^*_j we get the inequality $\int_E z_j f d\mu \geq \varepsilon$ and hence $\int_E h d\mu \geq \varepsilon$. Now we shall appeal to the theorem of Mazur. Let $a_1^m ... a_{k(m)}^m$, $m \in \mathbb{N}$ be non-negative numbers, such that $\sum_j a_j^m = 1$ and $\lim_m (\sum_j a_j^m z_{j+m}^* f) = h$ in $L_1(\mu)$. Without lost of generality, we may assume, that above convergence holds $\mu - a.e.$, Clearly, if z_0 is weakly* st-closter point of the sequence $\{\sum_j a_j^m z_{\leq j+m}^*$: m∈ℕ} then $h = z_0^* f$ µ-a.e. In particular ,we have

(iv) $$\int_E z_0^* f d\mu \geq \varepsilon$$

On the other hand, since each g_n is st-Pettis integrable, the functional $x^* \to \int_E x^* g_n d\mu$ is weakly* continues. Hence, if $\{\omega_{n,\alpha}^*\}$ is a subsequence of $\{\sum_j a^m z_{j+m}^* : m > n\}$ which converges st-weakly to z^*_0, then applay (i) we get :

$$0 = \lim_\alpha \int_E \omega_{n,\alpha}^* g_n d\mu = \lim_\alpha \omega_{n,\alpha}^* \int_E g_n d\mu = z_0^* \int_E g_n d\mu = \int_E z_0^* g_n d\mu$$

Since this holds for each $n \in \mathbb{N} \setminus A$,we see from (iii) that $\int_E z_0^* f d\mu = 0$. This contradicts the inequality (iv).

Theorem 20. Let (S, Σ, µ) be a measure space and $\Sigma_0 \subset \Sigma$. The function f : S→X f is st-Pettis integrable and st-weakly measurable with respect to a separable measure space (S, Σ_0, $\mu|_{\Sigma_0}$) if and only if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of X-valued simple functions on S such that

(a) The family $\{x^* f_n : n \in \mathbb{N}, x^* \in B(X^*)\}$ is uniformly integrable,

(b) Foe each x*∈X* $st - \lim_n x^* f_n = x^* f$ µ-a.e.

Proof. Since the simple functions are Pettis integrable, one direction of this is immediate from the theorem 19. Assume that f is weakly measurable with respect to a separable space (S, Σ_0, $\mu|_{\Sigma_0}$) and, let $\tilde{\Sigma} = \sigma(\{E_n, n \in \mathbb{N}\}) \subset \Sigma_0$ be a countable generated σ-algebra which is $\mu|_{\Sigma_0}$ -dense in Σ_0. Moreover, let Π_n be a partition of S generated by the sets $E_1,\ldots,E_n$. Put for each n the functions

$$f_n = \sum_{E \subset \Pi_n} \frac{\int_E f d\mu}{\mu(E)} \chi_E \qquad (\frac{0}{0} = 0)$$

It is well known that $\{f_n, \sigma(\Pi_n)\}_{n=1}^{\infty}$ is an X-valued martingale and $x^* f_n \to E(x^* f|_{\tilde{\Sigma}})$ a.a.n. is in $L_1(S, \tilde{\Sigma}, \mu|_{\Sigma_0})$ (cf. Neveu [14]) and µ-a.e. (Diestel [5]). Moreover the conditional expectation operator is a contraction on $L_1(\mu|_{\tilde{\Sigma}})$ and so we have

$$\int_E |x^* f_n| d\mu \leq \int_E |x^* f| d\mu \text{ a. a. } n \in \mathbb{N}.$$

As by the assumption is dense in Σ_0, we have $E(x^* f|_{\tilde{\Sigma}}) = x^* f$ µ-a. e. and so

$x^* f_n \to x^* f$ $\mu|_{\Sigma_0}$ -a. e. and a. a. n.

On the other hand, from the condition (b) we have that the sequence $\{x^* f_n : n \in \mathbb{N}\}$ is st-convergent to x*f weakly in $L_1(\mu)$. The above conditions means exactly that for each E∈Σ the sequence $\{\int_E f_n d\mu\}$ is st-convergent to $\int_E f d\mu$. Hence ν_n is contain in a weakly closure of the set $\bigcup_n \nu_n$ where ν_n is the indefinite Pettis integral of f_n. As each set $\nu_n(\Sigma)$ is finite dimensional, the union is weakly separable. According to the well known result of Mazur the weakly and norm separability in Banach space coincide.

Proposition 21. If the function f : S→X is st- Bochner integrable then it is also st- Pettis integrable and the equation holds

$$st-P-\int_E f(s)d\mu = st-Bs-\int_E f(s)d\mu$$

for every set E⊂V.

Proof. Since the function f(s) is st- Bochner integrable there exists a determinant sequence of simple functions f_n convergent almost everywhere uniformly and for almost every n to the function f. While the functions x* of X* are continuous, we have

$$|x^*(f_n)-x^*(f)| \leq \|x^*\|.\|f_n(s)-f(s)\| \to 0 \text{ a. a. n,}$$

so

$$\text{st-Bs-}\int_E |x^*(f_n)-x^*(f)|\,d\mu \leq \|x^*\|\int_E \|f_n - f\|\,d\mu \to 0.$$

This means that the sequence of functions $x^*(f_n)$ is statistically convergent to x*f. It follows that x*f is st-Bochner integrable as the real function. Considering once more the property of integration of simple functions we have

$$x^*\int_E f_n d\mu = \int_E x^* f_n d\mu \to \int_E x^* f d\mu \text{ for every x* of X*.}$$

On the other hand , the sequence which is st-weakly convergent has a unique limit. It implies that from statistical convergence of the sequence of integrals $\left\{\int_E f_n d\mu\right\}$ to the st- Bochner integral $\int_E fd\mu$ entails the convergence to the st-Bochner integral of the sequence

$$x^*\int_E f_n d\mu \to x^*\int_E fd\mu .$$

Consequently

$$\int_E x^* fd\mu = x^*\int_E fd\mu .$$

From the (2) and (3) we proved the existence of st- Pettis integral and its are equal.

Theorem 22. Let (S,Σ, μ) be a finite measure space, X a Banach space and f : S→X. Suppose there is a sequence $\{f_n\}$ of st-Pettis integrable functions from S to X such that $\lim_K x^* f_n = x^* f$ μ-a.e. for each x* in X*(the null set on which convergence fails may vary with x*). If there is a scalar function v(x) with $\| x^* f_n \| \leq v(x)$ μ-a.e. for each x* ∈X* and n∈K, then f is st-Pettis integrable and

$$st-\lim\int_E f_n d\mu = Ps-\int_E fd\mu .$$

Proof. By the domination theorem, the function x*f is st-Bohner integrable for every x*∈X* and

$$st-\lim\int_E x^* f_n d\mu = Bs-\int_E x^* fd\mu \text{, for } E\in\Sigma. \qquad (7)$$

We can write

$$\int_E x^* f_n d\mu = x^*\int_E f_n d\mu \qquad (8)$$

From (7) and (8) we have that the sequence $\{\int_E f_n d\mu\}$ is fundamental in X*. By the completeness of Banach space X* the above sequence has the limes y:

$$st-\lim x^*\int_E f_n d\mu = x^* y \text{ for every x*∈X*.}$$

This fulfills two conditions of st-Pettis integrability of function f:

$$Ps-\int_E fd\mu = y = st-\lim\int_E f_n d\mu .$$

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