ISSN 2347-1921

Existence and uniqueness of mild solutions of nonlinear difference-integrodifferential equation with nonlocal condition

[1]H. L. Tidke, [2]R. T. More

[1]Department of Matheamtics,

North Maharasjhtra University, Jalgaon-425 001, India

tharibhau@gmail.com

[2] Department of Mathematics,

Arts, Commerce and Science College, Bodwad, Jalgaon-425 310, India

rupeshmore9@yahoo.com

ABSTRACT

In this paper we investigate the existence, uniqueness and continuous dependence of solutions of the difference-integrodifferential equations. The results are obtained by using the well known Banach fixed point theorem, the theory of semigroups and the inequality established by B. G. Pachpatte.

Keywords:

Existence, uniqueness and continuous dependence; difference-integrodifferential equation;Bana ch fixed point theorem; Pachpatte's integral inequality; semigroup theory.

Subject Classification: 45N05, 47G20, 34K05, 47H10, 34A12

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 9, No 3

editor@cirworld.com

www.cirjam.com

ISSN 2347-1921

1. INTRODUCTION

Using Tychonov's fixed point theorem, the method of successive approximations, and the comparison method, S. Sugiyama [14] studied the existence and uniqueness of solutions of the following problem:

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-1)), \tag{1.1}$$

for $0 \le t \le t_1$, with the conditions

$$x(t-1) = \phi(t) \quad (0 \le t < 1), \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

where x and f represent n-dimensional vectors (see [14] for details) and Stokes [13] has discussed the same problems as above for nonlinear differential equations.

In [16], S. Sugiyama proved the existence, stability, and boundedness of solutions of the difference-differential problem (1.1)–(1.3) by making use of Tychonov's fixed point theorem with additional condition on f and we also refer the papers of S. Sugiyama [15, 17]. Subsequently some authors have been studied the problems of existence, uniqueness and other properties of solutions of (1.1) by using different techniques, see [1, 6, 9, 12] and the references cited therein. We also refer some papers and monographs [9, 14], [1, p. 342], [5, p.308], [7, p. 18].

Recently, in the interesting paper [11], B. G. Pachpatte has studied the existence, uniqueness and continuous dependence of solutions (1.1)–(1.3) with an infinite interval of t, by the well known Banach fixed point theorem and the Gronwall-Bellman integral inequality.

From the above works, we can see a fact, although the difference-differential problems have been investigated by some authors. However, to our knowledge, the difference-integrodifferential equation with nonlocal conditions and an infinitesimal generator of operators has not been discussed extensively. So motivated by all the works above, the aim of this paper is to prove the existence, uniqueness and continuous dependence of solutions of the difference-integrodifferential of the form:

$$x'(t) = Ax(t) + f\left(t, x(t), \int_0^t k(t, s, x(s))ds, x(t-1)\right), \tag{1.4}$$

for $t \in \mathbb{R}_+ = [0, \infty)$ under the conditions

$$x(t-1) = \phi(t) \quad (0 \le t < 1), \tag{1.5}$$

$$x(0) + g(x) = x_0, \tag{1.6}$$

where A is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, $f \in C(\mathbb{R}_+ \times X \times X \times X, X)$, $k \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times X, X)$, $g \in C(C(\mathbb{R}_+, X), X)$ and $\phi(t)$ is a continuous function for $0 \le t < 1$, $\lim_{t \to 1-0} \varphi(t)$ exists, for which we denote by $\phi(1-0) = c_0$. If we consider the solutions of (1.4) for $t \in \mathbb{R}_+$, we obtain a function $x(t-1)$ which is unable to define as solution for $0 \le t < 1$. Hence, we have to impose some condition, for example the condition (1.5). We note that, if $0 \le t < 1$, the problem is reduced to integrodifferential equation

$$x'(t) = Ax(t) + f\left(t, x(t), \int_0^t k(t, s, x(s))ds, \phi(t)\right),$$

with initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.4)–(1.6) for $0 \le t < \infty$.

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3 deals with existence and uniqueness of the solutions. Finally, in Section 4 we discuss results on continuous dependence of solutions on initial data, functions involved therein and parameters.

2. PRELIMINARIES AND HYPOTHESES

Before proceeding to the statement of our main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

ISSN 2347-1921

Let X be the Banach space with norm $\|\cdot\|$. Let S be the space of all continuous functions from R_+ into X and fulfill the condition

$$\|z(t)\| = O(\exp(\lambda t)), \tag{2.1}$$

for some positive constant $\lambda > 0$. In this space we define the norm (see [2, 8])

$$\|z\|_S = \sup_{t\in\mathrm{R}_+} [\|z(t)\|\exp(-\lambda t)]. \tag{2.2}$$

It is easy to see that S with the above norm is a Banach space. Note that condition (2.1) implies the existence of a nonnegative constant N such that $\| z(t)\| \le N\exp(\lambda t)$ for $t \in \mathrm{R}_+$. Using this fact in (2.2) we observe that

$$\|z\|_S \le N. \tag{2.3}$$

Definition 2.1 Let $-A$ is the infinitesimal generator of a C_0-semigroup $T(t), t \ge 0$, on a Banach space X. The function $x \in B$ given by

$$x(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \phi(s))ds, \tag{2.4}$$

for $0 \le t < 1$, and

$$x(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \phi(s))ds$$
$$+ \int_1^t T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, x(s-1))ds, \tag{2.5}$$

for $1 \le t < \infty$, is called the mild solution of the problem (1.4)–(1.6).

We require the following Lemma known as the Pachpatte's inequality in our further discussion.

Lemma 2.2 (,p. 152) *Let* $u, e, b \in C(\mathrm{R}_+, \mathrm{R}_+)$ *and for* $s \le t$ *;* $a(t,s), c(t,s) \in C(\mathrm{R}_+^2, \mathrm{R}_+)$. *If* $e(t)$ *and* $a(t,s)$ *be be nondecreasing in* $t \in \mathrm{R}_+$ *and*

$$u(t) \le e(t) + \int_0^t a(t,s)[b(s)u(s) + \int_0^s c(s,\tau)u(\tau)d\tau]ds,$$

for $t \in \mathrm{R}_+$, then

$$u(t) \le e(t)\exp(\int_0^t a(t,s)[b(s) + \int_0^s c(s,\tau)d\tau]ds), \text{ for } t \in \mathrm{R}_+.$$

We list the following hypotheses:

(H_1) A is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in X such that $\|T(t)\| \le N_0$, for some $N_0 \ge 1$.

(H_2) The function f in (1.4) satisfies the condition

$$\|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})\| \le p_1(t)[\|x-\bar{x}\| + \|y-\bar{y}\| + \|z-\bar{z}\|],$$

for $(t,x,y,z), (t,\bar{x},\bar{y},\bar{z}) \in \mathrm{R}_+ \times X \times X \times X$, where $p_1 \in C(\mathrm{R}_+, \mathrm{R}_+)$ and increasing function.

(H_3) The function k in (1.4) satisfies the condition

$$\|k(t,s,x) - k(t,s,\bar{x})\| \le q_1(t,s)[\|x-\bar{x}\|],$$

for $(t,s,x),(t,s,\bar{x}) \in R_+^2 \times X$, where $q_1 \in C(R_+^2, R_+)$.

(H4) There exists constant K such that

$$K = \max_{0 \le s \le t < \infty} \|k(t,s,0)\|.$$

(H5) There exist constants $G, G_1 \ge 0$ such that

$$\|g(x)\| \le G_1, \quad \text{and} \quad \|g(x) - g(\bar{x})\| \le G\|x - \bar{x}\|,$$

for every $x, \bar{x} \in C(R_+, X)$.

(H6) For λ as in (2.1):

(a) there exist nonnegative constants $\alpha < \dfrac{1 - N_0 G}{N_0(1+\gamma)}$, $N_0 G < 1$, $\gamma < 1$ such that

$$\int_0^t [p_1(s) + p_1(s+1)]\exp(\lambda s)ds \le \alpha \exp(\lambda t), \quad \text{for} \quad t \in \mathbb{R}_+,$$

and

$$\int_0^t q_1(s,\tau)\exp(\lambda\tau)d\tau \le \gamma \exp(\lambda t), \quad \text{for} \quad \tau \le s \le t \in \mathbb{R}_+.$$

(b) there exists a nonnegative constant ν such that

$$\|x_0\| + G_1 + \int_0^1 p_1(s)\|\phi(s)\|ds + \int_0^t \|f(s,0,0,0)\|ds + K\int_0^t sp_1(s)ds \le \nu \exp(\lambda t), \quad \text{for} \quad t \in \mathbb{R}_+.$$

3. EXISTENCE AND UNIQUENESS

We first prove the fundamental result.

Theorem 3.1 *Assume that hypotheses $(H_1)-(H_6)$ hold. Then the problem (1.4)– (1.6) has a unique mild solution on R_+ in S.*

Proof. Let $x(t) \in S$ and define the operator $F : S \to S$ by (see [14])

$$Fx(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \phi(s))ds, \qquad (3.1)$$

for $0 \le t < 1$, and

$$Fx(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \phi(s))ds$$
$$+ \int_1^t T(t-s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, x(s-1))ds, \qquad (3.2)$$

for $1 \le t < \infty$. First we shall show that Fx maps S into itself. Since all functions involved in (3.1) and (3.2) are continuous, therefore, Fx is continuous on R_+ and $Fx \in X$. To verify that (2.1) is fulfilled, we consider the following two cases.

Case 1: $0 \le t < 1$. By using the hypotheses $(H_1)-(H_6)$ and (2.3) in (3.1), then we have

$$\|Fx(t)\| \le N_0[\|x_0\| + G_1] + \int_0^t N_0[\|f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \phi(s)) - f(s,0,0,0)\|]ds + \int_0^t N_0\| f(s,0,0,0)\| ds$$

$$\le N_0[\| x_0\| + G_1] + \int_0^t N_0 p_1(s)[\|x(s) - 0\| + \|\int_0^s k(s,\tau,x(\tau))d\tau - 0\| + \| \phi(s) - 0\|]ds + \int_0^t N_0\|f(s,0,0,0)\| ds$$

$$\le N_0[\|x_0\| + G_1] + \int_0^t N_0 p_1(s)[\|x(s)\| + \int_0^s q_1(s,\tau)\|x(\tau) - 0\|d\tau$$

$$+ \int_0^s K d\tau + \| \phi(s)\|]ds + \int_0^t N_0\|f(s,0,0,0)\| ds$$

$$\le N_0[\|x_0\| + G_1] + \int_0^t N_0 p_1(s)[\|x(s)\| + \int_0^s q_1(s,\tau)\|x(\tau)\|d\tau$$

$$+ (s-0)K + \| \phi(s)\|]ds + \int_0^t N_0\|f(s,0,0,0)\| ds$$

$$\le N_0[\|x_0\| + G_1 + \int_0^t \|f(s,0,0,0)\| ds + K\int_0^t s p_1(s)ds + \int_0^1 p_1(s)\| \phi(s)\| ds]$$

$$+ \int_0^t N_0 p_1(s)[\| x(s)\| + \gamma \|x\|_S \exp(\lambda s)]ds$$

$$\le N_0 \nu \exp(\lambda t) + \alpha N_0\| x\|_S \exp(\lambda t) + \alpha\gamma N_0\|x\|_S \exp(\lambda t)]$$

$$\le N_0[\nu + \alpha N(1+\gamma)]\exp(\lambda t). \tag{3.3}$$

Case 2: $1 \le t < \infty$. From (3.2), using the hypotheses and (2.3), then looking at Case 1 immediately we have

$$\|Fx(t)\| \le N_0[(\|x_0\| + G_1) + \int_0^1 p_1(s)\| \phi(s)\| ds + \int_0^t \|f(s,0,0,0)\| ds + K\int_0^t s p_1(s)ds]$$

$$+ N_0 \int_1^t p_1(s)\|x(s-1)\| ds + \int_0^t N_0 p_1(s)[\|x(s)\| + \gamma \|x\|_S \exp(\lambda s)]ds$$

$$\le N_0[\nu \exp(\lambda t) + \alpha\gamma \|x\|_S \exp(\lambda t) + \int_0^t p_1(s)\|x(s)\| ds + \int_1^t p_1(s)\|x(s-1)\| ds]$$

$$\le N_0[\nu \exp(\lambda t) + \alpha\gamma \|x\|_S \exp(\lambda t) + \int_0^t p_1(s)\|x(s)\| ds + I_1], \tag{3.4}$$

where

$$I_1 = \int_1^t p_1(s)\|x(s-1)\| ds. \tag{3.5}$$

By making the change of variable, we obtain

$$I_1 = \int_0^{t-1} p_1(\sigma+1)\|x(\sigma)\| d\sigma \le \int_0^t p_1(\sigma+1)\|x(\sigma)\| d\sigma. \tag{3.6}$$

Using (3.6) in (3.4), we get

$$\|Fx(t)\| \le N_0[\nu \exp(\lambda t) + \alpha\gamma \|x\|_S \exp(\lambda t) + \int_0^t p_1(s)\|x(s)\| ds + \int_0^t p_1(s+1)\|x(s)\| ds]$$

$$\leq N_0[\nu\exp(\lambda t)+\alpha\gamma\,\|x\|_S\exp(\lambda t)+\int_0^t[p_1(s)+p_1(s+1)]\|x(s)\|ds]$$

$$\leq N_0[\nu\exp(\lambda t)+\alpha\gamma\,\|x\|_S\exp(\lambda t)+\|x\|_S\int_0^t[p_1(s)+p_1(s+1)]\exp(\lambda s)\,ds]$$

$$\leq N_0[\nu\exp(\lambda t)+\alpha\gamma\,\|x\|_S\exp(\lambda t)+\alpha\|x\|_S\exp(\lambda t)]$$

$$\leq N_0[\nu+\alpha N(1+\gamma)]\exp(\lambda t). \tag{3.7}$$

From (3.3) and this inequality, it follows that $Fx\in S$. This proves that F maps S into itself.

Next, we verify that the operator F is a contraction map. Let $x,y\in S$. We consider the following two cases.

Case 1: $0\leq t<1$. From (3.1) and using the hypotheses, we have

$$\|(Fx)(t)-(F)y(t)\|\leq N_0G\|x-y\|+N_0\int_0^t p_1(s)[\|x(s)-y(s)\|+\|\int_0^s k(s,\tau,x(\tau))-k(s,\tau,y(\tau)))d\tau\|$$

$$+\|\phi(s)-\phi(s)\|]ds$$

$$\leq N_0G\|x-y\|+N_0\int_0^t p_1(s)[\|x(s)-y(s)\|+\int_0^s q_1(s,\tau)\|x(\tau)-y(\tau)\|d\tau]ds$$

$$\leq N_0G\|x-y\|+N_0\int_0^t p_1(s)[\|x(s)-y(s)\|+\|x-y\|_S\int_0^s q_1(s,\tau)\exp(\lambda\tau)d\tau]ds$$

$$\leq N_0G\exp(\lambda t)\|x-y\|_S+N_0[\int_0^t p_1(s)\|x(s)-y(s)\|ds+\gamma\,\|x-y\|_S\int_0^t p_1(s)\exp(\lambda s)ds]$$

$$\leq N_0[G\exp(\lambda t)\|x-y\|_S+\|x-y\|_S\int_0^t p_1(s)\exp(\lambda s)ds+\gamma\,\|x-y\|_S\int_0^t p_1(s)\exp(\lambda s)ds]$$

$$\leq N_0[G\exp(\lambda t)\|x-y\|_S+\alpha\|x-y\|_S\exp(\lambda t)+\alpha\gamma\;\|x-y\|_S\exp(\lambda t)]$$

$$\leq N_0[G+\alpha(1+\gamma)]\|x-y\|_S\exp(\lambda t). \tag{3.8}$$

Case 2: $1\leq t<\infty$. From (3.2) and using the hypotheses, we have

$$\|(Fx)(t)-(Fy)(t)\|\leq N_0G\exp(\lambda t)\|x-y\|_S+\int_0^t N_0p_1(s)[\|x(s)-y(s)\|+\|x-y\|_S\int_0^s q_1(s,\tau)\exp(\lambda\tau)d\tau]ds$$

$$+\int_1^t N_0p_1(s)\|x(s-1)-y(s-1)\|ds$$

$$\leq N_0G\exp(\lambda t)\,\|x-y\|_S+\int_0^t N_0p_1(s)[\|x(s)-y(s)\|+\gamma\,\|x-y\|_S\exp(\lambda s)]ds$$

$$+\int_1^t N_0p_1(s)\|x(s-1)-y(s-1)\,\|ds$$

$$\leq N_0[G\exp(\lambda t)\|x-y\|_S+\int_0^t p_1(s)\|x(s)-y(s)\|ds+\gamma\,\|x-y\|_S\int_0^t p_1(s)\exp(\lambda s)ds$$

$$+\int_1^t p_1(s)\|x(s-1)-y(s-1)\|\,ds]$$

$$\leq N_0[G\exp(\lambda t)\|x-y\|_S+\int_0^t p_1(s)\|x(s)-y(s)\,\|ds+\alpha\gamma\,\|x-y\|_S\exp(\lambda t)+I_2], \tag{3.9}$$

where

$$I_2=\int_1^t p_1(s)\|x(s-1)-y(s-1)\|ds. \tag{3.10}$$

By making the change of variable, we obtain

$$I_2 \le \int_0^t p_1(s+1)\|x(s)-y(s)\|\,ds. \tag{3.11}$$

Using this inequality and (3.9), we get

$$\begin{aligned}\|Fx(t)-Fy(t)\| &\le N_0[G\exp(\lambda t)\|x-y\|_S + \|x-y\|_S\int_0^t[p_1(s)+p_1(s+1)]\exp(\lambda s)ds + \alpha\gamma\,\|x-y\|_S\exp(\lambda t)] \\ &\le N_0[G\exp(\lambda t)\|x-y\|_S + \alpha\|x-y\|_S\exp(\lambda t) + \alpha\gamma\|x-y\|_S\exp(\lambda t)] \\ &\le N_0[G+\alpha(1+\gamma)]\|x-y\|_S\exp(\lambda t), \end{aligned} \tag{3.12}$$

for all $x, y \in S$. From (3.8) and (3.12), we observe that

$$\|Fx-Fy\|_S \le N_0[G+\alpha(1+\gamma)]\|x-y\|_S.$$

By condition (a) of hypothesis (H_6), we have $N_0[G+\alpha(1+\gamma)]<1$ and hence, it follows from Banach fixed point theorem [4, p. 37] that F has a unique fixed in S. The fixed point of F is however a solution of (1.4)-(1.6). This completes the proof.

The following theorem shows the uniqueness of solutions to (1.4)-(1.6) without the existence part.

Theorem 3.2 *Suppose that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem (1.4)-(1.6) has at most one solution on* $\mathbb{R}_+$.

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of (1.4)-(1.6) and $u(t)=\|x_1(t)-x_2(t)\|$, $t\in\mathbb{R}_+$. We consider the following two cases.

Case 1: $0\le t<1$. From the hypothesis, we have

$$\begin{aligned} u(t) &\le N_0Gu(t)+\int_0^t N_0p_1(s)[u(s)+\int_0^s\|k(s,\tau,x_1(\tau))-k(s,\tau,x_2(\tau))\|d\tau]ds \\ &\le N_0Gu(t)+\int_0^t N_0p_1(s)[u(s)+\int_0^s q_1(s,\tau)\|x_1(\tau)-x_2(\tau)\|d\tau]ds \\ &\le N_0Gu(t)+\int_0^t N_0p_1(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\ &\le N_0Gu(t)+\int_0^t N_0p_1(s)u(s)ds+\int_0^t N_0p_1(s)[\int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \end{aligned}$$

which implies

$$u(t)\le\int_0^t\frac{N_0p_1(s)}{1-N_0G}[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds. \tag{3.13}$$

Now a suitable application of Lemma 2.2 (with $e(t)=0$), known as Pachpatte's inequality, yields

$$\|x_1(t)-x_2(t)\|\le 0\exp(\int_0^t\frac{N_0p_1(s)}{1-N_0G}[1+\int_0^s q_1(s,\tau)d\tau]ds)\le 0. \tag{3.14}$$

Case 2: $1\le t<\infty$. From the hypothesis and following ideas from the above case, we obtain

$$\begin{aligned} u(t) &\le N_0Gu(t)+\int_0^t N_0p_1(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds+\int_1^t N_0p_1(s)\|x_1(s-1)-x_2(s-1)\|ds \\ &\le N_0Gu(t)+\int_0^t N_0p_1(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds+N_0I_3, \end{aligned} \tag{3.15}$$

where

$$I_3 = \int_1^t p_1(s)\|x_1(s-1) - x_2(s-1)\|ds.$$

By making a change of variable, we observe that

$$I_3 \le \int_0^t p_1(s+1)\|x_1(s) - x_2(s)\|ds.$$

Using this inequality in (3.15), we obtain

$$\begin{aligned} u(t) &\le N_0Gu(t) + \int_0^t N_0 p_1(s)[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds + \int_0^t N_0 p_1(s+1)u(s)ds \\ &\le N_0Gu(t) + \int_0^t N_0 p_1(s)[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds + \int_0^t N_0 p_1(s+1)u(s)ds \\ &\le N_0Gu(t) + \int_0^t N_0 (p_1(s) + p_1(s+1))[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \end{aligned}$$

which gives

$$u(t) \le \int_0^t \frac{N_0(p_1(s) + p_1(s+1))}{1 - N_0G}[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds.$$

Now a suitable application of Lemma 2.2 (with $e(t) = 0$), known as Pachpatte's inequality, yields

$$\|x_1(t) - x_2(t)\| \le 0\exp(\int_0^t \frac{N_0(p_1(s) + p_1(s+1))}{1 - N_0G}[1 + \int_0^s q_1(s,\tau)d\tau]ds) \le 0. \tag{3.16}$$

From (3.14) and the inequality (3.16), we have $x_1(t) = x_2(t)$ for $t \in \mathrm{R}_+$. Thus there is at most one solution to (1.4)–(1.6) on R_+. This completes the proof.

4. CONTINUOUS DEPENDENCE

In this section we study the continuous dependence of solutions to (1.4) on the given initial data, and on the function f. Also we show the continuous dependence of solutions of equations of the form (1.4) on certain parameters.

First, we shall give the following theorem concerning the continuous dependence of solutions to (1.4) on the given initial data.

Theorem 4.1 *Suppose that the hypotheses* (H_1)-(H_2) *hold and let* $x_1(t), x_2(t)$ *be the solutions of (1.4) with the initial conditions*

$$x_1(t-1) = \phi_1(t) \quad (0 \le t < 1), \quad x_1(0) + g(x_1) = c_1, \tag{4.1}$$

$$x_2(t-1) = \phi_2(t) \quad (0 \le t < 1), \quad x_2(0) + g(x_2) = c_2, \tag{4.2}$$

respectively, where c_1, c_2 are elements of X. Then

$$\|x_1(t) - x_2(t)\| \le \frac{c}{1 - N_0G}\exp(\int_0^t \frac{N_0 p_1(s)}{(1 - N_0G)}[1 + \int_0^s q_1(s,\tau)d\tau]ds), \tag{4.3}$$

for $0 \le t < 1$ and

$$\|x_1(t) - x_2(t)\| \frac{c}{1 - N_0G}\exp(\int_0^t \frac{N_0(p_1(s) + p_1(s+1))}{(1 - N_0G)}[1 + \int_0^s q_1(s,\tau)d\tau]ds), \tag{4.4}$$

for $1 \le t < \infty$, where

$$c = N_0\|c_1 - c_2\| + \int_0^1 N_0 p_1(s)\|\phi_1(s) - \phi_2(s)\| ds. \tag{4.5}$$

Proof. Let $u(t) = \|x_1(t) - x_2(t)\|$ for $t \in \mathrm{R}_+$. We consider the following two cases.

Case 1: $0 \le t < 1$. From the hypotheses, it follows that

$$\begin{aligned} u(t) &\le N_0(\|c_1 - c_2\| + Gu(t)) + \int_0^t N_0 p_1(s)[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau + \|\phi_1(s) - \phi_2(s)\|] ds \\ &\le N_0(\|c_1 - c_2\| + Gu(t)) + \int_0^t N_0 p_1(s)u(s)ds + \int_0^t N_0 p_1(s)[\int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\ &\quad + \int_0^1 N_0 p_1(s)\|\phi_1(s) - \phi_2(s)\| ds \\ &\le N_0\|c_1 - c_2\| + N_0 Gu(t) + \int_0^1 N_0 p_1(s)\|\phi_1(s) - \phi_2(s)\| ds \\ &\quad + \int_0^t N_0 p_1(s)u(s)ds + \int_0^t N_0 p_1(s)[\int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\ &\le c + N_0 Gu(t) + \int_0^t N_0 p_1(s)[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \end{aligned}$$

which implies

$$u(t) \le \frac{c}{1 - N_0 G} + \int_0^t \frac{N_0 p_1(s)}{1 - N_0 G}[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds. \tag{4.6}$$

Now an application of Lemma 2.2 (with $e(t) = \dfrac{c}{1 - N_0 G}$), known as Pachpatte's inequality, to (4.6), yields (4.3).

Case 2: $1 \le t < \infty$. By following a similar arguments as in Case 2 of the proof of Theorem 3.2 and from the hypotheses, it follows that

$$\begin{aligned} u(t) &\le N_0\|c_1 - c_2\| + N_0 Gu(t) + \int_0^1 N_0 p_1(s)\|\phi_1(s) - \phi_2(s)\| ds \\ &\quad + \int_0^t N_0(p_1(s) + p_1(s+1))[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\ &\le c + N_0 Gu(t) + \int_0^t N_0(p_1(s) + p_1(s+1))[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \end{aligned}$$

which implies

$$u(t) \le \frac{c}{1 - N_0 G} + \int_0^t \frac{N_0(p_1(s) + p_1(s+1))}{1 - N_0 G}[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds. \tag{4.7}$$

Now an application of Lemma 2.2 (with $e(t) = \dfrac{c}{1 - N_0 G}$), known as Pachpatte's inequality, to (4.7), yields (4.4). From (4.3) and (4.4), it follows that the solutions of equation (1.4) depends on the given initial data. This completes the proof.

Now, we consider (1.4)–(1.6) and the corresponding initial-value problem

$$y'(t) = Ay(t) + \overline{f}(t, y(t), \int_0^t k(t,s,y(s))ds, y(t-1)), \tag{4.8}$$

for $t \in \mathrm{R}_+$ under the initial conditions

$$y(t-1)=\psi(t), \quad y(0)+\overline{g}(y)=y_0, \tag{4.9}$$

where $\overline{f}\in C(\mathbb{R}_+\times X\times X\times X,X)$, $\overline{g}\in C(C(\mathbb{R}_+,X),X)$, $k\in C(\mathbb{R}_+^2\times X,X)$, and $\psi(t)$ is a continuous function for which the limit $\lim_{t\to 1-0}\psi(t)$ exists.

The following theorem shows the continuous dependence of solutions to (1.4)–(1.6) on the function f and the closeness of the solutions of equations (1.4)–(1.6) and (4.8)–(4.9).

Theorem 4.2 *Suppose that the hypotheses* $(H_1)-(H_5)$ *hold and there exist constants* $\varepsilon_1>0,\delta_1>0,\delta_2>0$ *such that*

$$\|f(t,u,v,w)-\overline{f}(t,u,v,w)\|\le\varepsilon_1, \tag{4.10}$$

$$\|g(u)-\overline{g}(u)\|\le\delta_1, \|x_0-y_0\|\le\delta_2, \tag{4.11}$$

where x_0,g,ϕ,f and $y_0,\overline{g},\psi,\overline{f}$ are as in (1.4)–(1.6) and (4.8)–(4.9). Let $x(t)$ and $y(t)$ be respectively, solutions of (1.4)–(1.6) and (4.8)–(4.9) on $\mathbb{R}_+$. Then

$$\|x(t)-y(t)\|\le\frac{\overline{c}}{(1-N_0G)}\exp\Big(\int_0^t\frac{N_0p_1(s)}{(1-N_0G)}\Big[1+\int_0^s q_1(s,\tau)d\tau\Big]ds\Big),$$

for $0\le t<1$ and

$$\|x(t)-y(t)\|\le\frac{\overline{c}}{(1-N_0G)}\exp\Big(\int_0^t\frac{N_0(p_1(s)+p_1(s+1))}{(1-N_0G)}\Big[1+\int_0^s q_1(s,\tau)d\tau\Big]ds\Big),$$

for $1\le t<\infty$, where

$$\overline{c}=N_0(\delta_2+\delta_1+\varepsilon_1 t)+\int_0^1 N_0p_1(s)\|\phi(s)-\psi(s)\|ds.$$

Proof. Let $u(t)=\|x(t)-y(t)\|$ for $t\in\mathbb{R}_+$. We consider the following two cases.

Case 1: $0\le t<1$. From the hypotheses, we have

$$\begin{aligned}
u(t)&\le N_0\|x_0-y_0\|+N_0\|g(y)-\overline{g}(y)\|+N_0\|g(x)-g(y)\|\\
&\quad+\int_0^t N_0\Big\|f\Big(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau,\phi(s)\Big)-f\Big(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s)\Big)\Big\|ds\\
&\quad+\int_0^t N_0\Big\|f\Big(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s)\Big)-\overline{f}\Big(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s)\Big)\Big\|ds\\
&\le N_0\delta_2+N_0\delta_1+N_0G\|x-y\|+\int_0^1 N_0p_1(s)\|\phi(s)-\psi(s)\|ds\\
&\quad+\int_0^t N_0p_1(s)\Big[\|x(s)-y(s)\|+\int_0^s q_1(s,\tau)\|x(\tau)-y(\tau)\|d\tau\Big]ds+\int_0^t N_0\varepsilon_1 ds\\
&\le\overline{c}+N_0Gu(t)+\int_0^t N_0p_1(s)\Big[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau\Big]ds,
\end{aligned}$$

which implies

$$u(t)\le\frac{\overline{c}}{(1-N_0G)}+\int_0^t\frac{N_0p_1(s)}{(1-N_0G)}\Big[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau\Big]ds, \tag{4.12}$$

Now an application of Lemma 2.2 (with $e(t)=\frac{\bar{c}}{(1-N_0G)}$), known as Pachpatte's inequality, to (4.28), yields that for $0\le t<1$,

$$\|x(t)-y(t)\|\le \frac{\bar{c}}{(1-N_0G)}\exp(\int_0^t \frac{N_0p_1(s)}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds). \tag{4.13}$$

Case 2: $1\le t<\infty$**.** Following an arguments as in Case 2 of the proof of Theorem 3.2 and from the hypotheses, we obtain

$$\begin{aligned}
u(t)&\le N_0\|x_0-y_0\|+N_0\|g(y)-\bar{g}(y)\|+N_0\|g(x)-g(y)\|\\
&+\int_0^1 N_0\|f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau,\phi(s))-f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))\|ds\\
&+\int_0^1 N_0\|f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))-\bar{f}(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))\|ds\\
&+\int_1^t N_0\|f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau,x(s-1))-f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,y(s-1))\|ds\\
&+\int_1^t N_0\|f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,y(s-1))-\bar{f}(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,y(s-1))\|ds\\
&\le N_0\delta_2+N_0\delta_1+N_0G\|x-y\|+\int_0^1 N_0p_1(s)\|\phi(s)-\psi(s)\|ds\\
&+\int_0^t N_0(p_1(s)+p_1(s+1))[\|x(s)-y(s)\|+\int_0^s q_1(s,\tau)\|x(\tau)-y(\tau)\|d\tau]ds+\int_0^t N_0\varepsilon_1 ds\\
&\le \bar{c}+N_0Gu(t)+\int_0^t N_0(p_1(s)+p_1(s+1))[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds,
\end{aligned}$$

which implies

$$u(t)\le \frac{\bar{c}}{(1-N_0G)}+\int_0^t \frac{N_0(p_1(s)+p_1(s+1))}{(1-N_0G)}[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \tag{4.14}$$

Now an application of Lemma 2.2 (with $e(t)=\frac{\bar{c}}{(1-N_0G)}$), known as Pachpatte's inequality, to (4.30), yields that for $1\le t<\infty$,

$$\|x(t)-y(t)\|\le \frac{\bar{c}}{(1-N_0G)}\exp(\int_0^t \frac{N_0(p_1(s)+p_1(s+1))}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds). \tag{4.15}$$

From this inequality and (4.29), it follows that (1.4)–(1.6) depends continuously on the functions involved therein. The proof is completed.

Remark 4.3 The result given in Theorem 4.2 relates the solutions of IVP (1.4)–(1.6) and of IVP (4.8)–(4.9) in the sense that if f is close to $\bar{f}$, x_0 is close to y_0, ϕ is close to ψ and g is close to $\bar{g}$, then the solutions of IVPs (1.4)–(1.6) and (4.8)–(4.9) are also close together.

We consider the IVP (1.4)–(1.6) together with

$$y'(t)=Ay(t)+f_k(t,y(t),\int_0^t k(t,s,y(s))ds,y(t-1)), \tag{4.16}$$

$$y(t-1)=\psi_k(t), \quad y(0)+\overline{g}_k(y)=c_k, \tag{4.17}$$

for $k=1,2,\cdots,$ where $\overline{f}\in C(\mathbb{R}_+\times X\times X\times X,X)$, $\overline{g}_k\in C(C(\mathbb{R}_+,X),X)$, $k\in C(\mathbb{R}_+^2\times X,X)$, R) and for each $k=1,2,\cdots,$ $\psi_k(t)$ is a continuous function for which the limit $\lim_{t\to 1-0}\psi_k(t)$ exists.

As an immediate consequence of Theorem 4.2, we have the following corollary.

Corollary 4.4 *Suppose that the hypotheses* $(H_1)-(H_5)$ *hold and there exist nonnegative constants* $\varepsilon_k,\delta_k,\overline{\delta}_k \quad (k=1,2,\cdots)$ *such that*

$$\|f(t,u,v,w)-\overline{f}_k(t,u,v,w)\|\le\varepsilon_k, \tag{4.18}$$

$$\|g(u)-\overline{g}_k(u)\|\le\delta_k, \|x_0-c_k\|\le\overline{\delta}_k, \tag{4.19}$$

with $\varepsilon_k\to 0$ and $\delta_k,\overline{\delta}_k\to 0$ as $k\to\infty$, where x_0,g,ϕ,f,k and $c_k,\overline{g}_k,\psi_k,\overline{f}_k$ are as in (1.4)–(1.6) and (4.16)–(4.17). If $y_k(t)\,(k=1,2,\cdots)$ and $x(t)$ are respectively the solutions of (4.16)–(4.17) and (1.4)–(1.6) on $\mathbb{R}_+$, then $y_k(t)\to x(t)$ as $k\to\infty$ on $\mathbb{R}_+$.

Proof. For $k=1,2,\cdots$, the conditions of of Theorem 4.2 hold. As an application of of Theorem 4.2 and Lemma 2.2 yields

$$\|y_k(t)-x(t)\|\le\frac{\overline{c_k}}{(1-N_0G)}\exp\left(\int_0^t\frac{N_0p_1(s)}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds\right), \tag{4.20}$$

for $0\le t<1$ and

$$\|y_k(t)-x(t)\|\le\frac{\overline{c_k}}{(1-N_0G)}\exp\left(\int_0^t\frac{N_0(p_1(s)+p_1(s+1))}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds\right), \tag{4.21}$$

for $1\le t<\infty$, where

$$\overline{c}=N_0(\delta_k+\overline{\delta_k}+\varepsilon_k t)+\int_0^1 N_0p_1(s)\|\phi(s)-\psi_k(s)\|ds.$$

The required results follow from (4.20) and (4.21). It follows that (1.4)–(1.6) depends continuously on the functions involved therein. This completes the proof.

Remark 4.5 *The result obtained in Corollary 4.4 provide sufficient conditions that ensures solutions of IVPs (4.16)–(4.17) will converge to the solutions of IVP (1.4)–(1.6).*

Next, we consider the difference-differential equations

$$x'(t)=Ax(t)+\overline{f}(t,x(t),\int_0^t k(t,s,x(s))ds,x(t-1),\mu_1), \tag{4.22}$$

$$x'(t)=Ax(t)+\overline{f}(t,x(t),\int_0^t k(t,s,x(s))ds,x(t-1),\mu_2), \tag{4.23}$$

for $t\in\mathbb{R}_+$, where $\overline{f}\in C(\mathbb{R}_+\times X\times X\times X\times\mathbb{R},X)$, and with the initial conditions given by (1.5)–(1.6).

The following theorem states the continuous dependence of solutions to (4.22) and (4.23) with the initial conditions given by (1.5)–(1.6) on parameters.

Theorem 4.6 *Assume that hypotheses* $(H_1), (H_3)-(H_5)$ *and there exists an increasing function* $p_2 \in C(\mathrm{R}_+, \mathrm{R}_+)$ *such that*

$$\|\overline{f}(t,x,y,z,\mu_1)-\overline{f}(t,\bar{x},\bar{y},\bar{z},\mu)\| \le p_2(t)[\|x-\bar{x}\|+\|y-\bar{y}\|+\|z-\bar{z}\|+|\mu_1-\mu_2|].$$

Let $x_1(t)$ and $x_2(t)$ be the solutions of (4.22) with (1.5)–(1.6) and (4.23) with (1.5)–(1.6) respectively. Then

$$\|x(t)-y(t)\| \le \Big(\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^1 p_2(s)ds\Big)\exp\Big(\int_0^t \frac{N_0 p_2(s)}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds\Big),$$

for $0 \le t < 1$ and

$$\|x(t)-y(t)\| \le \Big(\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^t p_2(s)ds\Big)\exp\Big(\int_0^t \frac{N_0(p_2(s)+p_2(s+1))}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds\Big),$$

for $1 \le t < \infty$.

Proof. Let $u(t)=\|x_1(t)-x_2(t)\|$ for $t \in \mathrm{R}_+$. We consider the following two cases.

Case 1: $0 \le t < 1$. From the hypotheses, we have

$$\begin{aligned}
u(t) &\le \|T(t)\|g(x_1)-g(x_2)\|+\int_0^t\|T(t-s)\|\|[\overline{f}(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau,\phi(s),\mu_1) \\
&\quad -\overline{f}(s,x_2(s),\int_0^s k(s,\tau,x_2(\tau))d\tau,\phi(s),\mu_2)]\|ds \\
&\le N_0Gu(t)+\int_0^t N_0 p_2(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau+\|\phi(s)-\phi(s)\|+|\mu_1-\mu_2|]ds \\
&\le N_0Gu(t)+\int_0^t N_0 p_2(s)|\mu_1-\mu_2|ds+\int_0^t N_0 p_2(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds,
\end{aligned}$$

which implies

$$u(t) \le \frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^1 p_2(s)ds+\int_0^t \frac{N_0 p_2(s)}{(1-N_0G)}[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds. \tag{4.24}$$

Now an application of Lemma 2.2 (with $e(t)=\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^1 p_2(s)ds$), known as Pachpatte's inequality, to (4.24), yields

$$\|x(t)-y(t)\| \le \Big(\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^1 p_2(s)ds\Big)\exp\Big(\int_0^t \frac{N_0 p_2(s)}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds\Big), \tag{4.25}$$

for $0 \le t < 1$.

Case 2: $1 \le t < \infty$. By following the arguments in Case 2 of the proof of Theorem 3.2 and from the hypotheses, we have

$$\begin{aligned}
u(t) &\le N_0G\|x_1-x_2\|+\int_0^t N_0 p_2(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\
&\quad +\int_1^t N_0 p_2(s)\|x_1(s-1)-x_2(s-1)\|ds+N_0|\mu_1-\mu_2|\int_0^t p_2(s)ds \\
&\le N_0G\|x_1-x_2\|+N_0|\mu_1-\mu_2|\int_0^t p_2(s)ds+\int_0^t N_0(p_2(s)+p_2(s+1))[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds,
\end{aligned}$$

which implies

$$u(t) \le \frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^t p_2(s)ds+\int_0^t \frac{N_0(p_2(s)+p_2(s+1))}{(1-N_0G)}[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds. \tag{4.26}$$

Now an application of Lemma 2.2 (with $e(t)=\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^\beta p_2(s)ds$), known as Pachpatte's inequality, to (4.26), gives

$$\|x(t)-y(t)\|\leq(\frac{N_0|\mu_1-\mu_2|}{(1-N_0G)}\int_0^t p_2(s)ds)\exp(\int_0^t\frac{N_0(p_2(s)+p_2(s+1))}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds), \quad (4.27)$$

for $1\leq t<\infty$. From (4.25) and (4.27), it follows that the solutions (4.22) with (1.5)–(1.6) and (4.23) with (1.5)–(1.6) depend continuously on the parameters μ_1,μ_2. This completes the proof.

Remark 4.7 *The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.*

A slight variant of Theorem 4.2 is given in the following theorem

Theorem 4.8 *Suppose that*

$$\|f(t,x,y,z)-\overline{f}(t,\overline{x},\overline{y},\overline{z})\|\leq p_3(t)[\|x-\overline{x}\|+\|y-\overline{y}\|+\|z-\overline{z}\|],$$

where an increasing function $p_3\in C(\mathbb{R}_+,\mathbb{R}_+)$ and hypotheses $(H_1),(H_3)-(H_5)$ and condition 4.11 hold. Let $x(t)$ and $y(t)$ be respectively, solutions of (1.4)–(1.6) and (4.8)–(4.9) on $\mathbb{R}_+$. Then

$$\|x(t)-y(t)\|\leq(\frac{\overline{l}}{(1-N_0G)})\exp(\int_0^t\frac{N_0p_3(s)}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds),$$

for $0\leq t<1$ and

$$\|x(t)-y(t)\|\leq(\frac{\overline{l}}{(1-N_0G)})\exp(\int_0^t\frac{N_0(p_3(s)+p_3(s+1))}{(1-N_0G)}[1+\int_0^s q_1(s,\tau)d\tau]ds),$$

for $1\leq t<\infty$, where

$$\overline{l}=N_0[\delta_2+\delta_1]+\int_0^1 N_0p_3(s)\|\phi(s)-\psi(s)\|ds.$$

Proof. Let $u(t)=\|x(t)-y(t)\|$ for $t\in\mathbb{R}_+$. We consider the following two cases.

Case 1: $0\leq t<1$. From the hypotheses, we have

$$\begin{aligned}u(t)&\leq N_0\|x_0-y_0\|+N_0\|g(y)-\overline{g}(y)\|+N_0\|g(x)-g(y)\|\\&\quad+\int_0^t N_0\|f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau,\phi(s))-f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))\|ds\\&\quad+\int_0^t N_0\|f(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))-\overline{f}(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\psi(s))\|ds\\&\leq N_0\delta_2+N_0\delta_1+N_0G\|x-y\|+\int_0^1 N_0p_3(s)\|\phi(s)-\psi(s)\|ds\\&\quad+\int_0^t N_0p_3(s)[\|x(s)-y(s)\|+\int_0^s q_1(s,\tau)\|x(\tau)-y(\tau)\|d\tau]ds\\&\leq\overline{l}+N_0Gu(t)+\int_0^t N_0p_3(s)[u(s)+\int_0^s q_1(s,\tau)u(\tau)d\tau]ds,\end{aligned}$$

which implies

$$u(t) \le \frac{\bar{l}}{(1-N_0G)} + \int_0^t \frac{N_0 p_3(s)}{(1-N_0G)}[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \tag{4.28}$$

Now an application of Lemma 2.2 (with $e(t) = \dfrac{\bar{l}}{(1-N_0G)}$), known as Pachpatte's inequality, to (4.28), yields that for $0 \le t < 1$,

$$\|x(t) - y(t)\| \le (\frac{\bar{l}}{(1-N_0G)}) \exp(\int_0^t \frac{N_0 p_3(s)}{(1-N_0G)}[1 + \int_0^s q_1(s,\tau)d\tau]ds). \tag{4.29}$$

Case 2: $1 \le t < \infty$**.** Following an arguments as in Case 2 of the proof of Theorem 3.2 and from the hypotheses, we obtain

$$\begin{aligned} u(t) &\le N_0\delta_2 + N_0\delta_1 + N_0Gu(t) + \int_0^1 N_0 p_3(s)\|\phi(s) - \psi(s)\| ds \\ &+ \int_0^t N_0(p_3(s) + p_3(s+1))[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds \\ &\le \bar{l} + N_0Gu(t) + \int_0^t N_0(p_3(s) + p_3(s+1))[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \end{aligned}$$

which implies

$$u(t) \le \frac{\bar{l}}{(1-N_0G)} + \int_0^t \frac{N_0(p_3(s) + p_3(s+1))}{(1-N_0G)}[u(s) + \int_0^s q_1(s,\tau)u(\tau)d\tau]ds, \tag{4.30}$$

Now an application of Lemma 2.2 (with $e(t) = \dfrac{\bar{l}}{(1-N_0G)}$), known as Pachpatte's inequality, to (4.30), yields that for $1 \le t < \infty$,

$$\|x(t) - y(t)\| \le (\frac{\bar{l}}{(1-N_0G)}) \exp(\int_0^t \frac{N_0(p_3(s) + p_3(s+1))}{(1-N_0G)}[1 + \int_0^s q_1(s,\tau)d\tau]ds). \tag{4.31}$$

This completes the proof.

REFERENCES

[1] Bellman, R. and Cooke, K. L. , Differential-Difference Equations, *Academic Press, New York,* 1963..

[2] Bielecki, A.,Une remarque sur la méthod de Banach-Cacciopoli-Tikhnov dans la théorie des équations difféerentilles ordinaries, Bull. Acad. Polon. Sci. Sér. Sci. Math. Phys. Astr.,4(1965), 261-264.

[3] Corduneanu, C., Bielecki's method in the theory of integral equations, Ann.Univ. Mariae Curie-Sklodowska, Section A, 38(1984), 23-40.

[4] Corduneanu, C., Integral Equations and Applications, Cambridge University Press, 1991.

[5] Driver, R. D., Ordinary and Delay Differential Equations, Appl. Math. Sci. 20, Springer, New York, 1977.

[6] Hale, J. K., Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[7] Kung, Y., Delay Differential Equations with Applicationsin Population Dynamics, Academic Press, New York, 1993.

[8] Pachapatte, B. G., Inegral and Finite Difference Inequalities and Applications, North-Holland Mathematics Studies, Vol.205, Lesevier Science B. V., Amsterdam, (2006).

[9] Pachapatte, B. G., On a nonlinear Volterra integral-functional equation, Funkcialaj Ekvacioj, 26(1983), 1-9.

[10] Pachapatte, B. G., Inequalities for Differential and Integral Equations, Academic Press, New York, 1998.

[11] Pachapatte, B. G., Some basic theorems on difference-differential equations, Electronic Journal of Differential Equations, Vol. 2008(2008), No. 75, 1-11.

[12] Pachapatte, B. G., Existexne and uniqueness theorems on certain difference-differential equations, Electronic Journal of Differential Equations, Vol. 2009(2009), No. 49, 1-12.

[13] Stokes, A., The applications of a fixed point theorem to a variety of non-linear stability problems, Contributions to the Theory of Nonlinear Oscillations, V, Ann. Math. Studies, 45(1960), 173-184.

[14] Sugiyama, S., On the existence and uniqueness theorems of difference-differential equations, K dai Math. Sem. Rep.,12(1960), 179-190.

[15] Sugiyama, S., On the boundednes of solutions of difference-differential equations, Proc. Japan Acad., 36(1960), 456-460.

[16] Sugiyama, S., Existence theorems on difference-differential equations, Proc. Japan Acad., 38(1962), 145-149.

[17] Sugiyama, S., Dependence properties of solutions of the retardation and initial values in the theory of difference-differential equations, K dai Math. Sem. Rep.,15(1963), 67-78.