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The Total Open Monophonic Number of a Graph

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ABSTRACT

For a connected graph G of order $n \geq 2$, a subset S of vertices of G is a monophonic set of G if each vertex v in G lies on a x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is defined as the monophonic number of G, denoted by $m(G)$. A monophonic set of cardinality $m(G)$ is called a m-set of G. A set S of vertices of a connected graph G is an open monophonic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a minimum open monophonic set and this cardinality is the open monophonic number, $om(G)$. A connected open monophonic set of G is an open monophonic set S such that the subgraph $< S >$ induced by S is connected. The minimum cardinality of a connected open monophonic set of G is the connected open monophonic number of G and is denoted by $om_c(G)$. A total open monophonic set of G is an open monophonic set S such that the subgraph $< S >$ induced by S contains no isolated vertices. The minimum cardinality of a total open monophonic set of G is the total open monophonic number of G and is denoted by $om_t(G)$. A total open monophonic set of cardinality $om_t(G)$ is called a om_t-set of G. The total open monophonic numbers of certain standard graphs are determined. Graphs with total open monophonic number 2 are characterized. It is proved that if G is a connected graph such that $om_t(G) = 3$(or $om_c(G) = 3$), then $G = K_3$ or G contains exactly two extreme vertices. It is proved that for any integer $n \geq 3$, there exists a connected graph G of order n such that $om(G) = 2$, $om_t(G) = om_c(G) = 3$. It is proved that for positive integers r, d and $k \geq 4$ with $r \leq d \leq 2r$, there exists a connected graph G of radius r, diameter d and total open monophonic number k. It is proved that for positive integers a, b, n with $4 \leq a \leq b \leq n$, there exists a connected graph G of order n such that $om_t(G) = a$ and $om_c(G) = b$.

Keywords:

Monophonic number; open monophonic number; connected open monophonic number; total open monophonic number.

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1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an *u-v geodesic*. It is known that this distance is a metric on the vertex set V. For any vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad G* and the maximum eccentricity is its *diameter*, *diam G* of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. A *geodetic set* of G is a set $S \subseteq V$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. A vertex x is said to *lie* on a *u-v geodesic P* if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. A set S of vertices of a connected graph G is an *open geodetic set* of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number* $og(G)$. It is clear that every open geodetic set is a geodetic set so that $g(G) \leq og(G)$. The geodetic number of a graph was introduced and studied in [1, 2]. The open geodetic number of a graph was introduced and studied in [3, 5, 7] in the name open geodomination in graphs. A chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge u_iu_j with $j \geq i + 2$. For two vertices u and v in a connected graph G, a u-v path is called a *monophonic path* if it contains no chords. A set S of vertices in a connected graph G is a *monophonic set* of G if every vertex of G is contained in a monophonic path joining some pair of vertices in S. The *monophonic number* $m(G)$ of G is the cardinality of a minimum monophonic set. A set S of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of an x-y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* and this cardinality is the *open monophonic number* $om(G)$ of G. An open monophonic set of cardinality $om(G)$ is called a *om-set* of G. The open monophonic number of a graph was introduced and studied in [9]. The *connected open monophonic number* of a graph was introduced and studied in [8].

The following theorems are used in the sequal.

Theorem 1.1[9] Every extreme vertex of a connected graph G belongs to each open monophonic set of G. In particular, if the set S of all extreme vertices of G is an open monophonic set of G, then S is the unique minimum open monophonic set of G.

Theorem 1.2 [9] If G is a connected graph with no extreme vertices, then $om(G) \geq 3$.

Theorem 1.3 [9] If G is a connected graph with a cutvertex v, then every open monophonic set of G contains at least one vertex from each component of $G - v$.

Theorem 1.4 [9] Each cutvertex of a connected graph G belongs to every minimum connected open monophonic set of G.

2. TOTAL OPEN MONOPHONIC NUMBER OF A GRAPH

Definition 2.1 Let G be a connected graph with at least two vertices. A *total open monophonic set* of a graph G is an open monophonic set S such that the subgraph $< S >$ induced by S contains no isolated vertices. The minimum cardinality of a total open monophonic set of G is the *total open monophonic number* of G and is denoted by $om_t(G)$. A total open monophonic set of cardinality $om_t(G)$ is called om_t*-set* of G.

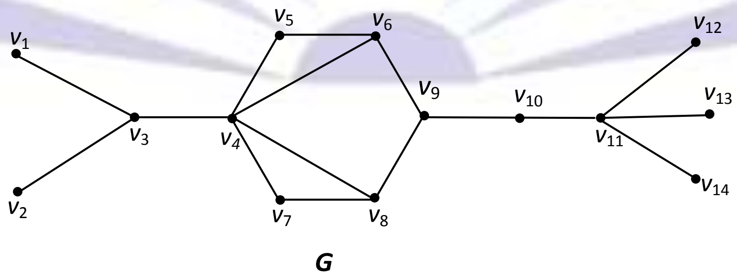

Figure 1 A graph G with $om_t(G)$ = 10.

Example 2.2 For the graph G given in Figure 1, it is clear that the set $S = \{v_1, v_2, v_5, v_7, v_{12}, v_{13}, v_{14}\}$ is the unique minimum open monophonic set of G so that $om(G) = 7$. It is easily verified that the set $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_{11}, v_{12}, v_{13}, v_{14}\}$ is the unique minimum total open monophonic set of G so that $om_t(G) = 10$. Also, it is clear that $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and $S_3 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ are the minimum connected open monophonic sets and so $om_c(G) = 13$. Thus the open monophonic number, total open monophonic number and the connected open monophonic number of a graph are different.

By Theorem 1.1, each extreme vertex belongs to every total open monophonic set. Since a total open monophonic set contains no isolated vertices, it follows that each support vertex of G also belongs to every total monophonic set of G.

Let S be the set of all extreme vertices and support vertices of G. Then every total open monophonic set of G contains S. If S is a total open monophonic set of G, then it follows that S is the unique minimum total open monophonic set of G. Thus we have the following theorem.

Theorem 2.3 Every total open monophonic set of a connected graph G contains all its extreme vertices and support vertices. If the set of all extreme vertices and support vertices form a total open monophonic set of G, then it is the unique minimum total open monophonic set of G.

Corollary 2.4 For the complete graph $K_n (n \geq 2)$, $om_t(K_n) = n$.

Theorem 2.5 For a connected graph G, $2 \leq om(G) \leq om_t(G) \leq om_c(G) \leq n$.

Proof. An open monophonic set needs at least two vertices and so $om(G) \geq 2$. Since every connected open monophonic set of G is a total open monophonic set, and every total open monophonic set is an open monophonic set, it follows that $om(G) \leq om_t(G) \leq om_c(G)$. Since the vertex set V is a connected open monophonic set of G, it is clear that $om_c(G) \leq n$. Hence $2 \leq om(G) \leq om_t(G) \leq om_c(G) \leq n$.

Corollary 2.6 If G is a connected graph such that $om_t(G) = 2$, then $om(G) = 2$.

Corollary 2.7 If G is a connected graph such that $om(G) = n$, then $om_t(G) = om_c(G) = n$.

Remark 2.8 For the complete graph K_n $(n \geq 2)$, $om_t(K_n) = n$ so that the total open monophonic number of a graph attains its least value 2 and largest value n. Also, all the inequalities in Theorem 2.5 can be strict.

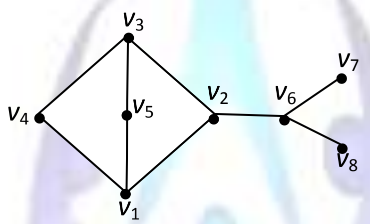

Figure 2 A Graph G with $2 < om(G) < om_t(G) < om_c(G) < n$.

For the graph G given in Figure 2, it is clear that the set $S = \{v_1, v_3, v_4, v_7, v_8\}$ is a minimum open monophonic set of G so that $om(G) = 5$. It is easily verified that the set $S_1 = \{v_1, v_3, v_4, v_6, v_7, v_8\}$ is a minimum total open monophonic set so that $om_t(G) = 6$. Also, it is clear that the set $S_2 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8\}$ is a minimum connected open monophonic set of G so that $om_c(G) = 7$. Thus $2 < om(G) < om_t(G) < om_c(G) < n$.

Also, we notice that for any path of order at least 4, the open monophonic number is 2, whereas the total open monophonic number is 4. This shows that the converse of Corollary 2.6 need not be true.

Theorem 2.9 For any non-trivial tree T, the set of all endvertices and support vertices of T is the unique minimum total open monophonic set of G.

Proof. Since the set of all endvertices and support vertices of T forms a total open monophonic set, the results follows from Theorem 2.3.

Theorem 2.10 If G is a connected graph with no extreme vertices, then $om_t(G) \geq 3$.

Proof. This follows from Theorems 1.2 and 2.5.

Theorem 2.11 Let G be a connected graph with cutvertices and let S be a total open monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.

Proof. Since every total open monophonic set is an open monophonic set, the result follows from Theorem 1.3.

Now, we characterize the graphs for which the total open monophonic number is 2.

Theorem 2.12 For any connected graph G, $om_t(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $om_t(G) = 2$. Conversely, if $om_t(G) = 2$, then $S = \{u, v\}$ is a minimum total open monophonic set of G. Then uv is an edge. It is clear that no vertex other than u and v lie on a $u - v$ monophonic path and so $G = K_2$.

Theorem 2.13 If G is connected graph such that $om_t(G) = 3$ (or $om_c(G) = 3$), then $G = K_3$ or G contains exactly two extreme vertices.

Proof. Let $om_t(G) = 3$. Let $S = \{v_1, v_2, v_3\}$ be a minimum total open monophonic set of G. Then it is clear that the subgraph $< S >$ induced by S is either K_3 or $K_{1,2}$. If $< S > = K_3$, then obviously $G = K_3$. Now, suppose that $< S > = K_{1,2}$.

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Without loss of generality, assume that v_2 and v_3 are nonadjacent. By Theorem 1.1, it follows that any vertex $v \neq v_1, v_2, v_3$ is non-extreme. We show that v_2 and v_3 are extreme in G. If v_2 is not extreme, then there exists a vertex v in S such that $v \neq v_1, v_2, v_3$ and v_2 lies as an internal vertex of a $v_3 - v$ monophonic path in G. This is not possible and so v_2 is an extreme vertex of G. Similarly, v_3 is also an extreme vertex of G. Thus G contains exactly two extreme vertices.

Corollary 2.14 If G is a connected graph such that $om_t(G) = 3$ (or $om_c(G) = 3$) and if $S = \{v_1, v_2, v_3\}$ is a minimum total(or connected) open monophonic set of G such that subgraph induced by S is $K_{1,2}$ with v_2 and v_3 nonadjacent, and if deg v_2 = deg v_3 = 1 in G, then $G = K_{1,2}$.

Proof. By Theorem 2.13, v_2 and v_3 are the only two extreme vertices of G. Since deg v_2 = deg v_3 = 1, a vertex $v \neq v_1, v_2, v_3$ cannot lie as an internal vertex of a $v_2 - v_3$ monophonic path. Hence it follows that $G = K_{1,2}$.

The following existence theorem is interesting.

Theorem 2.15 For any integer $n \geq 3$, there exists a connected graph G of order n such that $om(G) = 2$ and $om_t(G) = om_c(G) = 3$.

Proof. For $n = 3$, let $G = K_{1,2}$. Then, obviously $om(G) = 2$, $om_t(G) = om_c(G) = 3$. For $n = 4$, let G be the graph given in Figure 3.

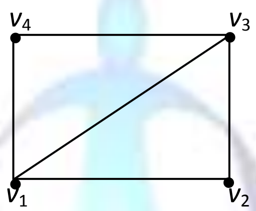

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Figure 3 A graph *G* of order 4 with $om_t(G) = om_c(G) = 3$.

Then $S = \{v_2, v_4\}$ is an open monophonic set of G and $S' = \{v_1, v_2, v_4\}$ is a minimum total as well as a connected open monophonic set of G so that $om_t(G) = om_c(G) = 3$.

For $n \geq 5$, let G be the graph obtained from the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ by adding the edges $v_1 v_3$ and $v_1 v_{n-1}$. The graph G is given in Figure 4. Then v_2 and v_n are the extreme vertices of G and $S = \{v_2, v_n\}$ is an open monophonic set of G so that $om(G) = 2$. Since the subgraph induced by S is not connected (not total also), and since $S' = \{v_1, v_2, v_n\}$ is a connected open monophonic set of G, it follows that $om_t(G) = om_c(G) = 3$. Thus the proof is complete.

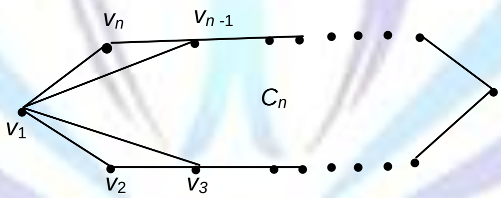

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Figure 4 A graph *G* of order *n* with $om_t(G) = om_c(G) = 3$.

In the following, we give another class of graphs of order $n \geq 4$ with $om(G) = 2$ and $om_t(G) = om_c(G) = 3$.

For $n \geq 4$, let $P_{n-1} : v_1, v_2, \ldots, v_{n-1}$ be a path of order $n - 1$. Let G be the graph in Figure 5 obtained from P_{n-1} by adding a new vertex v and joining the edges vv_i for each $i = 1, 2, \ldots, n - 1$. Then v_1 and v_{n-1} are the extreme vertices of G and $S = \{v_1, v_{n-1}\}$ is an open monophonic set of G so that $om(G) = 2$. Since the subgraph induced by S is not connected (not total also), and since $S' = \{v_1, v, v_{n-1}\}$ is a connected open monophonic set of G, it follows that $om_t(G) = om_c(G) = 3$.

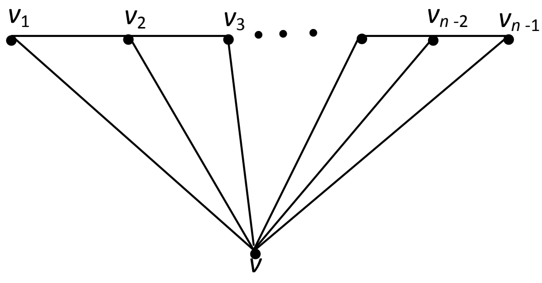

Figure 5 A graph *G* of order *n* with $om_t(G) = 3$

We leave the following problem as an open question.

Problem 2.16 Characterize the class of graphs G for which $om_t(G) = 3$ (or $om_c(G) = 3$).

Theorem 2.17 For any cycle $G = C_n (n \geq 4)$, $om_t(G) = 4$.

Proof. For $G = C_4$, it is clear that no 3-element subset of vertices is an open monophonic set of G. Hence it follows that $om_t(G) = 4$. For $G = C_5$, it is easily seen that no 3-element subset of vertices is an open monophonic set of G. Since $S = \{v_1, v_2, v_3, v_4\}$ is a total open monophonic set of G, it follows that $om_t(G) = 4$. Let the cycle $G = C_n (n \geq 6)$ be $C_n : v_1, v_2, \ldots, v_n, v_1$. Since G has no extreme vertices, it follows from Theorem 2.10 that $om_t(G) \geq 3$. It is easily seen that no 3-element subset of G is a total open monophonic set. Now, let v_t be a vertex of G such that $d(v_1, v_t) \geq 3$. Then it is clear that $S' = \{v_1, v_2, v_t, v_{t+1}\}$ is a total open monophonic set of G so that $om_t(G) = 4$. Thus the proof of the theorem is complete.

Theorem 2.18 For the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$, $om_t(G) = 4$.

Proof. Let $G = K_{r,s} (2 \leq r \leq s)$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of G. Since G contains no extreme vertices, it follows from Theorem 2.10 that $om_t(G) \geq 3$. It is clear that no 3-element subset of vertices of G is an open monophonic set of G so that $om_t(G) \geq 4$. Let S be any set of four vertices formed by taking two vertices from each of U and W. Then it is clear that S is a total open monophonic set of G so that $om_t(G) = 4$.

Theorem 2.19 For any wheel $W_n = K_1 + C_{n-1} (n \geq 5)$, $om_t(W_n) = 4$.

Proof. Let $W_n = K_1 + C_{n-1} (n \geq 5)$. Let $n \geq 7$. Since W_n has no extreme vertices, it follows from Theorem 2.10 that $om_t(W_n) \geq 3$. It is easily seen that no 3-element subset of W_n is a total open monophonic set. Now, let v_t be a vertex of C_{n-1} such that $d(v_1, v_t) \geq 3$ in C_{n-1}. Then it is clear that $S' = \{v_1, v_2, v_t, v_{t+1}\}$ is a total open monophonic set of W_n so that $om_t(W_n) = 4$. Now, let $W_n = K_1 + C_{n-1} (n = 5, 6)$. Since W_n has no extreme vertices, it follows from Theorem 2.10 that $om_t(W_n) \geq 3$. It is easily verified that no 3-element subset of vertices of W_n is an open monophonic set. Since $S = \{v_1, v_2, v_3, v_4\}$ is a total open monophonic set of W_n, it follows that $om_t(W_n) = 4$. Thus the proof is complete.

3. EXISTENCE RESULTS

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [6] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the total open monophonic number can also be prescribed, when $a \leq b \leq 2a$.

Theorem 3.1 For positive integers r, d and $k \geq 4$ with $r \leq d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $om_t(G) = k$.

Proof. If $r = 1$, then $d = 1$ or 2. For $d = 1$, let $G = K_k$. Then $om_t(G) = k$. For $d = 2$, $om_t(G) = k$, where $G = K_{1,k-1}$. For $r \geq 2$, we construct a graph G with the desired properties as follows:

Case 1. $r = d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph in Figure 6, obtained from C_{2r} by adding the new vertices $v_1, v_2, \ldots, v_{k-3}$ and joining each $v_i (1 \leq i \leq k-3)$ with u_1 and u_2 of C_{2r}, and also joining u_r and u_{r+2}. It is easily verified that the eccentricity of each vertex of G is r so that $rad\ G = diam\ G = r$.

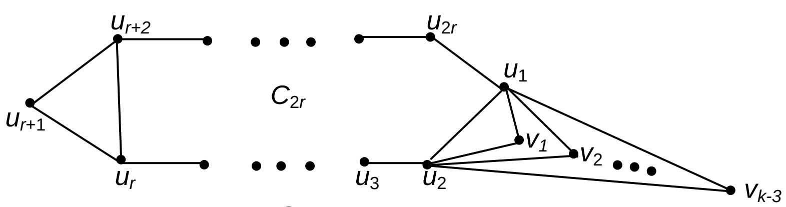

Figure 6 A graph *G* with *rad G = diam G = r and* $om_t(G) = k$.

Let $S = \{v_1, v_2, \ldots, v_{k-3}, u_{r+1}\}$ be the set of all extreme vertices of G. Then S is an open monophonic set of G, and it is not a total open monophonic set of G. By Theorem 2.3, every total open monophonic set of G contains S. It is clear that for any $x \notin S$, $S \cup \{x\}$ is not a total open monophonic set of G. It is easily verified that the set $S_1 = S \cup \{u_1, u_r\}$ is a minimum total open monophonic set of G so that $om_t(G) = k$.

Case 2. $r < d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, v_2, \ldots, v_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying the vertex v_0 of P_{d-r+1} and u_1 of C_{2r}. Now, let G be the graph obtained by adding the new vertices $w_1, w_2, \ldots, w_{k-4}$ to H and joining each vertex $w_i (1 \le i \le k - 4)$ with the vertex v_{d-r-1}, and also joining u_r and u_{r+2}. The graph G is shown in Figure 7 and has $rad\ G = r$ and $diam\ G = d$.

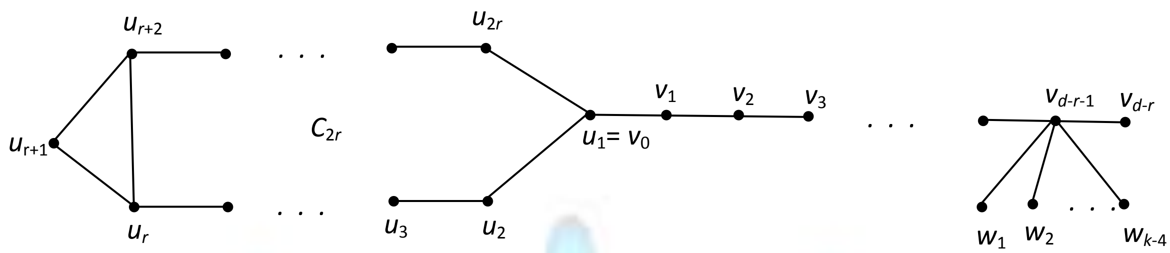

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Figure 7 A graph G with *rad G =r*, *diam G = d and om$_t$*(G) = *k.*

Let $S = \{w_1, w_2, \ldots, w_{k-4}, v_{d-r}, u_{r+1}, v_{d-r-1}\}$ be the set of all extreme vertices and support vertices of G. By Theorem 2.3, every total open monophonic set of G contains S. Since $S \cup \{u_r\}$ is a total open monophonic set of G, it follows that $om_t(G) = k$.

Theorem 3.2 For positive integers r, d and $k = 3$ with $r \le d \le 2r$ and $d = r + 1$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $om_t(G) = k$.

Proof. If $r = 1$, then $d = 1$ or 2. For $d = 1$, let $G = K_k$. Then $om_t(G) = k$. For $d = 2$, $om_t(G) = k$, where $G = K_{1,k-1}$. For $r \ge 2$, we construct a graph G with the desired properties as follows:

Case 1. $r = d$. For $r = 2$, let G be the graph shown in Figure 8. Then it is clear that $d = 2$ and $om_t(G) = 3$.

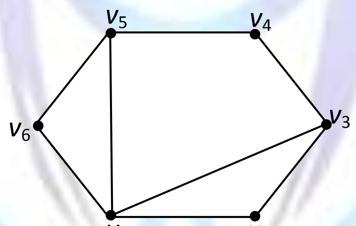

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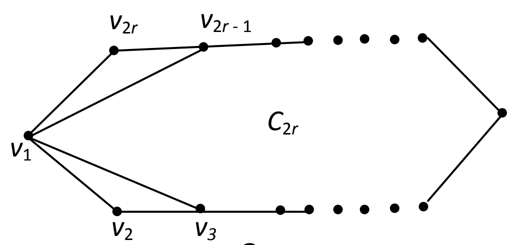
Figure 8 A graph *G* with *rad G = diam G* = 2 and *om$_t$*(*G*) = 3.

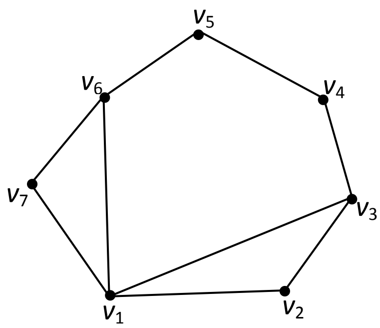
Now, let $r \ge 3$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$. Let G be the graph in Figure 9, obtained by adding the edges v_1v_3 and v_1v_{2r-1}. It is easily verified that the eccentricity of each vertex of G is r so that $rad\ G = diam\ G = r$. Also v_2 and v_{2r} are the extreme vertices of G and $S = \{v_2, v_{2r}\}$ is an open monophonic set of G so that $om(G) = 2$. Since the subgraph induced by S is not total, and since $S' = \{v_1, v_2, v_{2r}\}$ is a total open monophonic set of G, it follows that $om_t(G) = 3$.



G

Figure 9 A graph *G* with *rad G = diam G* and *om$_t$*(*G*) = 3.

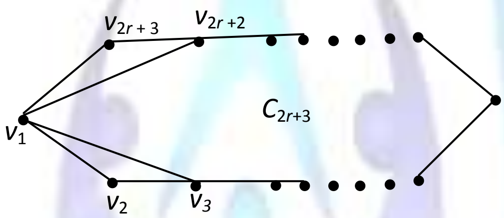
Case 2. $r < d$ and $d = r + 1$. For $r = 2$, let G be the graph shown Figure 10. Then it is clear that $d = 3$ and $om_t(G) = 3$.



G

Figure 10 A graph *G* with *rad G* =2, *diam G* = 3 and *om*$_t$(*G*) = 3.

Now, let $r \geq 3$. Let C_{2r+3} : $v_1, v_2, \ldots, v_{2r+3}, v_1$ be a cycle of order $2r + 3$. Let G be the graph in Figure 11 obtained by adding the edges v_1v_3 and v_1v_{2r+2}. It is easily verified that the eccentricity of each vertex of G is r so that $rad\ G = r$ and $diam\ G = r + 1$. Also v_2 and v_{2r+3} are the extreme vertices of G and $S = \{v_2, v_{2r+3}\}$ is an open monophonic set of G so that $om(G) = 2$. Since the subgraph induced by S is not total, and since $S' = \{v_1, v_2, v_{2r+3}\}$ is a total open monophonic set of G, it follows that $om_t(G) = 3$.



G

Figure 11 A graph *G* with *rad G* = *r*, *diam G* = *r* + 1 and *om*$_t$(*G*) = 3.

We leave the following problem as an open question.

Problem 3.3 For positive integers r, d and $k = 3$ with $r \leq d \leq 2r$, does there exist a connected graph G with $rad\ G = r$, $diam\ G = r + l$ with $2 \leq l \leq r$ and $om_t(G) = k$?

Remark 3.4 For $k = 2$, by Theorem 2.12, $om_t(G) = 2$ if and only if $G = K_2$. Hence for $k = 2$, a graph exists only when $r = d = 1$.

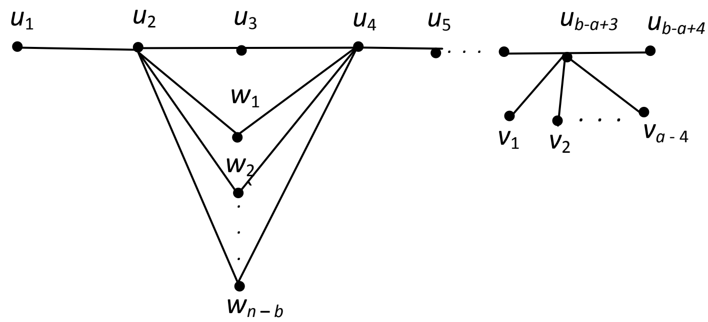
In the view of Theorem 2.5, we have the following realization theorem.

Theorem 3.5 For positive integers a, b and n with $4 \leq a \leq b \leq n$, there exists a connected graph G of order n, with $om_t(G) = a$ and $om_c(G) = b$.

Proof. We prove this theorem by considering four cases.

Case 1. $a = b = n$. By Theorem 1.1, $om_c(G) = om_t(G) = n$ for $G = K_n$.

Case 2. $a < b < n$. Let P_{b-a+4}: $u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b - a + 4$. Let G be the graph of order n in Figure 12, obtained from P_{b-a+4} by adding the new vertices $w_1, w_2, \ldots, w_{n-b}$; $v_1, v_2, \ldots, v_{a-4}$ to P_{b-a+4} and joining $w_1, w_2, \ldots, w_{n-b}$ with both u_2 and u_4; and also joining each $v_i (1 \leq i \leq a - 4)$ with u_{b-a+3}.

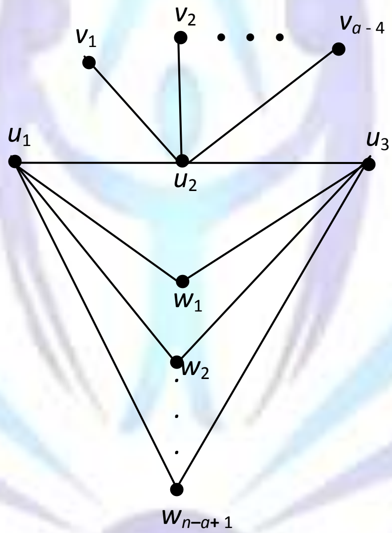


G

Figure 12 A graph *G* with $om_t(G)$ = a and $om_c(G)$ = b for $a < b < n$.

Let $S_1 = \{u_1, u_{b-a+4}, v_1, v_2, \ldots, v_{a-4}\}$, $S_2 = \{u_2, u_{b-a+3}\}$ and $S_3 = \{u_2, u_4, u_5, \ldots, u_{b-a+3}\}$ denote the sets of all extreme vertices, support vertices and cutvertices, respectively. Since $S_1 \cup S_2$ is a total open monophonic set of *G*, it follows from Theorem 2.3 that $om_t(G) = a$. By Theorems 1.1 and 1.4, every connected open monophonic set contains $S_1 \cup S_3$. Since the subgraph induced by $S_1 \cup S_3$ is not connected, and since $S_1 \cup S_3 \cup \{u_3\}$ is a connected open monophonic set of *G*, it follows that $om_c(G) = b$.

Case 3. $a = b < n$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let *G* be the graph of order *n* in Figure 13, obtained from P_3 by adding the new vertices $v_1, v_2, \ldots, v_{a-4}$ and joining each $v_i (1 \le i \le a - 4)$ with u_2; and also adding the new vertices $w_1, w_2, \ldots, w_{n-a+1}$ and joining each $w_i (1 \le i \le n - a + 1)$ with u_1 and u_3.



G

Figure 13 A graph *G* with $om_t(G) = om_c(G)$ = a for $a < b < n$

First, let $a > 4$. Let $S = \{v_1, v_2, \ldots, v_{a-4}, u_2\}$. By Theorem 2.3, every total open monophonic set of *G* contains *S*. It is easily verified that for any vertex $w_i (1 \le i \le n - a + 1)$, $S_i = S \cup \{u_1, u_3, w_i\}$ is a minimum total open monophonic set of *G* so that $om_t(G) = a$. Since S_i is also minimum connected open monophonic set of *G*, we have $om_c(G) = a$. Thus $om_t(G) = om_c(G) = a = b$. Next, let $a = 4$. Then it is clear that for any vertex $w_i (1 \le i \le n - a + 1)$, $T_i = \{u_1, u_2, u_3, w_i\}$ is a minimum total open monophonic set as well as a minimum connected open monophonic set of *G* so that $om_t(G) = om_c(G) = 4 = a = b$.

Case 4. $a < b = n$. Let $P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b - a + 4$. Let *G* be the graph of order *n* in Figure 14, obtained from P_{b-a+4} by adding the new vertices $v_1, v_2, \ldots, v_{a-4}$ and joining each $v_i (1 \le i \le a - 4)$ with u_{b-a+3}.

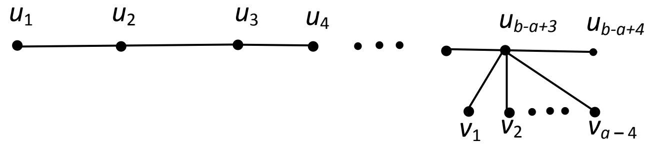


Figure 14 A graph G with $om_t(G) = a$ and $om_c(G) = b = n$ for $a < b = n$.

Let $S = \{u_1, u_{b-a+4}, v_1, v_2, \ldots, v_{a-4}, u_2, u_{b-a+3}\}$ be the set of all extreme vertices and support vertices of G. It follows from Theorem 2.3 that $om_t(G) = a$. Let $S_1 = \{u_2, u_3, \ldots, u_{b-a+3}\}$ be the set of all cutvertices of G. Since $S \cup S_1$ is a connected open monophonic set of G, it follows from Theorems 1.1 and 1.4 that $om_c(G) = b = n$.

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