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EIGENVALUE PROBLEM WITH MOVING DISCONTINUITY POINTS

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ABSTRACT

In this paper, we present a Sturm Liouville problem which has discontinuities in the neighborhood of the midpoint of an interval and a boundary condition depending on an eigenparameter. We derive operator theoretic formulation in suitable Hilbert space, give some properties of the eigenvalues and obtain asymptotic formulas for the eigenvalues and the corresponding eigenfunctions.

Keywords

Eigenvalue and Eigenfunction; Discontinuous Sturm Liouville Problem; Moving Discontinuity Points.

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1 INTRODUCTION

The Sturm Liouville theory is one of the most actual and extensively developing fields of theoretical and applied mathematics. In recent years, highly important results in this field have been obtained for classic Sturm Liouville problems (see [1-4]) and discontinuous Sturm Liouville problems which has transmission conditions at one, two or several points of discontinuity (see [5-8], [9-11] and [12-14], respectively].

We consider the boundary value problem

$$\tau(u) := -u''(x) + q(x)u(x) = \lambda u(x), \qquad x \in I\,, \tag{1.1}$$

with boundary conditions

$$B_a(u) := \beta_1 u(a) + \beta_2 u'(a) = 0\,, \tag{1.2}$$

$$B_b(u) := \lambda\left(\alpha_1' u(b) - \alpha_2' u'(b)\right) + \alpha_1 u(b) - \alpha_2 u'(b) = 0\,, \tag{1.3}$$

and transmission conditions

$$T_{-\varepsilon}(u) := u(\theta_{-\varepsilon}-) - \delta u(\theta_{-\varepsilon}+) = 0\,, \tag{1.4}$$

$$T_{-\varepsilon}'(u) := u'(\theta_{-\varepsilon}-) - \delta u'(\theta_{-\varepsilon}+) = 0\,, \tag{1.5}$$

$$T_{+\varepsilon}(u) := \delta u(\theta_{+\varepsilon}-) - \gamma u(\theta_{+\varepsilon}+) = 0\,, \tag{1.6}$$

$$T_{+\varepsilon}'(u) := \delta u'(\theta_{+\varepsilon}-) - \gamma u'(\theta_{+\varepsilon}+) = 0\,, \tag{1.7}$$

where $I = [a, \theta_{-\varepsilon}) \cup (\theta_{-\varepsilon}, \theta_{+\varepsilon}) \cup (\theta_{+\varepsilon}, b]$; λ is a complex spectral parameter; $q(x)$ is a given real-valued function, which is continuous in $[a, \theta_{-\varepsilon})$, $(\theta_{-\varepsilon}, \theta_{+\varepsilon})$ and $(\theta_{+\varepsilon}, b]$ and has a finite limit $q(\theta_{-\varepsilon}\pm) := \lim_{x\to\theta_{-\varepsilon}\pm} q(x)$ and $q(\theta_{+\varepsilon}\pm) := \lim_{x\to\theta_{+\varepsilon}\pm} q(x)$; $\beta_i, \alpha_i, \alpha_i', \delta, \gamma$ $(i = 1,2)$ are real numbers such that $\delta \neq 0, \gamma \neq 0$; ε is a parameter such that $0 < \varepsilon < (b-a)/2$; $\theta = (a+b)/2$; $\theta_{\pm\varepsilon}\pm := (\theta \pm \varepsilon) \pm 0$ and

$$\rho := \left(\alpha_1' \alpha_2 - \alpha_1 \alpha_2'\right) > 0\,. \tag{1.8}$$

In this paper, we present a problem which has points of discontinuity in the neighborhood of the midpoint of an interval to different from the studies in the literature. ε is a parameter controling the change of neighborhood process (it can be called tuning parameter) and by using the change of this ε parameter it's possible to determine points of discontinuity. That is, two points of discontinuity can be determined in the interval $[a,b]$ for each ε value in the interval $0 < \varepsilon < (b-a)/2$. For example, let $a = -1, b = 4$ so that θ and ε parameter's interval are $\theta = 3/2$ and $0 < \varepsilon < 5/2$; points of discontinuity are $\theta_{-\varepsilon} = 1$ and $\theta_{+\varepsilon} = 2$ for $\varepsilon = 1/2$, points of discontinuity are $\theta_{-\varepsilon} = -1/2$ and $\theta_{+\varepsilon} = 7/2$ for $\varepsilon = 2$, etc. The main result is that points of discontinuity can be moved by changing ε parameter, so that they can be called moving discontinuity points. In the special case for our problem when $a = 0, b = \pi$ and $\varepsilon = d$ $(0 < d < \pi/2)$ is derived in [15,16] (the problems in these works do not contain an eigenparameter in the boundary conditions).

Firstly, we derive operator theoretic formulation in suitable Hilbert space such a way that the considered problem can be interpreted as the eigenvalue problem of this operator, then we give some properties of the eigenvalues and the eigenfunctions and finally we obtain asymptotic formulas for the eigenvalues and the corresponding eigenfunctions depending on ε parameter.

2 OPERATOR-THEORETIC FORMULATION

To formulate a theoretic approach to the problem (1.1)-(1.7) we define the Hilbert space $H := L_2(a,b) \oplus \mathbb{C}$ with an inner product

$$\langle F(.),G(.)\rangle_H := \int_a^{\theta_{-\varepsilon}} f(x)\overline{g}(x)dx + \delta^2 \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} f(x)\overline{g}(x)dx + \gamma^2 \int_{\theta_{+\varepsilon}}^{b} f(x)\overline{g}(x)dx + \frac{\gamma^2}{\rho} h\overline{k}, \tag{2.1}$$

where $F(x)=\begin{pmatrix} f(x) \\ h \end{pmatrix}, G(x)=\begin{pmatrix} g(x) \\ k \end{pmatrix} \in H$, $f(.),g(.)\in L_2(a,b)$ and $h,k\in\mathbb{C}$. For convenience we put

$$R(u)=\alpha_1 u(b)-\alpha_2 u'(b),\ R'(u)=\alpha_1' u(b)-\alpha_2' u'(b). \tag{2.2}$$

For function $f(x)$, which is defined on $[a,\theta_{-\varepsilon})\cup(\theta_{-\varepsilon},\theta_{+\varepsilon})\cup(\theta_{+\varepsilon},b]$ and has finite limit $f(\theta_{-\varepsilon}\pm):=\lim_{x\to\theta_{-\varepsilon}\pm} f(x)$ and $f(\theta_{+\varepsilon}\pm):=\lim_{x\to\theta_{+\varepsilon}\pm} f(x)$, by $f_{(1)}(x), f_{(2)}(x)$ and $f_{(3)}(x)$ we denote the functions

$$f_{(1)}(x):=\begin{cases} f(x), & x\in[a,\theta_{-\varepsilon}), \\ f(\theta_{-\varepsilon}-), & x=\theta_{-\varepsilon}, \end{cases} \qquad f_{(3)}(x):=\begin{cases} f(x), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}), \\ f(\theta_{+\varepsilon}-), & x=\theta_{+\varepsilon}, \end{cases}$$

$$f_{(2)}(x):=\begin{cases} f(\theta_{-\varepsilon}+), & x=\theta_{-\varepsilon}, \\ f(x), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}), \end{cases} \qquad f_{(4)}(x):=\begin{cases} f(\theta_{+\varepsilon}+), & x=\theta_{+\varepsilon}, \\ f(x), & x\in(\theta_{+\varepsilon},b], \end{cases}$$

which are defined on $I_1:=[a,\theta_{-\varepsilon}],\ I_2:=[\theta_{-\varepsilon},\theta_{+\varepsilon}]$ and $I_3:=[\theta_{+\varepsilon},b]$, respectively.

Let $D(A)\subseteq H$ be the set of all $F(x)=\begin{pmatrix} f(x) \\ R'(f) \end{pmatrix}\in H$ such that $f_{(i)}(.),\ f_{(i)}'(.)$ are absolutely continuous in $I_i\,(i=1,2,3),\ \tau(f)\in L_2(a,b)$, $h=R'(f)$ and $B_a(f)=0,\ T_{\pm\varepsilon}(f)=T_{\pm\varepsilon}'(f)=0$.

Define the operator $A:D(A)\to H$ by

$$A\begin{pmatrix} f(x) \\ R'(f) \end{pmatrix}=\begin{pmatrix} \tau(f) \\ -R(f) \end{pmatrix},\ \begin{pmatrix} f(x) \\ R'(f) \end{pmatrix}\in D(A). \tag{2.3}$$

The eigenvalues and the eigenfunctions of the problem (1.1)-(1.7) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A, respectively.

Theorem 2.1 The operator A in H is symmetric.

Proof. For $F(.),G(.)\in D(A)$

$$\langle AF(.),G(.)\rangle_H = \int_a^{\theta_{-\varepsilon}} \tau(f(x))\overline{g}(x)dx + \delta^2 \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \tau(f(x))\overline{g}(x)dx + \gamma^2 \int_{\theta_{+\varepsilon}}^{b} \tau(f(x))\overline{g}(x)dx - \frac{\gamma^2}{\rho} R(f)R'(\overline{g}). \tag{2.4}$$

By two partial integration we obtain

$$\langle AF(.),G(.)\rangle_H = \langle F(.),AG(.)\rangle_H + W(f,\overline{g};\theta_{-\varepsilon}-) - W(f,\overline{g};a) + \delta^2 W(f,\overline{g};\theta_{+\varepsilon}-) - \delta^2 W(f,\overline{g};\theta_{-\varepsilon}+) + \gamma^2 W(f,\overline{g};b) - \gamma^2 W(f,\overline{g};\theta_{+\varepsilon}+) - \frac{\gamma^2}{\rho}\left(R(f)R'(\overline{g}) - R'(f)R(\overline{g})\right), \tag{2.5}$$

where, as usual, by $W(f,g;x)$ we denote the Wronskian of the functions $f(x)$ and $g(x)$

$$W(f,g;x)=f(x)g'(x)-f'(x)g(x). \tag{2.6}$$

Since $f(x)$ and $\overline{g}(x)$ are satisfied the boundary condition (1.2),(1.3) and the transmission conditions (1.4)-(1.7), we get

$$W\left(f,\overline{g};a\right)=0, \tag{2.7}$$

$$W\left(f,\overline{g};\theta_{-\varepsilon}-\right)=\delta^2 W\left(f,\overline{g};\theta_{-\varepsilon}+\right), \tag{2.8}$$

$$W\left(f,\overline{g};\theta_{+\varepsilon}-\right)=\frac{\gamma^2}{\delta^2}W\left(f,\overline{g};\theta_{+\varepsilon}+\right), \tag{2.9}$$

$$\frac{\gamma^2}{\rho}\left(R(f)R'(\overline{g})-R'(f)R(\overline{g})\right)=\gamma^2 W\left(f,\overline{g};b\right). \tag{2.10}$$

Finally substituting (2.7)-(2.10) in (2.5) then we have

$$\left\langle AF(.),G(.)\right\rangle_H=\left\langle F(.),AG(.)\right\rangle_H, \tag{2.11}$$

thus the operator A is Hermitian. The symmetry of A arises from the well-known fact that $D(A)$ is dense in H.

Corollary 2.2 All eigenvalues of the problem (1.1)-(1.7) are real.

We can now assume that all eigenfunctions of the problem (1.1)-(1.7) are real valued.

Corollary 2.3 Let λ_1 and λ_2 be two different eigenvalues of the problem (1.1)-(1.7). Then the corresponding eigenfunctions u_1 and u_2 of this problem are orthogonal in the sense of

$$\int_a^{\theta_{-\varepsilon}} u_1(x)u_2(x)dx+\delta^2\int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} u_1(x)u_2(x)dx+\gamma^2\int_{\theta_{+\varepsilon}}^{b} u_1(x)u_2(x)dx+\frac{\gamma^2}{\rho}R'(u_1)R'(u_2)=0. \tag{2.12}$$

3 CONSTRUCTION OF FUNDAMENTAL

Now we will construct a special fundamental system of solutions of the equation (1.1). Let us consider the next initial value problem:

$$-u''(x)+q(x)u(x)=\lambda u(x),\quad x\in(a,\theta_{-\varepsilon}), \tag{3.1}$$

$$u(a)=\beta_2,\qquad u'(a)=-\beta_1. \tag{3.2}$$

By virtue of Theorem 1.5. in [17], this problem has a unique solution $u=\phi_{-\varepsilon,\lambda}(x)=\phi_{-\varepsilon}(x,\lambda)$, which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[a,\theta_{-\varepsilon}]$. Similarly, employing the same method as in proof of Theorem 1.5.in [17], we see that the problem

$$-u''(x)+q(x)u(x)=\lambda u(x),\quad x\in(\theta_{+\varepsilon},b), \tag{3.3}$$

$$u(b)=\lambda\alpha_2'+\alpha_2,\qquad u'(b)=\lambda\alpha_1'+\alpha_1, \tag{3.4}$$

has a unique solution $u=\chi_{+\varepsilon,\lambda}(x)=\chi_{+\varepsilon}(x,\lambda)$ which is an entire function of parameter λ for each fixed $x\in[\theta_{+\varepsilon},b]$.

Now the function $\phi_{\varepsilon,\lambda}(x)$ is defined in terms of $\phi_{-\varepsilon,\lambda}(x)$ as follows: the initial-value problem,

$$-u''(x)+q(x)u(x)=\lambda u(x),\quad x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}), \tag{3.5}$$

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$$u(\theta_{-\varepsilon}) = \delta^{-1}\phi_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-), \qquad u'(\theta_{-\varepsilon}) = \delta^{-1}\phi'_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-), \tag{3.6}$$

which contains the entire functions of eigenparameter λ, has unique solution $u=\phi_{\varepsilon,\lambda}(x)=\phi_{\varepsilon}(x,\lambda)$ for each $\lambda\in\mathbb{C}$. Also the function $\phi_{+\varepsilon,\lambda}(x)$ is defined in terms of $\phi_{\varepsilon,\lambda}(x)$ as follows: the initial value problem

$$-u''(x)+q(x)u(x)=\lambda u(x), \quad x\in(\theta_{+\varepsilon},b), \tag{3.7}$$

$$u(\theta_{+\varepsilon}) = \frac{\delta}{\gamma}\phi_{\varepsilon,\lambda}(\theta_{+\varepsilon}-), \qquad u'(\theta_{+\varepsilon}) = \frac{\delta}{\gamma}\phi'_{\varepsilon,\lambda}(\theta_{+\varepsilon}-), \tag{3.8}$$

which contains the entire functions of eigenparameter λ, has unique solution $u=\phi_{+\varepsilon,\lambda}(x)=\phi_{+\varepsilon}(x,\lambda)$ for each $\lambda\in\mathbb{C}$.

Similarly, the function $\chi_{\varepsilon,\lambda}(x)$ is defined in terms of $\chi_{+\varepsilon,\lambda}(x)$ as follows: the initial-value problem,

$$-u''(x)+q(x)u(x)=\lambda u(x), \quad x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}), \tag{3.9}$$

$$u(\theta_{+\varepsilon}) = \frac{\gamma}{\delta}\chi_{+\varepsilon,\lambda}(\theta_{+\varepsilon}+), \qquad u'(\theta_{+\varepsilon}) = \frac{\gamma}{\delta}\chi'_{+\varepsilon,\lambda}(\theta_{+\varepsilon}+), \tag{3.10}$$

has unique solution $u=\chi_{\varepsilon,\lambda}(x)=\chi_{\varepsilon}(x,\lambda)$ for each $\lambda\in\mathbb{C}$. And the function $\chi_{-\varepsilon,\lambda}(x)$ is defined in terms of $\chi_{\varepsilon,\lambda}(x)$ as follows: the initial-value problem,

$$-u''(x)+q(x)u(x)=\lambda u(x), \quad x\in(a,\theta_{-\varepsilon}), \tag{3.11}$$

$$u(\theta_{-\varepsilon}) = \delta\chi_{\varepsilon,\lambda}(\theta_{-\varepsilon}+), \qquad u'(\theta_{-\varepsilon}) = \delta\chi'_{\varepsilon,\lambda}(\theta_{-\varepsilon}+), \tag{3.12}$$

has unique solution $u=\chi_{-\varepsilon,\lambda}(x)=\chi_{-\varepsilon}(x,\lambda)$ for each $\lambda\in\mathbb{C}$.

Let us construct two basic solutions of equation (1.1) as

$$\phi_{\lambda}(x)=\begin{cases}\phi_{-\varepsilon,\lambda}(x), & x\in[a,\theta_{-\varepsilon}),\\ \phi_{\varepsilon,\lambda}(x), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ \phi_{+\varepsilon,\lambda}(x), & x\in(\theta_{+\varepsilon},b],\end{cases} \qquad \chi_{\lambda}(x)=\begin{cases}\chi_{-\varepsilon,\lambda}(x), & x\in[a,\theta_{-\varepsilon}),\\ \chi_{\varepsilon,\lambda}(x), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ \chi_{+\varepsilon,\lambda}(x), & x\in(\theta_{+\varepsilon},b].\end{cases} \tag{3.13}$$

Since the Wronskians $W(\phi_{\varepsilon,\lambda},\chi_{\varepsilon,\lambda};x)$ and $W(\phi_{\pm\varepsilon,\lambda},\chi_{\pm\varepsilon,\lambda};x)$ are independent of variable $x\in I_i\ (i=1,2,3)$ and $\phi_{\varepsilon,\lambda}(x),\chi_{\varepsilon,\lambda}(x),\phi_{\pm\varepsilon,\lambda}(x),\chi_{\pm\varepsilon,\lambda}(x)$ are the entire functions of the parameter λ for each $x\in I_i\ (i=1,2,3)$, the the functions

$$\begin{aligned}\omega_{-\varepsilon}(\lambda) &:= W(\phi_{-\varepsilon,\lambda},\chi_{-\varepsilon,\lambda};x), \ x\in[a,\theta_{-\varepsilon}],\\ \omega_{\varepsilon}(\lambda) &:= W(\phi_{\varepsilon,\lambda},\chi_{\varepsilon,\lambda};x), \ x\in[\theta_{-\varepsilon},\theta_{+\varepsilon}],\\ \omega_{+\varepsilon}(\lambda) &:= W(\phi_{+\varepsilon,\lambda},\chi_{+\varepsilon,\lambda};x), \ x\in[\theta_{+\varepsilon},b],\end{aligned} \tag{3.14}$$

are the entire functions of parameter λ.

After short calculation we see that $\omega_{-\varepsilon}(\lambda)=\delta^2\omega_{\varepsilon}(\lambda)=\gamma^2\omega_{+\varepsilon}(\lambda)$. Now we may introduce characteristic function $\omega(\lambda)$ as

$$\omega(\lambda):=\omega_{-\varepsilon}(\lambda)=\delta^2\omega_{\varepsilon}(\lambda)=\gamma^2\omega_{+\varepsilon}(\lambda). \tag{3.15}$$

Theorem 3.1 The eigenvalues of the problem (1.1)-(1.7) are coincided zeros of the function $\omega(\lambda)$.

Proof. Let $\omega(\lambda_0)=0$. Then $W\left(\phi_{-\varepsilon,\lambda_0},\chi_{-\varepsilon,\lambda_0};x\right)=0$ and so the functions $\phi_{-\varepsilon,\lambda_0}(x)$ and $\chi_{-\varepsilon,\lambda_0}(x)$ are linearly dependent, that is,

$$\chi_{-\varepsilon,\lambda_0}(x)=k\phi_{-\varepsilon,\lambda_0}(x),\quad x\in[a,\theta_{-\varepsilon}],\ \text{for some } k\neq 0. \tag{3.16}$$

Consequently, $\chi_{\lambda_0}(x)$ satisfied the boundary condition (1.2), so the function $\chi_{\lambda_0}(x)$ is an eigenfunction of the problem (1.1)-(1.7) corresponding to the eigenvalue λ_0.

Now let $u_0(x)$ be any eigenfunction corresponding to the eigenvalue λ_0, but $\omega(\lambda_0)\neq 0$.

Then the functions $\phi_{\varepsilon,\lambda_0}(x)$, $\chi_{\varepsilon,\lambda_0}(x)$ and $\phi_{\pm\varepsilon,\lambda_0}(x)$, $\chi_{\pm\varepsilon,\lambda_0}(x)$ are linearly independent on $I_i\ (i=1,2,3)$. Thus, $u_0(x)$ may be represented as in the form

$$u_0(x)=\begin{cases} c_1\phi_{-\varepsilon,\lambda_0}(x)+c_2\chi_{-\varepsilon,\lambda_0}(x), & x\in[a,\theta_{-\varepsilon}),\\ c_3\phi_{\varepsilon,\lambda_0}(x)+c_4\chi_{\varepsilon,\lambda_0}(x), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ c_5\phi_{+\varepsilon,\lambda_0}(x)+c_6\chi_{+\varepsilon,\lambda_0}(x), & x\in(\theta_{+\varepsilon},b], \end{cases} \tag{3.17}$$

where at least one of the constants $c_i\left(i=\overline{1,6}\right)$ is not zero. Considering the equations

$$B_a(u_0(x))=0,\ B_b(u_0(x))=0,\ T_{\pm\varepsilon}(u_0(x))=0,\ T'_{\pm\varepsilon}(u_0(x))=0, \tag{3.18}$$

as the homogenous system of linear equations of the variables $c_i\left(i=\overline{1,6}\right)$ and taking into account (3.6), (3.8), (3.10) and (3.12), it follows that the determinant of this system is

$$\begin{vmatrix} 0 & \omega_{-\varepsilon}(\lambda_0) & 0 & 0 & 0 & 0\\ \phi_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & \chi_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & -\phi_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & -\chi_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & 0 & 0\\ \phi'_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & \chi'_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & -\phi'_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & -\chi'_{\varepsilon,\lambda_0}(\phi_{-\varepsilon}+) & 0 & 0\\ 0 & 0 & \phi_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & \chi_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & -\phi_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & -\chi_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+)\\ 0 & 0 & \phi'_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & \chi'_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & -\phi'_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+) & -\chi'_{+\varepsilon,\lambda_0}(\phi_{+\varepsilon}+)\\ 0 & 0 & 0 & 0 & \omega_{+\varepsilon}(\lambda_0) & 0 \end{vmatrix}$$

$$=-\omega_{-\varepsilon}(\lambda_0)\,\omega_{\varepsilon}(\lambda_0)\,\omega^2_{+\varepsilon}(\lambda_0)\neq 0.$$

Thus, the system (3.18) has only trivial solution $c_i=0,\ \left(i=\overline{1,6}\right)$ and so we get contradiction which completes the proof.

Lemma 3.2 All eigenvalues λ_n are simple zeros of $\omega(\lambda)$.

Proof. Using the well-known Lagrange's formula it can be shown that

$$(\lambda-\lambda_n)\left(\int_a^{\theta_{-\varepsilon}}\phi_\lambda(x)\phi_{\lambda_n}(x)dx+\delta^2\int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}}\phi_\lambda(x)\phi_{\lambda_n}(x)dx+\gamma^2\int_{\theta_{+\varepsilon}}^{b}\phi_\lambda(x)\phi_{\lambda_n}(x)dx\right)=\gamma^2 W\left(\phi_\lambda,\phi_{\lambda_n};b\right), \tag{3.19}$$

for any λ. Since

$$\chi_{\lambda_n}(x)=k_n\phi_{\lambda_n}(x),\ x\in[a,\theta_{-\varepsilon})\cup(\theta_{-\varepsilon},\theta_{+\varepsilon})\cup(\theta_{+\varepsilon},b],\ \text{for some } k_n\neq 0, \tag{3.20}$$

then

$$\begin{aligned} W\left(\phi_\lambda,\phi_{\lambda_n};b\right) &= \frac{1}{k_n}W\left(\phi_\lambda,\chi_{\lambda_n};b\right) \\ &= \frac{1}{k_n}\left(\lambda_n R'\left(\phi_\lambda\right)+R\left(\phi_\lambda\right)\right) \\ &= \frac{1}{k_n}\left(\omega(\lambda)-\left(\lambda-\lambda_n\right)R'\left(\phi_\lambda\right)\right) \\ &= \frac{1}{k_n}\left(\lambda-\lambda_n\right)\left(\frac{\omega(\lambda)}{\left(\lambda-\lambda_n\right)}-R'\left(\phi_\lambda\right)\right) \end{aligned} \quad (3.21)$$

substituting (3.21) in (3.19) and letting $\lambda \to \lambda_n$ we get

$$\int_a^{\theta_{-\varepsilon}}\left(\phi_{\lambda_n}(x)\right)^2dx+\delta^2\int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}}\left(\phi_{\lambda_n}(x)\right)^2dx+\gamma^2\int_{\theta_{+\varepsilon}}^{b}\left(\phi_{\lambda_n}(x)\right)^2dx=\frac{\gamma^2}{k_n}\left(\omega'\left(\lambda_n\right)-R'\left(\phi_{\lambda_n}\right)\right). \quad (3.22)$$

Now putting

$$R'\left(\phi_{\lambda_n}\right)=\frac{1}{k_n}R'\left(\chi_{\lambda_n}\right)=\frac{\rho}{k_n}\ , \quad (3.23)$$

in (3.22) it yields $\omega'\left(\lambda_n\right)\neq 0$, which completes the proof.

4 ASYMPTOTIC FORMULAS FOR EIGENVALUES AND EIGENFUNCTIONS

Now we derive asymptotic formulas of the eigenvalues and eigenfunctions similar to the classical techniques of [3,6,9,17]. We begin by proving some lemmas.

Lemma 4.1 Let $\phi_\lambda(x)$ be the solutions of equation (1.1) defined in Section 3. Then the following integral equations hold for $k=0$ and $k=1$:

$$\frac{dk}{dx^k}\phi_{-\varepsilon,\lambda}(x)=\beta_2\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-a)-\frac{\beta_1}{\sqrt{\lambda}}\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-a)+\frac{1}{\sqrt{\lambda}}\int_a^x\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-y)q(y)\phi_{-\varepsilon,\lambda}(y)dy, \quad (4.1)$$

$$\frac{dk}{dx^k}\phi_{\varepsilon,\lambda}(x)=\frac{1}{\delta}\phi_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-)\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-\theta_{-\varepsilon})+\frac{1}{\delta\sqrt{\lambda}}\phi'_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-)\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-\theta_{-\varepsilon})+\frac{1}{\sqrt{\lambda}}\int_{\theta_{-\varepsilon}}^x\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-y)q(y)\phi_{\varepsilon,\lambda}(y)dy, \quad (4.2)$$

$$\frac{dk}{dx^k}\phi_{+\varepsilon,\lambda}(x)=\frac{\delta}{\gamma}\phi_{\varepsilon,\lambda}(\theta_{+\varepsilon}-)\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-\theta_{+\varepsilon})+\frac{\delta}{\gamma\sqrt{\lambda}}\phi'_{\varepsilon,\lambda}(\theta_{+\varepsilon}-)\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-\theta_{+\varepsilon})+\frac{1}{\sqrt{\lambda}}\int_{\theta_{+\varepsilon}}^x\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-y)q(y)\phi_{+\varepsilon,\lambda}(y)dy. \quad (4.3)$$

Proof. For proving it is enough substitute $\lambda\phi_{-\varepsilon,\lambda}(y)+\phi''_{-\varepsilon,\lambda}(y)$, $\lambda\phi_{\varepsilon,\lambda}(y)+\phi''_{\varepsilon,\lambda}(y)$ and $\lambda\phi_{+\varepsilon,\lambda}(y)+\phi''_{+\varepsilon,\lambda}(y)$ instead of $q(y)\phi_{-\varepsilon,\lambda}(y)$, $q(y)\phi_{\varepsilon,\lambda}(y)$ and $q(y)\phi_{+\varepsilon,\lambda}(y)$ in the integral terms of the (4.1)-(4.3), respectively, and integrate by parts twice.

Lemma 4.2 Let $\operatorname{Im}\sqrt{\lambda}=t$. Then the functions $\phi_{\pm\varepsilon,\lambda}(x)$ and $\phi_{\varepsilon,\lambda}(x)$ have the following asymptotic representations for $|\lambda|\to\infty$, which hold uniformly for $x\in I_i\ (i=1,2,3)$:

$$\frac{dk}{dx^k}\phi_{-\varepsilon,\lambda}(x)=\beta_2\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-1}e^{|t|(x-a)}\right), \tag{4.4}$$

$$\frac{dk}{dx^k}\phi_{\varepsilon,\lambda}(x)=\frac{\beta_2}{\delta}\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-1}e^{|t|(x-a)}\right), \tag{4.5}$$

$$\frac{dk}{dx^k}\phi_{+\varepsilon,\lambda}(x)=\frac{\beta_2}{\gamma}\frac{dk}{dx^k}\cos\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-1}e^{|t|(x-a)}\right), \tag{4.6}$$

if $\beta_2\neq 0$,

$$\frac{dk}{dx^k}\phi_{-\varepsilon,\lambda}(x)=-\frac{\beta_1}{\sqrt{\lambda}}\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-2}e^{|t|(x-a)}\right), \tag{4.7}$$

$$\frac{dk}{dx^k}\phi_{\varepsilon,\lambda}(x)=-\frac{\beta_1}{\delta\sqrt{\lambda}}\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-2}e^{|t|(x-a)}\right), \tag{4.8}$$

$$\frac{dk}{dx^k}\phi_{+\varepsilon,\lambda}(x)=-\frac{\beta_1}{\gamma\sqrt{\lambda}}\frac{dk}{dx^k}\sin\sqrt{\lambda}(x-a)+O\left(\left(\sqrt{\lambda}\right)^{k-2}e^{|t|(x-a)}\right), \tag{4.9}$$

if $\beta_2=0$.

Proof. The proof of the formulas is identical to Titchmarsh's proof of similar results [6,9,17].

Lemma 4.3 Let $\operatorname{Im}\sqrt{\lambda}=t$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations:

Case 1. $\beta_2\neq 0$, $\alpha_1^{'}\neq 0$:

$$\omega(\lambda)=\lambda\sqrt{\lambda}\alpha_1^{'}\beta_2\gamma\sin\sqrt{\lambda}(b-a)+O\left(\lambda e^{|t|(b-a)}\right), \tag{4.10}$$

Case 2. $\beta_2\neq 0$, $\alpha_1^{'}=0$:

$$\omega(\lambda)=\lambda\alpha_2^{'}\beta_2\gamma\cos\sqrt{\lambda}(b-a)+O\left(\sqrt{\lambda}e^{|t|(b-a)}\right), \tag{4.11}$$

Case 3. $\beta_2=0$, $\alpha_1^{'}\neq 0$:

$$\omega(\lambda)=\lambda\alpha_1^{'}\beta_1\gamma\cos\sqrt{\lambda}(b-a)+O\left(\sqrt{\lambda}e^{|t|(b-a)}\right), \tag{4.12}$$

Case 4. $\beta_2=0$, $\alpha_1^{'}=0$:

$$\omega(\lambda)=-\sqrt{\lambda}\alpha_2^{'}\beta_1\gamma\sin\sqrt{\lambda}(b-a)+O\left(e^{|t|(b-a)}\right), \tag{4.13}$$

Proof. The proof is immediate by substituting (4.6) and (4.9) into the representation

$$\omega(\lambda)=\gamma^2\left(\left(\lambda\alpha_2^{'}+\alpha_2\right)\phi_{+\varepsilon,\lambda}(b)-\left(\lambda\alpha_1^{'}+\alpha_1\right)\phi_{+\varepsilon,\lambda}^{'}(b)\right). \tag{4.14}$$

Corollary 4.4 The eigenvalues of the problem (1.1)-(1.7) are bounded below.

Now we can obtain the asymptotic approximation formula for the eigenvalues of the considered problem (1.1)-(1.7). Since the eigenvalues coincide with the zeros of the entire function $\omega(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollaries 2.2 and 4.4 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0\le\lambda_1\le\lambda_2,\ldots,$ listed according to their multiplicity.

Theorem 4.5 The eigenvalues $\lambda_n, n=0,1,2,\ldots,$ of the problem (1.1)-(1.7) have the following asymptotic representation for $n\to\infty$:

Case 1. $\beta_2 \neq 0$, $\alpha_1^{'} \neq 0$:

$$\sqrt{\lambda_n} = (n-1)\frac{\pi}{(b-a)} + O\left(\frac{1}{n}\right), \tag{4.15}$$

Case 2. $\beta_2 \neq 0$, $\alpha_1^{'} = 0$:

$$\sqrt{\lambda_n} = \left(n - \frac{1}{2}\right)\frac{\pi}{(b-a)} + O\left(\frac{1}{n}\right), \tag{4.16}$$

Case 3. $\beta_2 = 0$, $\alpha_1^{'} \neq 0$:

$$\sqrt{\lambda_n} = \left(n - \frac{1}{2}\right)\frac{\pi}{(b-a)} + O\left(\frac{1}{n}\right), \tag{4.17}$$

Case 4. $\beta_2 = 0$, $\alpha_1^{'} = 0$:

$$\sqrt{\lambda_n} = n\frac{\pi}{(b-a)} + O\left(\frac{1}{n}\right). \tag{4.18}$$

Proof. We will only consider the first case. We will apply the well-known Rouche theorem, which asserts that if $f(\lambda)$ and $g(\lambda)$ are analytic inside and on a closed contour C and $|f(\lambda)| > |g(\lambda)|$ on C, then $f(\lambda)$ and $f(\lambda) + g(\lambda)$ have the same number of zeros inside C, provided that each zero is counted according to its multiplicity. It follows that $\omega(\lambda)$ has the same number of zeros inside the contour as the leading term in (4.10). If $\lambda_0 \leq \lambda_1 \leq \lambda_2, \ldots,$ are the zeros of $\omega(\lambda)$, we have

$$\sqrt{\lambda_n} = (n-1)\frac{\pi}{(b-a)} + \xi_n \tag{4.19}$$

where $|\xi_n| < \dfrac{\pi}{2(b-a)}$ for sufficiently large n. By putting in (4.10) we have $\xi_n = O\left(\dfrac{1}{n}\right)$, so the proof is completed for case 1. The proof for the other cases is similar.

Then from (4.4)-(4.9) $(\text{for } k = 0)$ and the above theorem, the asymptotic behaviour of the eigenfunctions

$$\phi_{\lambda_n}(x) = \begin{cases} \phi_{-\varepsilon,\lambda_n}(x), & x \in [a, \theta_{-\varepsilon}), \\ \phi_{\varepsilon,\lambda_n}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \phi_{+\varepsilon,\lambda_n}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases} \tag{4.20}$$

of the problem (1.1)-(1.7) is given by

$$\phi_{\lambda_n}(x) = \begin{cases} \beta_2 \cos\left(\dfrac{(n-1)\pi}{(b-a)}(x-a)\right) + O\left(\dfrac{1}{n}\right), & x \in [a, \theta_{-\varepsilon}), \\ \dfrac{\beta_2}{\delta}\cos\left(\dfrac{(n-1)\pi}{(b-a)}(x-a)\right) + O\left(\dfrac{1}{n}\right), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \dfrac{\beta_2}{\gamma}\cos\left(\dfrac{(n-1)\pi}{(b-a)}(x-a)\right) + O\left(\dfrac{1}{n}\right), & x \in (\theta_{+\varepsilon}, b], \end{cases} \quad \text{if } \beta_2 \neq 0,\ \alpha_1^{'} \neq 0,$$

$$\phi_{\lambda_n}(x)=\begin{cases}\beta_2\cos\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in[a,\theta_{-\varepsilon}),\\ \frac{\beta_2}{\delta}\cos\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ \frac{\beta_2}{\gamma}\cos\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{+\varepsilon},b],\end{cases} \quad \text{if } \beta_2\neq 0,\ \alpha_1'=0,$$

$$\phi_{\lambda_n}(x)=\begin{cases}-\frac{\beta_1(b-a)}{(n-1/2)\pi}\sin\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in[a,\theta_{-\varepsilon}),\\ -\frac{\beta_1(b-a)}{(n-1/2)\pi\delta}\sin\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ -\frac{\beta_1(b-a)}{(n-1/2)\pi\gamma}\sin\left(\frac{(n-1/2)\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{+\varepsilon},b],\end{cases} \quad \text{if } \beta_2=0,\ \alpha_1'\neq 0,$$

$$\phi_{\lambda_n}(x)=\begin{cases}-\frac{\beta_1(b-a)}{n\pi}\sin\left(\frac{n\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in[a,\theta_{-\varepsilon}),\\ -\frac{\beta_1(b-a)}{n\pi\delta}\sin\left(\frac{n\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{-\varepsilon},\theta_{+\varepsilon}),\\ -\frac{\beta_1(b-a)}{n\pi\gamma}\sin\left(\frac{n\pi}{(b-a)}(x-a)\right)+O\left(\frac{1}{n}\right), & x\in(\theta_{+\varepsilon},b],\end{cases} \quad \text{if } \beta_2=0,\ \alpha_1'=0.$$

All these asymptotic formulas hold uniformly for x.

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