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A plane boundary value problem of Thermo- magnetoelasticity for two parallel DC-bus bars by a boundary integral method

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ABSTRACT

A boundary integral method previously introduced by two of the authors [1] is properly extended to investigate the plane problem of linear, uncoupled thermo-magnetoelasticity for two parallel, infinite, circular cylindrical electric conductors carrying steady, axial currents and placed a distance apart in an external medium kept at a constant temperature. Such a setting allows disregarding the thermal interaction between the two cylinders, leaving only the magnetic interaction.

The basic equations and boundary conditions are briefly mentioned as in [1] and the solution of the problem is obtained for all quantities of physical interest. Numerical results are given for the so-called magnetic displacements occurring in the representation of the mechanical displacements and a detailed discussion of these results is provided.

Indexing terms/Keywords

Thermo-magnetoelasticity; DC-Bus bars; bipolar coordinates; Boundary integral method.

Academic Discipline And Sub-Disciplines

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1. INTRODUCTION

In a previous paper [1], two of the authors (MSA and AFG) introduced a boundary integral method to deal with the plane problem of linear, uncoupled thermo-magnetoelasticity for isotropic, homogeneous media in simply connected regions. The method relies on the use of harmonic functions in real variables, which allows taking advantage of the representations of harmonic functions in different systems of coordinates. As an illustration of the method, El-Dhaba [2] treated the problem of an infinite electric conductor of elliptic cross-section carrying a steady, axial current.

The present paper investigates the problem of two parallel, infinite, circular cylindrical electric conductors of the same material, carrying steady, axial currents and placed a distance apart in free space kept at a constant temperature. The conductors deform under the combined action of Joule heat and the magnetic field distribution. For most materials, the deformations due to the magnetic field are usually much smaller than those produced by heat. For this reason,we have used a thermal setting which prohibits thermal interaction between the two cylinders, leaving only the magnetic interaction. Within the present formulation, the magnetic field is derived from a magnetic potential in the quasi-static approximation and the heat problem is solved independently of the magneto-mechanical problem. The only coupling that is considered is the dependence of the magnetic permeability on strain, more precisely magnetistriction. Under some assumptions, this coupling still allows for the magnetic problem to be solved independently of the mechanical problem. For further details, the reader may refer to [1]. More general formulations of the equations of Magneto-thermoelasticity may be found elsewhere [3, 4].

The problem is solved using the above-mentioned method and formulae are presented for all the quantities of physical interest. For the numerical part, however, we have preferred to focus only on the so-called "magnetic displacements" occurring in the representation of the mechanical displacement.

2. DESCRIPTION OF THE PHYSICAL PROBLEM

We obtain the deformation occurring in two parallel DC-busbars placed a distance apart from each other, in an ambient free space kept at the constant, reference temperature. Thus, the only interaction between the cylinders is through the magnetic field.

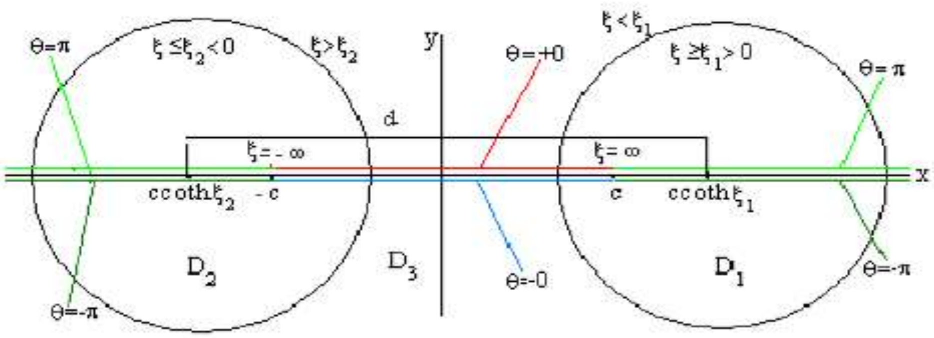
The problem is solved within the linear uncoupled theory of Magneto-thermoelasticity, using a variation of a boundary integral method previously introduced by two of the authors (AFG and MSA) for simply connected domains. The dependence of the magnetic permeability of the body on magnetostriction is taken in consideration through two material parameters. Under certain restrictions, such a dependence does not prevent the uncoupling of the magnetostatic problem from the mechanical one. First, the magnetostatic problem is solved to find the vector potential everywhere in space, from which one deduces the magnetic field distribution. Then, the solution for the uncoupled heat problem is obtained under uniform bulk heating and radiation condition at the boundaries. Four important functions of position, the so-called "magnetic displacements" and "thermal displacements" are then calculated through path-independent line integrals. Finally, the elastic problem is solved in stresses using Airy's stress function.

Let the two elastic busbars carry uniform, axial currents of densities J_1 and J_2. These currents may flow in the same sense or else be in opposite senses. The cylinders are placed in an external medium with given constant ambient temperature T_e, measured from a reference temperature T_r.

When there is no electric current in one of the cylinders, one simply sets the corresponding current density to zero.

Let the cylinders have radii a (for the right cylinder) and b (for the left cylinder) and let the distance between their centers be d. The domain inside the right cylinder is denoted D_1, the one inside the left cylinder is denoted D_2 and the external region to the cylinders is denoted D_3. The boundaries of D_1 and D_2 are denoted C_1 and C_2 respectively.

We use a system of bipolar coordinates (ξ, θ) associated with a system of orthogonal Cartesian coordinates (x, y, z) as usual. The two cylinders are described in the system of bipolar coordinates by the equations $\xi = \xi_1$ and $\xi = \xi_2$.



3. THE BIPOLAR COORDINATES

Let the cylinders have radii a (for the right cylinder) and b (for the left cylinder) and let the distance between their centers be d. The domain inside the right cylinder is denoted D_1, the one inside the left cylinder is denoted D_2 and the external region to the cylinders is denoted D_3 (see figure). The boundaries of D_1 and D_2 are denoted C_1 and C_2 respectively

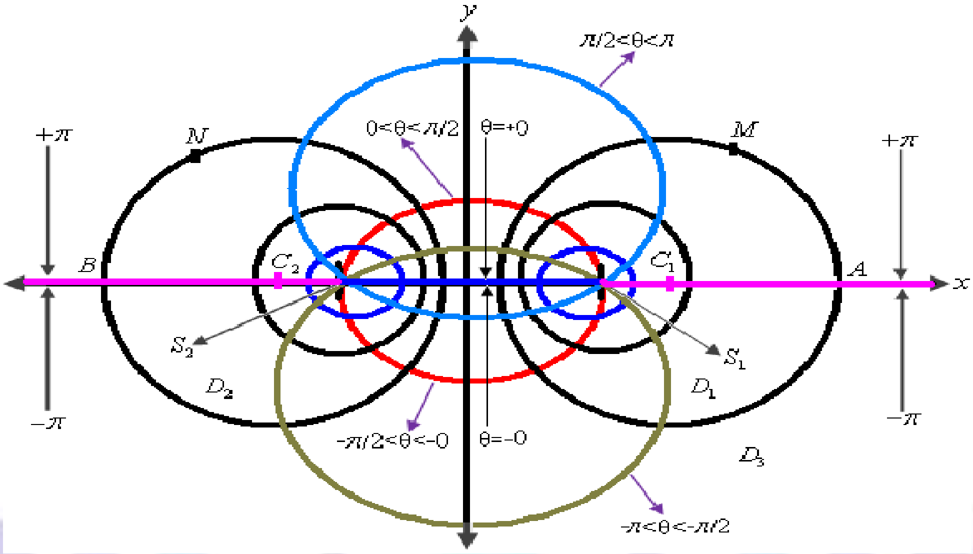


Fig. 1. Two eccentric circular cylinders

We use a system of bipolar coordinates (ξ, θ) associated with a system of orthogonal Cartesian coordinates (x, y) as shown on the figure. The arrows point at arcs of circles $\theta = \text{constant}$ In the limiting case $\theta = 0,$ or $\theta = \pm\pi$, these arcs degenerate into segments on the x-axis.

$$x = \frac{c \sinh \xi}{\cosh \xi + \cos \theta}, \quad y = \frac{c \sin \theta}{\cosh \xi + \cos \theta}, \quad -\pi \le \theta < \pi, \quad -\infty < \xi < \infty, \qquad (1)$$

The two cylinders are described in the system of bipolar coordinates by the equations $\xi = \xi_1$ and $\xi = \xi_2$, and the following relations hold:

$$a = c \operatorname{csch} \xi_1, \qquad b = c \operatorname{csch} \xi_2, \qquad (2)$$

by using the well-known relation between the inverse hyper-geometric functions and logarithmic function, one deduces that

$$\xi_1 = -\ln a + \ln\left(c + \sqrt{a^2 + c^2}\right), \qquad \xi_2 = -\ln b + \ln\left(-c + \sqrt{b^2 + c^2}\right), \qquad (3)$$

The distance d between the two centers of the circles

$$d = \sqrt{c^2 \left(\coth \xi_1 - \coth \xi_2\right)^2} = c\left(\coth \xi_1 - \coth \xi_2\right),$$

use equation (2), then

$$d = \sqrt{a^2 + c^2} + \sqrt{b^2 + c^2}, \qquad (4)$$

It is easily shown that

$$c = \frac{1}{2d}\sqrt{\left(d^2 - (a+b)^2\right)\left(d^2 - (a-b)^2\right)}. \qquad (5)$$

Equations (1) may be written as [5]

$$x = c\,\mathrm{sgn}(\xi)\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n|\xi|}\cos n\theta\right), \quad y = -2c\sum_{n=0}^{\infty}(-1)^n e^{-n|\xi|}\sin n\theta, \quad \forall\ \xi, \qquad (6)$$

Also,

$$r^2 \equiv x^2 + y^2 = c^2\frac{\cosh\xi - \cos\theta}{\cosh\xi + \cos\theta}, \qquad (7)$$

Use equation (6), to get

$$r^2 = c^2\left(\mathrm{sgn}(\xi)\,2\coth\xi - 1\right) + \mathrm{sgn}(\xi)\,4c^2\coth\xi\sum_{n=1}^{\infty}(-1)^n e^{-\mathrm{sgn}(\xi)n\xi}\cos n\theta, \quad \forall\ \xi \qquad (8)$$

Let each of the boundaries C_1 and C_2 have the parametric representation

$$x = x(s), \qquad y = y(s), \qquad (9)$$

with $x(s)$ and $y(s)$ twice continuously differentiable functions of their argument.

Here, (x, y, z) denote orthogonal Cartesian coordinates in space with origin O and unit vectors $\underline{i}, \underline{j}, \underline{k}$ respectively and s -the arc length as measured on each boundary separately in the positive sense associated with D_1 and D_2, from fixed points Q_1 and Q_2 on C_1 and C_2 respectively to a general boundary point Q.

Let $\underline{\tau}$ and $\underline{n}$ be the unit vector tangent and the unit vector normal to C_1 or C_2 at Q in the sense of increase of s. One has

$$\underline{\tau} = \left(\dot{x}, \dot{y}\right), \qquad \underline{n} = \left(\dot{y}, -\dot{x}\right), \qquad (10)$$

and the "dot" over a symbol denotes differentiation w.r.to s.

All unknown functions are assumed to depend only on the two coordinates (x, y). We shall quote without proof the general equations of static, linear uncoupled Magneto-thermoelasticity as in [1] to be used throughout the text.

Equation of heat conduction

In the steady state, the temperature T in each cylinder, as measured from the reference temperature T_r, satisfies Poisson's equation

$$\nabla^2 T = -\frac{J^2}{\sigma K}, \qquad (11)$$

where J is the electric current density, σ is the electric conductivity and K is the coefficient of heat conduction. The general solution of equation (11) is taken as

$$T = T_h + T_p, \qquad (12)$$

T_h being the harmonic part of T and the particular solution is

$$T_p = -\frac{J^2}{4\sigma K}\left(x^2 + y^2\right). \qquad (13)$$

On the boundary of each cylinder, the following thermal radiation condition takes place:

$$\frac{\partial T(s)}{\partial n} = -\frac{Bi}{K}\left(T(s) - T_e\right). \qquad (14)$$

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4. EQUATIONS OF MAGNETOSTATICS

The SI system of units is used throughout. Within the quasi-static approximation, and neglecting all effects due to the electric field, the magnetic induction vector $\underline{B}$ inside each cylinder derives from a magnetic vector potential $\underline{A}$ according to the relation

$$\underline{B} = \underline{\nabla} \times \underline{A}, \qquad (15)$$

and, in view of the geometry of the problem, this vector potential will be directed parallel to the cylinders' axes:

$$\underline{A} = A(x, y)\underline{k}. \qquad (16)$$

The magnetic constitutive relations read

$$B_i = \mu^* \mu_{ij} H_j, \qquad i, j = 1,2,3, \qquad (17)$$

H_i being the components of the magnetic field vector and μ_{ij} the components of the magnetic permeability tensor of the body, assumed to depend linearly on strain according to the rule

$$\mu_{ij} = \mu_0 \delta_{ij} + \mu_1 I_1 \delta_{ij} + \mu_2 \varepsilon_{ij}, \quad i, j = 1,2,3, \quad (18)$$

where μ_0, μ_1 and μ_2 are constants with obvious physical meaning, I_1 is the first invariant of the strain tensor with components ε_{ij} and δ_{ij} denote the Kronecker delta symbols. Constant μ^* refers to the magnetic permeability of vacuum with value $\mu^* = 10^{-7}\ H.m^{-1}$. An electric analogue of (18) for the dielectric tensor components may be found in [6] and [7].

Since we are assuming a quadratic dependence of strain on the magnetic field (magnetostriction), upon substitution of (18) into (17) one may neglect, as an approximation, the third and higher order terms in the magnetic field compared to the first order term and write [8, 9]

$$\underline{B} = \mu^* \mu_0 \underline{H}, \qquad (19)$$

The function A satisfies the well-known Poisson's equation

$$\nabla^2 A = -\mu^* \mu_0 J, \qquad (20)$$

In the free space (referred to by $*$) surrounding the cylinders, the equations of Magnetostatics hold with $\mu_0 = \varepsilon = 1$ and $\mu_1 = \mu_2 = J = 0$. In particular, equation (20) is replaced by Taking into account the irrotationality condition for the electric field outside the body and the continuity of its tangential component across the surface of the cylindrical body, one may write

$$\nabla^2 A^* = 0, \qquad (21)$$

Inside each cylinder, the solution of (20) is looked for in the form

$$A = A_h - \frac{1}{4}\mu^* \mu_0 J \left(x^2 + y^2\right), \qquad (22)$$

while

$$A^* = A_\infty + A_r^*, \qquad (23)$$

Here, A_h is the harmonic part of function A, A_r^* is the harmonic part of A^* which has a regular behavior at infinity and A_∞ is a known function which satisfies Laplace's equation but does not vanish at infinity. Functions A_h and A_r^* represent the modification of the magnetic vector potential due to the presence of the body.

In addition, the following radiation condition must take place:

$$\left|A_r^*\right| = O\left(r^{-\delta}\right), \quad \delta > 0 \text{ as } r = \left(x^2 + y^2\right)^{\frac{1}{2}} \to \infty,$$

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by virtue of which the arbitrary additive constant intervening in the definition of the magnetic vector potential has been determined.

It is worth noting that the function A_∞ has to be precised in each individual case under consideration. As an example, if the magnetic field is due to a uniform electric current of volume density J and intensity I flowing in a region D ($I = J\Sigma_D$, where D_Σ is the area of D), then

$$A_\infty = -\frac{\mu^* I}{2}\ln\frac{r}{l}, \qquad (24)$$

where l is a characteristic dimension of the region D.

In the absence of surface electric currents, the equations of Magnetostatics are complemented by the magnetic boundary conditions expressing the continuity of: (i) the vector potential; (ii) the tangential component of the magnetic field. Thus

$$A_h - A_r^* = A_\infty + \frac{1}{4}\mu^*\mu_0 J r^2, \qquad (25)$$

$$\frac{1}{\mu_0}\frac{\partial}{\partial n}\left(A_h - A_r^*\right) = \frac{\partial}{\partial n}\left(A_\infty + \frac{1}{4}\mu^*\mu_0 J r^2\right), \qquad (26)$$

These conditions, together with the vanishing of A_r^* at infinity, are sufficient for the complete determination of the harmonic functions A_h in both cylinders, together with A_r^*.

5. EQUATIONS OF ELASTICITY

5.1 Equations of equilibrium

In the absence of body forces of non-electromagnetic origin, the equations of mechanical equilibrium in the plane for each of the two cylinders read

$$\nabla_j \sigma_{ij} = 0, \; i, j = 1, 2, 3, \qquad (27)$$

where σ_{ij} are the components of the "total" stress tensor and ∇_j denotes covariant differentiation.

Equations (27) are satisfied if the only identically non-vanishing stress components σ_{xx} , σ_{yy} and σ_{xy} are defined through the stress function U by the relations

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \qquad (28)$$

5.2 The constitutive relations

The generalized Hooke's law may be derived consistently for an appropriate form of the free energy of the medium, using the general principles of Continuum Mechanics. It reads (see [8, 9], [6] and also [7] for the electric analogue):

$$\sigma_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)} I_1 \delta_{ij} + \frac{\nu E}{1+\nu}\varepsilon_{ij} - \frac{\alpha E}{1-2\nu} T\delta_{ij} + \mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right) H_i H_j$$
$$-\frac{1}{2}\mu^*\left(\mu_0 + \mu_2\right) H^2 \delta_{ij}, \qquad (29)$$

where $H^2 = H_i H_i$ is the squared magnitude of the magnetic field, E, ν and α are Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion respectively for the considered elastic medium. In components, equation (29) yields

$$\sigma_{xx}=\frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\nu E}{1+\nu}\frac{\partial u}{\partial x}-\frac{\alpha E}{1-2\nu}T+\frac{1}{2}\mu^{*}\left(\mu_0-\mu_1-\mu_2\right)H_x^2 -\frac{1}{2}\mu^{*}\left(\mu_0+\mu_2\right)H_y^2, \qquad (30a)$$

$$\sigma_{yy}=\frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\nu E}{1+\nu}\frac{\partial v}{\partial y}-\frac{\alpha E}{1-2\nu}T-\frac{1}{2}\mu^{*}\left(\mu_0+\mu_2\right)H_x^2 +\frac{1}{2}\mu^{*}\left(\mu_0-\mu_1-\mu_2\right)H_y^2, \qquad (30b)$$

$$\sigma_{xy}=\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu^{*}\left(\mu_0-\frac{1}{2}\mu_1\right)H_xH_y. \qquad (30b)$$

It is worth noting that the contribution of the electric field to the stress tensor components is usually negligibly small as compared to the magnetic terms (as may be verified from dimension analysis) and has therefore been omitted from the generalized Hooke's law.

5.3 THE KINEMATICAL RELATIONS

These are the relations between the strain tensor components ε_{ij} and the displacement vector components u_i .

$$\varepsilon_{ij}=\frac{1}{2}\left(\nabla_i u_j+\nabla_j u_i\right), \quad i,j=1,2, \qquad (31a)$$

or in Cartesian components

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}, \qquad \varepsilon_{yy}=\frac{\partial v}{\partial y}, \qquad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \qquad (31b)$$

where u and v stand for u_1 and u_2 respectively.

5.4 THE COMPATIBILITY CONDITION

The condition of solvability of equations (31b) is

$$2\frac{\partial^2 e_{xy}}{\partial x\partial y}=\frac{\partial^2 e_{yy}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial y^2}, \qquad (32)$$

These equations are complemented with the proper boundary conditions, to be discussed in detail later on.

6. EQUATION FOR THE STRESS FUNCTION

The following equation for the stress function may be obtained from the general field equations by standard procedure [9]

$$\nabla^4 U=\frac{\alpha E J^2}{(1-\nu)\sigma K}-\frac{(1-2\nu)\mu^{*}}{2(1-\nu)}\left(\frac{1}{2}\mu_1+\mu_2\right)\nabla^2 H^2+\frac{1}{1-\nu}\mu^{*}\left(\mu_0-\frac{1}{2}\mu_1\right)J^2. \qquad (33)$$

This is the same as the biharmonic equation given in [9], taking in consideration the differences in the used systems of units, but is different from that produced by Yuan [10] because different stress functions and different stresses are involved in the latter.

The general solution of (33) is

$$U=x\Phi+y\Phi^{c}+\Psi+U_p, \qquad (34)$$

where Φ and Ψ are two harmonic functions belonging to the class of functions $C^2(D_i) \cap C^1(\bar{D}_i)$, $\bar{D}_i$ denotes the closure of D, $i = 1, 2$, and superscript `c' denotes the harmonic conjugate. Function U_p is any particular solution of the equation

$$\nabla^2 U_p = -\frac{\alpha E}{1-\nu} T_p - \frac{(1-2\nu)\mu^*}{2(1-\nu)} \left(\frac{1}{2}\mu_1 + \mu_2\right) H^2 + \frac{1}{4(1-\nu)} \mu^* \left(\mu_0 - \frac{1}{2}\mu_1\right) J^2 \left(x^2 + y^2\right), \qquad (35)$$

and may be expressed in the form of Newton's potential after the functions T_p and H^2 on the R.H.S. have been determined.

It follows from (33), (34) and (35) that

$$\nabla^2 U = 4\frac{\partial \Phi}{\partial x} + \nabla^2 U_p = 4\frac{\partial \Phi^c}{\partial y} + \nabla^2 U_p. \qquad (36)$$

7. A REPRESENTATION FOR THE MECHANICAL DISPLACEMENT VECTOR COMPONENTS

The set of equations of Elasticity and Magnetostatics, after lengthy manipulations, yield the following important representation of the mechanical displacement components [9]:

$$\frac{E}{1+\nu} u = -\frac{\partial U}{\partial x} + 4(1-\nu)\Phi + \frac{E}{1+\nu}(u_T + u_H), \qquad (37a)$$

$$\frac{E}{1+\nu} v = -\frac{\partial U}{\partial y} + 4(1-\nu)\Phi^c + \frac{E}{1+\nu}(v_T + v_H), \qquad (37b)$$

where

$$\left.\begin{aligned} u_T &= \alpha(1+\nu)\int_{M_0}^{M} \left(T_h dx - T_h^c dy\right), \\ v_T &= \alpha(1+\nu)\int_{M_0}^{M} \left(T_h^c dx + T_h dy\right), \end{aligned}\right\}, \qquad (38a)$$

and

$$\left.\begin{aligned} u_H &= \int_{M_0}^{M} (M_H dx + N_H dy), \\ v_H &= \int_{M_0}^{M} (R_H dx + S_H dy), \end{aligned}\right\}, \qquad (38b)$$

the line integrations being taken along a path inside any of the regions D_i, joining a fixed point M_i (which may be arbitrarily chosen in D_i) to a general field point M in D_i, ($i = 1, 2$). In fact, all four integrals are shown to be path-independent. The integration constants due to the arbitrariness of the points M_i are absorbed into the functions Φ and Φ^c which are yet to be determined. Also,

$$\left.\begin{aligned} M_H &= W\left(H_y^2 - H_x^2 + \frac{J^2}{2}\left(x^2 + y^2\right)\right), \\ S_H &= W\left(H_x^2 - H_y^2 + \frac{J^2}{2}\left(x^2 + y^2\right)\right), \end{aligned}\right\}, \qquad (39a)$$

and

$$\left.\begin{aligned} N_H &= -2W\left(H_x H_y + \frac{1}{\mu^* \mu_0} J A_h^c\right), \\ R_H &= -2W\left(H_x H_y - \frac{1}{\mu^* \mu_0} J A_h^c\right), \end{aligned}\right\}, \qquad (39b)$$

with $W = \frac{1}{2}(1+\nu)\frac{\mu^*}{E}\left(\mu_0 - \frac{1}{2}\mu_1\right)$.

It can be easily verified using the equations of Magnetostatics, that

$$\frac{\partial M_H}{\partial y} = \frac{\partial N_H}{\partial x}, \qquad \frac{\partial R_H}{\partial y} = \frac{\partial S_H}{\partial x}, \qquad (40)$$

and that these relations are not affected by the addition of an arbitrary constant to the function A_h^c.

$$M_H = \frac{\partial u_H}{\partial x}, \qquad N_H = \frac{\partial u_H}{\partial y}, \qquad R_H = \frac{\partial v_H}{\partial x}, \qquad S_H = \frac{\partial v_H}{\partial y}.$$

The mechanical displacement components u and v in (37a, b) are single-valued functions in each of D_1 and D_2, since the line integrals in (38a) are path-independent due to the Cauchy-Riemann conditions satisfied by the functions T_h and T_h^c, and the line integrals in (38b) are path-independent in view of relations (40).

It is important to note that each of Φ^c, T_h^c and A_h^c is defined up to an arbitrary additive constant in each of D_1 and D_2. These functions will be completely determined once their values have been specified some given point in each domain.

8. CONDITIONS FOR A UNIQUE SOLUTION

We now turn to the conditions to be satisfied in order to determine the unknown harmonic functions Φ and Ψ in an unambiguous manner. These are three types of conditions:

1) Conditions for eliminating the rigid body translation

We require the centers of both cross-sections of the cylinders to be fixed.

2) Conditions for eliminating the rigid body rotation

These conditions identically satisfied in view of the symmetry of the problem.

3) Additional simplifying conditions

In order to be able to determine the totality of the integration constants appearing throughout

the solution process, we require the following four supplementary conditions to be satisfied at two arbitrarily chosen points $Q_i \in C_i$, $(i = 1, 2)$. For convenience, these are taken to correspond to the value $s = 0$ of the boundary parameter. The additional conditions have no physical implications:

$$U(Q_i) = \frac{\partial U}{\partial x}(Q_i) = \frac{\partial U}{\partial y}(Q_i) = 0, \qquad i = 1, 2, \qquad (41)$$

or, equivalently,

$$U(Q_i) = \frac{\partial U}{\partial s}(Q_i) = \frac{\partial U}{\partial n}(Q_i) = 0, \qquad i = 1, 2, \qquad (42)$$

and

$$x(Q_i)\Phi^c(Q_i) - y(Q_i)\Phi(Q_i) + \Psi^c(Q_i) = 0, \qquad i = 1, 2, \qquad (43)$$

This last condition amounts to determining the value of Ψ^c at $Q_i,\ i=1,2$ and is chosen in conformity with [2].

9. THE BOUNDARY CONDITIONS OF ELASTICITY

For the problem under consideration, the magnetic force distribution on the boundaries C_1 and C_2 of the domains D_1 and D_2. Let

$$\underline{f_H} = f_x \underline{i} + f_y \underline{j} = f_\tau \underline{\tau} + f_n \underline{n}$$

denote the magnetic force per unit length of the boundary. Then, at a general boundary point $Q(s)$ on C_1 or C_2, the stress vector satisfies the condition of continuity

$$\sigma \underline{n} = \underline{f}, \qquad (44)$$

or, in components

$$\sigma_{xx} n_x + \sigma_{xy} n_y = f_x, \qquad \text{and} \qquad \sigma_{xy} n_x + \sigma_{yy} n_y = f_y, \qquad (45)$$

The force $\underline{f_H}$ may be expressed in terms of the Maxwell stress tensor σ^* as

$$\underline{f_H} = \sigma^* \underline{n}, \qquad (46)$$

with

$$\sigma_{ij} = \mu^* \left(H_i^* H_j^* - \frac{1}{2} H^{*2} \delta_{ij} \right), \qquad (47)$$

Substituting into (45) for σ_{xx}, σ_{xy} and σ_{yy} in terms of the stress function U and for n_x, n_y from the second of equations (10) and taking the simplifying conditions into account, one finally gets

$$\frac{\partial U}{\partial x}(s) = -\int_0^s f_y'(s')\,ds' = -Y(s), \qquad (48)$$

$$\frac{\partial U}{\partial y}(s) = -\int_0^s f_x'(s')\,ds' = X(s). \qquad (49)$$

Also,

$$\frac{\partial U}{\partial s}(s) = -\dot{x}Y(s) + \dot{y}X(s), \qquad \frac{\partial U}{\partial n}(s) = -\dot{y}Y(s) - \dot{x}X(s), \qquad (50)$$

or, in terms of the unknown harmonic functions

$$x\dot{\Phi} + y\dot{\Phi}^c + \dot{\Psi} + \dot{x}\Phi + \dot{y}\Phi^c = -\dot{x}Y + \dot{y}X - \frac{\partial U_p}{\partial s}, \qquad (51)$$

$$x\dot{\Phi}^c - y\dot{\Phi} + \dot{\Psi}^c + \dot{y}\Phi - \dot{x}\Phi^c = -\dot{y}Y - \dot{x}X - \frac{\partial U_p}{\partial n}. \qquad (52)$$

Solution for two infinite, parallel circular cylindrical conductors carrying steady, uniform, axial currents

For convenience, the parameters for region D_2 are labeled with a `dash', those for $D_1$ are undashed, while the external medium is labeled with a `star'.

10. SOLUTION FOR THE TEMPERATURE

Let us introduce the dimensionless Biot constant $B = \frac{lBi}{K}$ and the dimensionless parameters

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$$\beta_1 = \frac{\alpha l^2 J_1^2}{4K\sigma}, \qquad \beta_2 = \frac{\alpha l^2 J_2^2}{4K\sigma}, \tag{53}$$

where l is a characteristic length. The thermal radiation boundary condition for $\xi = \xi_1$ and $\xi = \xi_2$ reads

$$\frac{1}{h}\frac{\partial}{\partial \xi}\left(T_h(\theta) + T_p(\theta)\right) = \frac{Bi}{K}\left(T_h(\theta) + T_p(\theta) - T_e\right), \tag{54}$$

$$\frac{1}{h}\frac{\partial}{\partial \xi}\left(T_h^{'}(\theta) + T_p^{'}(\theta)\right) = -\frac{Bi}{K}\left(T_h^{'}(\theta) + T_p^{'}(\theta) - T_e\right), \tag{55}$$

The particular solutions for the temperatures in D_1 and D_2 are respectively given by

$$\alpha T_p = -\frac{c^2}{l^2}\beta_1 \frac{\cosh \xi - \cos\theta}{\cosh \xi + \cos\theta}, \qquad \xi > \xi_1, \tag{56}$$

$$\alpha T_p^{'} = -\frac{c^2}{l^2}\beta_2 \frac{\cosh \xi - \cos\theta}{\cosh \xi + \cos\theta}, \qquad \xi < \xi_2, \tag{57}$$

The general solutions for Laplace's equations in bipolar coordinates for D_1 and D_2 are taken as

$$\alpha T_h(\xi,\theta) = (\alpha A_0) + \sum_{n=1}^{\infty}(\alpha A_n)e^{-n(\xi-\xi_1)}\cos n\theta, \qquad \xi > \xi_1, \tag{58}$$

$$\alpha T_h^{'}(\xi,\theta) = (\alpha A_0^{'}) + \sum_{n=1}^{\infty}(\alpha A_n^{'})e^{n(\xi-\xi_2)}\cos n\theta, \qquad \xi < \xi_2. \tag{58}$$

Using the thermal boundary conditions (54) and (55), the expressions (56) and (57) and some orthogonality properties of the trigonometric functions, one finally obtains (2):

$$\alpha A_0 = \alpha T_e + \frac{2}{B}\frac{c}{l}\beta_1 e^{-\xi_1} + \frac{c^2}{l^2}\beta_1\left(2\coth \xi_1 - 1\right) - \frac{1}{2B}\frac{l}{c}\alpha A_1,$$

$$\alpha A_0^{'} = \alpha T_e + \frac{2}{B}\frac{c}{l}\beta_2 e^{\xi_2} - \frac{c^2}{l^2}\beta_2\left(2\coth \xi_2 + 1\right) - \frac{1}{2B}\frac{l}{c}\alpha A_1^{'},$$

while the coefficients αA_i and $\alpha A_i^{'}$, $i = 1, 2 \ldots$ are the solutions of the infinite system of linear algebraic equations

$$M_{ij}\left(\alpha A_{ij}\right) = C_i, \qquad M_{ij}^{'}\left(\alpha A_{ij}^{'}\right) = C_i^{'},$$

with

$$M_{i,i+1} = \frac{i+1}{2}, \quad M_{i,i} = \frac{c}{l}B + i\cosh \xi_1, \quad M_{i,i-1} = \frac{i-1}{2}, \quad M_{i,j} = 0, \ \text{otherwise}, \ i = 1, 2, \ldots,$$

$$M_{i,i+1}^{'} = \frac{i+1}{2}, \quad M_{i,i}^{'} = \frac{c}{l}B + i\cosh \xi_2, \quad M_{i,i-1}^{'} = \frac{i-1}{2}, \quad M_{i,j}^{'} = 0, \ \text{otherwise}, \ i = 1, 2, \ldots,$$

$$C_i = 4(-1)^i \frac{c^2}{l^2}\beta_1\left(\cosh \xi_1 + \frac{c}{l}B\coth \xi_1\right)e^{-i\xi_1}, \quad C_i^{'} = 4(-1)^i \frac{c^2}{l^2}\beta_2\left(\cosh \xi_2 - \frac{c}{l}B\coth \xi_2\right)e^{i\xi_2}, \ i = 1, 2, \ldots .$$

Using the Cauchy- Riemann relations

$$\frac{\partial T_h}{\partial \theta} = -\frac{\partial T_h^c}{\partial \xi}, \qquad \frac{\partial T_h}{\partial \xi} = \frac{\partial T_h^c}{\partial \theta},$$

the conjugate functions T_h^c and $T_h^{'c}$ in D_1 and D_2 respectively are obtained by

$$\alpha T_h^c = -\sum_{n=1}^{\infty}(\alpha A_n)e^{-n(\xi-\xi_1)}\sin n\theta, \qquad \alpha T_h^{'c} = \sum_{n=1}^{\infty}(\alpha A_n^{'})e^{n(\xi-\xi_2)}\sin n\theta, \qquad (59)$$

and the expressions for the temperature in D_1 and D_2 are

$$\alpha T_h^c(\xi,\theta) = (\alpha A_0) + \sum_{n=1}^{\infty}(\alpha A_n)e^{-n(\xi-\xi_1)}\cos n\theta - \beta_1\frac{c^2}{l^2}\frac{\cosh\xi-\cos\theta}{\cosh\xi+\cos\theta}, \qquad \xi>\xi_1>0, \qquad (60)$$

$$\alpha T_h^{'c}(\xi,\theta) = (\alpha A_0^{'}) + \sum_{n=1}^{\infty}(\alpha A_n^{'})e^{n(\xi-\xi_2)}\cos n\theta - \beta_2\frac{c^2}{l^2}\frac{\cosh\xi-\cos\theta}{\cosh\xi+\cos\theta}, \qquad \xi<\xi_2<0, \qquad (61)$$

In obtaining the above results, use has been made of the integrals

$$\int_{-\pi}^{\pi}\frac{\cos\theta}{\cosh\xi+\cos\theta}d\theta = -2\pi\left(\operatorname{sgn}(\xi)\coth\xi-1\right),$$

$$\int_{-\pi}^{\pi}\frac{\cosh\xi-\cos\theta}{\cosh\xi+\cos\theta}d\theta = 2\pi\left(\operatorname{sgn}(\xi)2\coth\xi-1\right),$$

$$\int_{-\pi}^{\pi}\frac{\cos\theta}{\cosh\xi+\cos\theta}\cos n\theta d\theta = -\operatorname{sgn}(\xi)(-1)^n 2\pi\coth\xi e^{-\operatorname{sgn}(\xi)n\xi},$$

$$\int_{-\pi}^{\pi}\frac{\cosh\xi-\cos\theta}{\cosh\xi+\cos\theta}\cos n\theta d\theta = \operatorname{sgn}(\xi)(-1)^n 4\pi\coth\xi e^{-\operatorname{sgn}(\xi)n\xi}, \quad \forall\xi, \ n=1,2,....$$

11. THE THERMAL DISPLACEMENTS

The "displacements" due to temperature are calculated from equations (38a) for general points $M(\xi,\theta)\in D_1$ and $N(\xi,\theta)\in D_2$. For definiteness, these points are taken in the half-plane $y>0$. The line integration in each domain is carried out as follows: (i) For the point $M(\xi,\theta)\in D_1$, take a path formed by the directed segment of a straight line C_1A joining the center C_1 of the first cylinder to the point of intersection A of the circle of constant ξ through M with the x-axis, and the directed arc AM of this circle, as shown on the figure; (ii) For the point $N(\xi,\theta)\in D_2$, take a path formed by the directed segment of a straight line C_2B joining the center C_2 of the second cylinder to the point of intersection B of the circle of constant ξ through N with the x-axis, and the directed arc BN of this circle, as shown on the figure. One gets

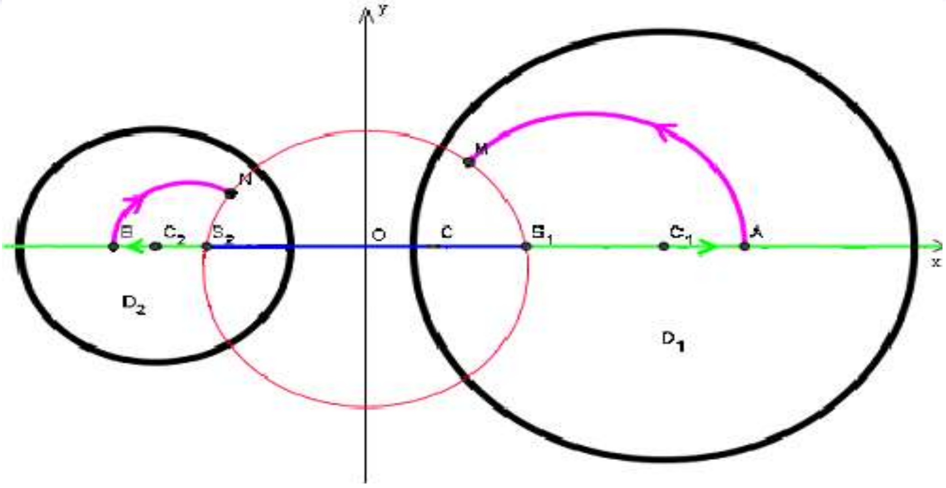

Fig. 2. Integration paths

$$\frac{1}{1+\nu}\frac{u_T}{l}=\int_{2\xi_1}^{\xi}\alpha T_h\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi+\int_{\pi}^{\theta}\left(\alpha T_h\frac{d}{d\theta}\left(\frac{x}{l}\right)-\alpha T_h^c\frac{d}{d\theta}\left(\frac{y}{l}\right)\right)d\theta,$$

$$\frac{1}{1+\nu}\frac{v_T}{l}=\int_{2\xi_1}^{\xi}\alpha T_h^c\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi+\int_{\pi}^{\theta}\left(\alpha T_h^c\frac{d}{d\theta}\left(\frac{x}{l}\right)+\alpha T_h\frac{d}{d\theta}\left(\frac{y}{l}\right)\right)d\theta,$$

and

$$\frac{1}{1+\nu}\frac{u_T^{'}}{l}=\int_{2\xi_1}^{\xi}\alpha T_h^{'}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi+\int_{\pi}^{\theta}\left(\alpha T_h^{'}\frac{d}{d\theta}\left(\frac{x}{l}\right)-\alpha T_h^{'c}\frac{d}{d\theta}\left(\frac{y}{l}\right)\right)d\theta,$$

$$\frac{1}{1+\nu}\frac{v_T^{'}}{l}=\int_{2\xi_1}^{\xi}\alpha T_h^{'c}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi+\int_{\pi}^{\theta}\left(\alpha T_h^{'c}\frac{d}{d\theta}\left(\frac{x}{l}\right)+\alpha T_h^{'}\frac{d}{d\theta}\left(\frac{y}{l}\right)\right)d\theta,$$

The following formulae are finally obtained for the "thermal" displacements in D_1 and D_2 :

$$\left.\begin{aligned}
\frac{1}{1+\nu}\frac{u_T}{l}&=\frac{c}{l}\left(\frac{\sinh\xi}{\cosh\xi+\cos\theta}-\frac{\sinh 2\xi_1}{\cosh 2\xi_1-1}\right)\alpha A_0\\
&\quad+2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^n\frac{j}{n+j}\alpha A_n\left(e^{-(n+j)\xi}-e^{-2(n+j)\xi_1}\right)e^{n\xi_1}\\
&\quad+2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^j\frac{j}{n+j}\alpha A_n e^{-n(\xi-\xi_1)}e^{-j\xi}\left(\cos(n+j)\theta-(-1)^{n+j}\right),\\
\frac{1}{1+\nu}\frac{v_T}{l}&=\frac{c}{l}\frac{\sin\theta}{\cosh\xi+\cos\theta}\alpha A_0-2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^j\frac{j}{n+j}e^{-n(\xi-\xi_1)}e^{-j\xi}\alpha A_n\sin(n+j)\theta,
\end{aligned}\right\}, \qquad (62)$$

and

$$\left.\begin{aligned}
\frac{1}{1+\nu}\frac{u_T^{'}}{l}&=\frac{c}{l}\left(\frac{\sinh\xi}{\cosh\xi+\cos\theta}-\frac{\sinh 2\xi_2}{\cosh 2\xi_2-1}\right)\alpha A_0\\
&\quad-2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^n\frac{j}{n+j}\alpha A_n^{'}\left(e^{(n+j)\xi}-e^{2(n+j)\xi_2}\right)e^{-n\xi_2}\\
&\quad-2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^j\frac{j}{n+j}\alpha A_n^{'}e^{n(\xi-\xi_2)}e^{j\xi}\left(\cos(n+j)\theta-(-1)^{n+j}\right),\\
\frac{1}{1+\nu}\frac{v_T^{'}}{l}&=\frac{c}{l}\frac{\sin\theta}{\cosh\xi+\cos\theta}\alpha A_0^{'}-2\frac{c}{l}\sum_{n,j=1}^{\infty}(-1)^j\frac{j}{n+j}\alpha A_n^{'}e^{n(\xi-\xi_2)}e^{j\xi}\sin(n+j)\theta,
\end{aligned}\right\}, \qquad (63)$$

12. SOLUTION FOR THE MAGNETIC VECTOR POTENTIAL

The general solutions of the equations (20) and (21) in D_1 , D_2 and D_3 are expressed as

$$A=A_h+A_p,\qquad \xi\geq\xi_1>0,\qquad (64a)$$

$$A'=A_h^{'}+A_p^{'},\qquad \xi\leq\xi_2>0,\qquad (64b)$$

$$A^{*}=A_r^{*}+A_\infty^{*},\qquad \xi_2<\xi<\xi_1,\qquad (64c)$$

with

$$\frac{1}{\mu^* l^2 J_1} A_p = -\frac{\mu_0}{4}\frac{r^2}{l^2}, \qquad \frac{1}{\mu^* l^2 J_1} A_p^{'} = -J\frac{\mu_0}{4}\frac{r^2}{l^2}, \qquad \frac{1}{\mu^* l^2 J_1} A_\infty^* = \frac{1}{2}\frac{I}{l^2 J_1}\ln\frac{r}{l}, \qquad (65)$$

where

$$I = \pi a^2 J_1 + \pi b^2 J_2, \qquad J = \frac{J_2}{J_1}, \qquad l = a+b, \qquad (66)$$

The solutions for Laplace's equations in bipolar cylindrical coordinates, compatible with the setting of the problem, are

$$\frac{1}{\mu^* l^2 J_1} A_h = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-n(\xi-\xi_1)} \cos n\theta, \qquad (67a)$$

$$\frac{1}{\mu^* l^2 J_1} A_h^{'} = \alpha_0^{'} + \sum_{n=1}^{\infty} \alpha_n^{'} e^{n(\xi-\xi_2)} \cos n\theta, \qquad (67b)$$

$$\frac{1}{\mu^* l^2 J_1} A_r^* = \alpha_0^* + \sum_{n=1}^{\infty} \left(\alpha_n^* e^{n(\xi-\xi_1)} + \beta_n^* e^{-n(\xi-\xi_2)}\right) \cos n\theta, \qquad (67c)$$

the constants to be determined from the magnetic boundary conditions. The radiation condition $A_r^*(\xi \to 0, \theta \to 0) = 0$ yields

$$\alpha_0^* = -\sum_{n=1}^{\infty} \left(\alpha_n^* e^{-n\xi_1} + \beta_n^* e^{n\xi_2}\right).$$

The solutions for the vector potential component in D_1, D_2 and D_3 are

$$\frac{1}{\mu^* l^2 J_1} A = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-n(\xi-\xi_1)} \cos n\theta - \frac{\mu_0}{4}\frac{r^2}{l^2}, \qquad (68a)$$

$$\frac{1}{\mu^* l^2 J_1} A' = \alpha_0^{'} + \sum_{n=1}^{\infty} \alpha_n^{'} e^{n(\xi-\xi_2)} \cos n\theta - J\frac{\mu_0}{4}\frac{r^2}{l^2}, \qquad (68b)$$

$$\frac{1}{\mu^* l^2 J_1} A^* = \alpha_0^* + \sum_{n=1}^{\infty} \left(\alpha_n^* e^{n(\xi-\xi_1)} + \beta_n^* e^{-n(\xi-\xi_2)}\right) \cos n\theta - \lambda \ln\frac{r}{l}, \qquad (68c)$$

and

$$\lambda = \frac{\pi}{2}\frac{a^2}{l^2}\left(1+q^2 J\right), \qquad q = \frac{b}{a}. \qquad (69)$$

Apply the magnetic boundary conditions (25) and (26) and use the orthogonality property of trigonometric functions to get

$$\alpha_0 = \alpha_0^* + \frac{\mu_0}{4}\frac{c^2}{l^2}\left(2\coth\xi_1 - 1\right) - \lambda \ln\frac{c}{l},$$

$$\alpha_0^{'} = \alpha_0^* - J\frac{\mu_0}{4}\frac{c^2}{l^2}\left(2\coth\xi_2 - 1\right) - \lambda \ln\frac{c}{l},$$

$$\alpha_p^* = -\frac{1}{p}\frac{(-1)^p(\mu_0-1)^2 e^{-p\xi_1}}{(\mu_0-1)^2 e^{-2p(\xi_1-\xi_2)}-(\mu_0+1)^2}\left[\frac{\mu_0}{\mu_0-1}\left(\frac{\mu_0+1}{\mu_0-1}\frac{a^2}{l^2}+J\frac{b^2}{l^2}e^{2p\xi_2}\right)-\left(1-(-1)^p\right)\lambda\left(\frac{\mu_0+1}{\mu_0-1}+e^{2p\xi_2}\right)\right],$$

$$\beta_p^* = -\frac{1}{p}\frac{(-1)^p(\mu_0-1)^2 e^{p\xi_2}}{(\mu_0-1)^2 e^{2p(\xi_2-\xi_1)}-(\mu_0+1)^2}\left[\frac{\mu_0}{\mu_0-1}\left(\frac{a^2}{l^2}e^{-2p\xi_1}+J\frac{b^2}{l^2}\frac{\mu_0+1}{\mu_0-1}\right)-\left(1-(-1)^p\right)\lambda\left(\frac{\mu_0+1}{\mu_0-1}+e^{-2p\xi_1}\right)\right],$$

$$\alpha_p' = \frac{1}{p}\frac{\mu_0}{\mu_0-1}\left[2p\beta_p^*+(-1)^p J\left((\mu_0-1)\,p\frac{c^2}{l^2}\coth\xi_2+\frac{b^2}{l^2}\right)e^{p\xi_2}\right],$$

$$\alpha_p = \frac{1}{p}\frac{\mu_0}{\mu_0-1}\left[2p\alpha_p^*+(-1)^p\left((\mu_0-1)\,p\frac{c^2}{l^2}\coth\xi_1-\frac{a^2}{l^2}\right)e^{-p\xi_1}\right].$$

The harmonic conjugate functions may now be calculated using the Cauchy-Riemann relations.

Taking A_h^c and $A_h^{'c}$ to vanish at the foci ($\xi=\infty,\ \theta=0$) and ($\xi=-\infty,\ \theta=0$) respectively, one gets

$$\frac{1}{\mu^* l^2 J_1}A_h^c = -\sum_{n=1}^{\infty}\alpha_n e^{-n(\xi-\xi_1)}\sin n\theta,\qquad \frac{1}{\mu^* l^2 J_1}A_h^{'c}\,' = \sum_{n=1}^{\infty}\alpha_n' e^{n(\xi-\xi_2)}\sin n\theta,$$

13. THE MAGNETIC FIELD VECTOR

In this section, we determine the magnetic field inside and outside the two cylinders, due to the electric currents J_1 and J_2. In view of (15) and (19), the magnetic field vector may be expressed in bipolar cylindrical coordinates as

$$\frac{1}{l^2J_1}H_\xi = \frac{1}{\mu_0}\frac{1}{h}\frac{\partial}{\partial\theta}\left(\frac{1}{\mu^* l^2 J_1}A\right),\qquad \frac{1}{l^2J_1}H_\theta = \frac{1}{\mu_0}\frac{1}{h}\frac{\partial}{\partial\xi}\left(\frac{1}{\mu^* l^2 J_1}A\right),\qquad (70a)$$

$$\frac{1}{l^2J_1}H_\xi' = \frac{1}{\mu_0}\frac{1}{h}\frac{\partial}{\partial\theta}\left(\frac{1}{\mu^* l^2 J_1}A'\right),\qquad \frac{1}{l^2J_1}H_\theta' = \frac{1}{\mu_0}\frac{1}{h}\frac{\partial}{\partial\xi}\left(\frac{1}{\mu^* l^2 J_1}A'\right),\qquad (70b)$$

Set equations (68a, b, c), with their derivatives into equations (70a, b) to get

(1)- The magnetic field vector components for the right cylinder (region D_1) are

$$\frac{1}{l^2J_1}H_\xi = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n\frac{\alpha_n}{\mu_0}e^{-n(\xi-\xi_1)}\sin n\theta+\frac{1}{2}\frac{c^2}{l^2}\frac{\cosh\xi\sin\theta}{(\cosh\xi+\cos\theta)^2}\right),\qquad (71a)$$

$$\frac{1}{l^2J_1}H_\theta = \frac{l}{h}\left(\sum_{n=1}^{\infty} n\frac{\alpha_n}{\mu_0}e^{-n(\xi-\xi_1)}\cos n\theta+\frac{1}{2}\frac{c^2}{l^2}\frac{\sinh\xi\cos\theta}{(\cosh\xi+\cos\theta)^2}\right).\qquad (71b)$$

(2)- The magnetic field vector components for the left cylinder (region D_2) are

$$\frac{1}{l^2J_1}H_\xi' = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n\frac{\alpha_n'}{\mu_0}e^{n(\xi-\xi_2)}\sin n\theta+\frac{1}{2}\frac{c^2}{l^2}J\frac{\cosh\xi\sin\theta}{(\cosh\xi+\cos\theta)^2}\right),\qquad (71c)$$

$$\frac{1}{l^2J_1}H_\theta' = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n\frac{\alpha_n'}{\mu_0}e^{n(\xi-\xi_2)}\cos n\theta-\frac{1}{2}\frac{c^2}{l^2}J\frac{\sinh\xi\cos\theta}{(\cosh\xi+\cos\theta)^2}\right).\qquad (71d)$$

(3)- The magnetic field vector components outside the two cylinders (region D_3) are

$$\frac{1}{l^2 J_1} H_\xi^* = \frac{l}{h} \sum_{n=1}^{\infty} L_n(\xi) \sin n\theta - \lambda \frac{l}{c} \frac{\cosh \xi \sin \theta}{\cosh \xi - \cos \theta}, \qquad (71e)$$

$$\frac{1}{l^2 J_1} H_\theta^* = \frac{l}{h} \sum_{n=1}^{\infty} R_n(\xi) \cos n\theta + \lambda \frac{l}{c} \frac{\sinh \xi \cos \theta}{\cosh \xi - \cos \theta}. \qquad (71f)$$

With

$$L_n(\xi) = -n\left(\alpha_n^* e^{n(\xi-\xi_1)} + \beta_n^* e^{-n(\xi-\xi_2)}\right), \qquad R_n(\xi) = -n\left(\alpha_n^* e^{n(\xi-\xi_1)} - \beta_n^* e^{-n(\xi-\xi_2)}\right).$$

Using equations (8), the magnetic field vector for D_1 and D_2 may be expressed as

$$\frac{1}{l^2 J_1} H_\xi = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n \frac{\alpha_n}{\mu_0} e^{-n(\xi-\xi_1)} \sin n\theta - \frac{c^2}{l^2} \coth \xi \sum_{n=1}^{\infty} n(-1)^n e^{-n\xi} \sin n\theta\right), \qquad (72a)$$

$$\frac{1}{l^2 J_1} H_\theta = \frac{l}{h}\left(\sum_{n=1}^{\infty} n \frac{\alpha_n}{\mu_0} e^{-n(\xi-\xi_1)} \cos n\theta - \frac{1}{2}\frac{c^2}{l^2} \operatorname{csch}^2 \xi \right.$$
$$\left. - \frac{c^2}{l^2} \sum_{n=1}^{\infty} (-1)^n \left(\operatorname{csch}^2 \xi + n \coth \xi\right) e^{-n\xi} \cos n\theta\right), \qquad (72b)$$

and

$$\frac{1}{l^2 J_1} H_\xi^{'} = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n \frac{\alpha_n^{'}}{\mu_0} e^{n(\xi-\xi_2)} \sin n\theta + J \frac{c^2}{l^2} \coth \xi \sum_{n=1}^{\infty} n(-1)^n e^{-n\xi} \sin n\theta\right), \qquad (72c)$$

$$\frac{1}{l^2 J_1} H_\theta^{'} = -\frac{l}{h}\left(\sum_{n=1}^{\infty} n \frac{\alpha_n^{'}}{\mu_0} e^{n(\xi-\xi_2)} \cos n\theta - \frac{1}{2}\frac{c^2}{l^2} J \operatorname{csch}^2 \xi \right.$$
$$\left. - J \frac{c^2}{l^2} \sum_{n=1}^{\infty} (-1)^n \left(\operatorname{csch}^2 \xi - n \coth \xi\right) e^{n\xi} \cos n\theta\right). \qquad (72d)$$

14. THE MAXWELLIAN STRESS TENSOR

The Maxwellian stress tensor σ^* for region D_3 has components

$$\frac{\sigma_{\xi\xi}^*}{E} = \frac{l^2}{h^2} \sum_{n,m=1}^{\infty} B_{n,m}(\xi) \cos(n-m)\theta - \frac{l^2}{h^2} \sum_{n,m=1}^{\infty} A_{n,m}(\xi) \cos(n+m)\theta$$
$$-\frac{l}{h} \sum_{n=1}^{\infty} C_n(\xi) \frac{\cos(n-1)\theta}{\cosh \xi - \cos \theta} + \frac{l}{h} \sum_{n=1}^{\infty} D_n(\xi) \frac{\cos(n+1)\theta}{\cosh \xi - \cos \theta} + \frac{1}{2}\frac{l^2}{c^2} \beta_1 \gamma \lambda^2, \qquad (73a)$$

$$\frac{\sigma_{\theta\theta}^*}{E} = -\frac{l^2}{h^2} \sum_{n,m=1}^{\infty} B_{n,m}(\xi) \cos(n-m)\theta + \frac{l^2}{h^2} \sum_{n,m=1}^{\infty} A_{n,m}(\xi) \cos(n+m)\theta$$
$$+\frac{l}{h} \sum_{n=1}^{\infty} C_n(\xi) \frac{\cos(n-1)\theta}{\cosh \xi - \cos \theta} - \frac{l}{h} \sum_{n=1}^{\infty} D_n(\xi) \frac{\cos(n+1)\theta}{\cosh \xi - \cos \theta} - \frac{1}{2}\frac{l^2}{c^2} \beta_1 \gamma \lambda^2, \qquad (73b)$$

$$\frac{\sigma_{\xi\theta}^*}{E} = \frac{l^2}{h^2} \sum_{n,m=1}^{\infty} E_{n,m}(\xi) \sin(n-m)\theta + \frac{l^2}{h^2} \sum_{n,m=1}^{\infty} E_{n,m}(\xi) \sin(n+m)\theta$$
$$+\frac{l}{h} \sum_{n=1}^{\infty} H_n(\xi) \frac{\sin(n-1)\theta}{\cosh \xi - \cos \theta} - \frac{l}{h} \sum_{n=1}^{\infty} F_n(\xi) \frac{\sin(n+1)\theta}{\cosh \xi - \cos \theta} - \frac{1}{4}\frac{l^2}{c^2} \beta_1 \gamma \lambda^2 \frac{\sinh 2\xi \sin 2\theta}{(\cosh \xi - \cos \theta)^2}. \qquad (73a)$$

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where

$$A_{n,m}(\xi)=\frac{1}{4}\beta_1\gamma\left(L_n(\xi)L_m(\xi)+R_n(\xi)R_m(\xi)\right),$$

$$B_{n,m}(\xi)=\frac{1}{4}\beta_1\gamma\left(L_n(\xi)L_m(\xi)-R_n(\xi)R_m(\xi)\right),$$

$$E_{n,m}(\xi)=\frac{1}{4}\beta_1\gamma\left(L_n(\xi)R_m(\xi)+L_n(\xi)R_m(\xi)\right),$$

$$C_n(\xi)=\frac{1}{2}\beta_1\gamma\frac{l}{c}\lambda\left(\cosh\xi L_n(\xi)+\sinh\xi R_n(\xi)\right),$$

$$D_n(\xi)=\frac{1}{2}\beta_1\gamma\frac{l}{c}\lambda\left(\cosh\xi L_n(\xi)-\sinh\xi R_n(\xi)\right),$$

$$H_n(\xi)=\frac{1}{2}\beta_1\gamma\frac{l}{c}\lambda\left(\cosh\xi R_n(\xi)+\sinh\xi L_n(\xi)\right),$$

$$F_n(\xi)=\frac{1}{2}\beta_1\gamma\frac{l}{c}\lambda\left(\cosh\xi R_n(\xi)-\sinh\xi L_n(\xi)\right),$$

and

$$\gamma=\frac{4K\sigma\mu^*}{\alpha E}. \qquad (74)$$

15. THE DISPLACEMENTS DUE TO THE MAGNETIC FIELD

In this section, we calculate the "displacement" vector components due to the magnetic field for two regions D_1 and D_2 by substitution of equations (39a, b) into equations (38b).

(1)- For D_1

$$\frac{u_H}{l}=W_1\left(K_1(\xi)+K_2(\xi,\theta)+K_3(\xi,\theta)-2K_4(\xi,\theta)\right), \qquad (75a)$$

$$\frac{v_H}{l}=W_1\left(-2K_5(\xi)+2K_6(\xi,\theta)+K_7(\xi,\theta)+K_8(\xi,\theta)\right), \qquad (75b)$$

with

$$K_1(\xi)=\int_{2\xi_1}^{\xi}\frac{H_y^2-H_x^2}{l^2J_1^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi,$$

$$K_2(\xi,\theta)=\frac{1}{2}\int_{\pi}^{\theta}\frac{r^2}{l^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta\Bigg|+\frac{1}{2}\int_{2\xi_1}^{\xi}\frac{r^2}{l^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi,$$

$$K_3(\xi,\theta)=\int_{\pi}^{\theta}\frac{H_y^2-H_x^2}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta-2\int_{\pi}^{\theta}\frac{H_xH_y}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta,$$

$$K_4(\xi,\theta)=\frac{1}{\mu_0}\int_{\pi}^{\theta}\frac{A_h^c}{\mu^*l^2J_1}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta,\qquad K_5(\xi)=\int_{2\xi_1}^{\xi}\frac{H_xH_y}{l^2J_1^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=\pi}d\xi,$$

$$K_6(\xi,\theta)=\frac{1}{\mu_0}\int_{\pi}^{\theta}\frac{A_h^c}{\mu^*l^2J_1}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta,\qquad K_7(\xi,\theta)=\frac{1}{2}\int_{\pi}^{\theta}\frac{r^2}{l^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta\Bigg|,$$

$$K_8(\xi,\theta)=\int_{\pi}^{\theta}\frac{H_x^2-H_y^2}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta-2\int_{\pi}^{\theta}\frac{H_xH_y}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta,$$

(2) For D_2

$$\frac{u_H^{'}}{l}=W_1\left(K_1^{'}(\xi)+J^2K_2^{'}(\xi,\theta)+K_3^{'}(\xi,\theta)-2JK_4^{'}(\xi,\theta)\right),\qquad (76a)$$

$$\frac{v_H^{'}}{l}=W_1\left(-2K_5^{'}(\xi)+2JK_6^{'}(\xi,\theta)+J^2K_7^{'}(\xi,\theta)+K_8^{'}(\xi,\theta)\right),\qquad (76b)$$

with

$$K_1^{'}(\xi)=\int_{2\xi_2}^{\xi}\frac{H_y^{'2}-H_x^{'2}}{l^2J_1^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=-\pi}d\xi,$$

$$K_2^{'}(\xi,\theta)=\frac{1}{2}\int_{\pi}^{\theta}\frac{r^2}{l^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta\Bigg|+\frac{1}{2}\int_{2\xi_2}^{\xi}\frac{r^2}{l^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=-\pi}d\xi,$$

$$K_3^{'}(\xi,\theta)=\int_{-\pi}^{\theta}\frac{H_y^{'2}-H_x^{'2}}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta-2\int_{-\pi}^{\theta}\frac{H_x^{'}H_y^{'}}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta,$$

$$K_4^{'}(\xi,\theta)=\frac{1}{\mu_0}\int_{-\pi}^{\theta}\frac{A_h^{'c}}{\mu^*l^2J_1}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta,\qquad K_5^{'}(\xi)=\int_{2\xi_2}^{\xi}\frac{H_x^{'}H_y^{'}}{l^2J_1^2}\frac{d}{d\xi}\left(\frac{x}{l}\right)\Bigg|_{\theta=-\pi}d\xi,$$

$$K_6^{'}(\xi,\theta)=\frac{1}{\mu_0}\int_{-\pi}^{\theta}\frac{A_h^{'c}}{\mu^*l^2J_1}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta,\qquad K_7^{'}(\xi,\theta)=\frac{1}{2}\int_{-\pi}^{\theta}\frac{r^2}{l^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta\Bigg|,$$

$$K_8^{'}(\xi,\theta)=\int_{-\pi}^{\theta}\frac{H_x^{'2}-H_y^{'2}}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{y}{l}\right)d\theta-2\int_{-\pi}^{\theta}\frac{H_x^{'}H_y^{'}}{l^2J_1^2}\frac{d}{d\theta}\left(\frac{x}{l}\right)d\theta,$$

and

$$W_1=\frac{1}{2}(1+\nu)\beta_1\gamma\left(\mu_0-\frac{1}{2}\mu_1\right).$$

In these formulae, (H_x, H_y) and (H_ξ, H_θ) are the components of the magnetic field in Cartesian and in bipolar coordinates respectively.

16. THE PARTICULAR SOLUTIONS

The particular solutions of the biharmonic equations U_p and $U_p^{'}$ for two regions D_1 and D_2 should be satisfy equation (35). By use the particular solutions for the temperature problem, then equation (35) may be written for two regions D_1 and D_2 as

$$\nabla^2\left(\frac{1}{E}U_p\right)=\frac{\beta_1}{1-\nu}\left(1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)\right)\frac{r^2}{l^2}-\frac{1-2\nu}{2(1-\nu)}\frac{\mu^*}{E}\left(\frac{1}{2}\mu_1+\mu_2\right)H^2, \qquad (78a)$$

$$\nabla^2\left(\frac{1}{E}U_p^{'}\right)=\frac{\beta_2}{1-\nu}\left(1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)\right)\frac{r^2}{l^2}-\frac{1-2\nu}{2(1-\nu)}\frac{\mu^*}{E}\left(\frac{1}{2}\mu_1+\mu_2\right)H^{'2}, \qquad (78b)$$

take ∇² for both sides and use $\nabla^2\frac{r^2}{l^2}=\frac{4}{l^2}$

to write equations (78a, b) as

$$\nabla^4\left(\frac{1}{E}U_p\right)=\frac{4}{l^2}\frac{\beta_1}{1-\nu}\left(1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)\right)-\frac{1-2\nu}{2(1-\nu)}\frac{\mu^*}{E}\left(\frac{1}{2}\mu_1+\mu_2\right)\nabla^2H^2, \qquad (79a)$$

$$\nabla^4\left(\frac{1}{E}U_p^{'}\right)=\frac{4}{l^2}\frac{\beta_2}{1-\nu}\left(1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)\right)-\frac{1-2\nu}{2(1-\nu)}\frac{\mu^*}{E}\left(\frac{1}{2}\mu_1+\mu_2\right)\nabla^2H^{'2}, \qquad (79b)$$

But, from equations (71a, b, c, d), one has

$$\nabla^2H^2=l^4J_1^2\nabla^4y+J_1^2 \qquad \nabla^2H^{'2}=l^4J_1^2\nabla^4y'+J_2^2, \qquad (80)$$

with

$$y=\frac{1}{4}\frac{1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta,$$

$$y'=\frac{1}{4}\frac{1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n^{'}\alpha_m^{'} e^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta.$$

Substitute from equations (80) into equations (79a,b) and rearrange:

$$\nabla^4\left[\left(\frac{U_p}{l^2E}\right)+\omega_1 y\right]=\frac{1}{l^4}D_1, \qquad \nabla^4\left[\left(\frac{U_p^{'}}{l^2E}\right)+\omega_1 y'\right]=\frac{1}{l^4}D_1^{'},$$

with

$$D_1=\frac{4\beta_1}{1-\nu}\left[1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)-\frac{\gamma}{8}(1-2\nu)\left(\frac{1}{2}\mu_1+\mu_2\right)\right],$$

$$D_1^{'}=\frac{4\beta_2}{1-\nu}\left[1+\frac{\gamma}{4}\left(\mu_0-\frac{1}{2}\mu_1\right)-\frac{\gamma}{8}(1-2\nu)\left(\frac{1}{2}\mu_1+\mu_2\right)\right], \qquad \omega_1=\frac{\beta_1\gamma}{2}\frac{1-2\nu}{1-\nu}\left(\frac{1}{2}\mu_1+\mu_2\right).$$

Since $\nabla^4r^4=64.$

The particular solution for two regions D_1 and D_2 are

$$\frac{U_p}{l^2E}=\frac{D_1}{64}\frac{r^4}{l^4}-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta, \qquad (81a)$$

$$\frac{U_p^{'}}{l^2E}=\frac{D_1^{'}}{64}\frac{r^4}{l^4}-\frac{1}{4}\frac{1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n^{'}\alpha_m^{'} e^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta. \qquad (81b)$$

The tangential and normal derivatives of function U_{p} are

$$\frac{\partial U_p}{\partial n}=-\frac{1}{h}\frac{\partial U_p}{\partial \xi},\qquad \frac{\partial U_p}{\partial s}=-\frac{1}{h}\frac{\partial U_p}{\partial \theta}\qquad (82a)$$

$$\frac{\partial U_p^{'}}{\partial n}=\frac{1}{h}\frac{\partial U_p^{'}}{\partial \xi},\qquad \frac{\partial U_p^{'}}{\partial s}=-\frac{1}{h}\frac{\partial U_p^{'}}{\partial \theta}\qquad (82b)$$

Also, the Cauchy-Riemann conditions are

$$\frac{\partial \Phi}{\partial \xi}=\frac{\partial \Phi^c}{\partial \theta},\qquad \frac{\partial \Phi}{\partial \theta}=-\frac{\partial \Phi^c}{\partial \xi}.$$

The representations of harmonic functions which satisfy the Cauchy-Riemann conditions and the finiteness condition for regions D_1 and D_2 are

$$\frac{\Phi}{lE}=a_0+\sum_{n=1}^{\infty}a_n e^{-n(\xi-\xi_1)}\cos n\theta,\qquad \frac{\Phi^c}{lE}=b_0-\sum_{n=1}^{\infty}a_n e^{-n(\xi-\xi_1)}\sin n\theta,\qquad (83a)$$

$$\frac{\Psi}{l^2E}=c_0+\sum_{n=1}^{\infty}c_n e^{-n(\xi-\xi_1)}\cos n\theta,\qquad \frac{\Psi^c}{l^2E}=d_0-\sum_{n=1}^{\infty}c_n e^{-n(\xi-\xi_1)}\sin n\theta,\qquad (83b)$$

and

$$\frac{\Phi'}{lE}=a_0^{'}+\sum_{n=1}^{\infty}a_n^{'} e^{n(\xi-\xi_2)}\cos n\theta,\qquad \frac{\Phi'^c}{lE}=b_0^{'}+\sum_{n=1}^{\infty}a_n^{'} e^{n(\xi-\xi_2)}\sin n\theta,\qquad (84a)$$

$$\frac{\Psi'}{l^2E}=c_0^{'}+\sum_{n=1}^{\infty}c_n^{'} e^{n(\xi-\xi_2)}\cos n\theta,\qquad \frac{\Psi'^c}{l^2E}=d_0^{'}+\sum_{n=1}^{\infty}c_n^{'} e^{n(\xi-\xi_2)}\sin n\theta.\qquad (84b)$$

Where the coefficients are to be determined using the mechanical boundary conditions.

The following boundary conditions on the stresses

For D_1

$$\frac{\sigma_{\xi\xi}}{E}=\frac{\sigma_{\xi\xi}^{*}}{E},\qquad \frac{\sigma_{\xi\theta}}{E}=\frac{\sigma_{\xi\theta}^{*}}{E},\qquad \text{at } \xi=\xi_1,\qquad (85a)$$

For D_2

$$\frac{\sigma_{\xi\xi}^{'}}{E}=\frac{\sigma_{\xi\xi}^{*}}{E},\qquad \frac{\sigma_{\xi\theta}^{'}}{E}=\frac{\sigma_{\xi\theta}^{*}}{E},\qquad \text{at } \xi=\xi_2.\qquad (85b)$$

17. THE STRESS FUNCTION

Substitute from equations (1), (81a), (81b), (83a), (83b) (84a), and (84b), into equation (34) to get the stress functions for the regions D_1 and D_2

$$\frac{U}{l^2E}=c_0+\frac{c}{l}\frac{\sinh\xi}{\cosh\xi+\cos\theta}a_0+\frac{c}{l}\frac{\sin\theta}{\cosh\xi+\cos\theta}b_0+\frac{c}{l}\sum_{n=1}^{\infty}a_n e^{-n(\xi-\xi_1)}\frac{\sinh\xi}{\cosh\xi+\cos\theta}\cos n\theta$$
$$-\frac{c}{l}\sum_{n=1}^{\infty}a_n e^{-n(\xi-\xi_1)}\frac{\sin\theta}{\cosh\xi+\cos\theta}\sin n\theta+\sum_{n=1}^{\infty}c_n e^{-n(\xi-\xi_1)}\cos n\theta+\frac{D_1}{64}\frac{r^4}{l^4}$$
$$-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta,\qquad (86a)$$

$$\frac{U}{l^2E}=c_0'+\frac{c}{l}\frac{\sinh\xi}{\cosh\xi+\cos\theta}a_0'+\frac{c}{l}\frac{\sin\theta}{\cosh\xi+\cos\theta}b_0'+\frac{c}{l}\sum_{n=1}^{\infty}a_n'e^{n(\xi-\xi_2)}\frac{\sinh\xi}{\cosh\xi+\cos\theta}\cos n\theta$$
$$+\frac{c}{l}\sum_{n=1}^{\infty}a_n'e^{n(\xi-\xi_2)}\frac{\sin\theta}{\cosh\xi+\cos\theta}\sin n\theta+\sum_{n=1}^{\infty}c_n'e^{n(\xi-\xi_2)}\cos n\theta+\frac{D_1'}{64}\frac{r^4}{l^4}$$
$$-\frac{1}{4}\frac{1}{\mu_0^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n'\alpha_m'e^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta. \qquad (86b)$$

18. THE STRESS TENSOR COMPONENTS

The stress tensor components in bipolar coordinates are

$$\frac{\sigma_{\xi\xi}}{E}=\frac{l}{h}\left[\frac{\partial}{\partial\theta}\left(\frac{l}{h}\frac{\partial}{\partial\theta}\left(\frac{U}{l^2E}\right)\right)-\frac{\partial}{\partial\xi}\left(\frac{l}{h}\right)\frac{\partial}{\partial\xi}\left(\frac{U}{l^2E}\right)\right], \qquad (87a)$$

$$\frac{\sigma_{\theta\theta}}{E}=\frac{l}{h}\left[\frac{\partial}{\partial\xi}\left(\frac{l}{h}\frac{\partial}{\partial\xi}\left(\frac{U}{l^2E}\right)\right)-\frac{\partial}{\partial\theta}\left(\frac{l}{h}\right)\frac{\partial}{\partial\theta}\left(\frac{U}{l^2E}\right)\right], \qquad (87b)$$

$$\frac{\sigma_{\xi\theta}}{E}=-\frac{l}{h}\left[\frac{\partial}{\partial\xi}\left(\frac{l}{h}\frac{\partial}{\partial\theta}\left(\frac{U}{l^2E}\right)\right)+\frac{\partial}{\partial\theta}\left(\frac{l}{h}\right)\frac{\partial}{\partial\xi}\left(\frac{U}{l^2E}\right)\right], \qquad (87c)$$

Substitute equations (86a, b) into equations (87a, b, c), to get the stress tensor components for D_1 and D_2

1) Stress components for D_1

$$\frac{\sigma_{\xi\xi}}{E}=\frac{l}{h}\frac{l}{c}\left\{\frac{D_1}{16}\frac{c^4}{l^4}\frac{\cosh\xi-\cos\theta}{(\cosh\xi+\cos\theta)^2}+\frac{D_1}{32}\frac{c^4}{l^4}\frac{\cosh^2\xi\sin^2\theta}{(\cosh\xi+\cos\theta)^3}-\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos(n-2)\theta}{\cosh\xi+\cos\theta}\right.$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c_n-\frac{c}{l}na_n\right)e^{-n(\xi-\xi_1)}\cos(n-1)\theta-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c_n+\frac{c}{l}na_n\right)e^{-n(\xi-\xi_1)}\cos(n+1)\theta$$
$$+\sum_{n=1}^{\infty}n\left((\sinh\xi-n\cosh\xi)c_n-n\frac{c}{l}\sinh\xi a_n\right)e^{-n(\xi-\xi_1)}\cos n\theta-\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}e^{\xi}\frac{\cos(n-1)\theta}{\cosh\xi+\cos\theta}$$
$$-\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}e^{-\xi}\frac{\cos(n+1)\theta}{\cosh\xi+\cos\theta}-\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos(n+2)\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{c}{l}\left(\cosh^2\xi-\frac{5}{2}\right)\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos n\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)(n-m+1)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m+1)\theta$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)(n-m-1)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m-1)\theta$$
$$\left.+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n-m)^2\cosh\xi-(n+m)\sinh\xi\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta\right\}, \qquad (88a)$$

$$\frac{\sigma_{\theta\theta}}{E}=\frac{l}{h}\frac{l}{c}\left\{\frac{D_1}{16}\frac{c^4}{l^4}\frac{\cosh\xi-\cos\theta}{(\cosh\xi+\cos\theta)^2}+\frac{D_1}{32}\frac{c^4}{l^4}\frac{\sinh^2\xi\cos^2\theta}{(\cosh\xi+\cos\theta)^3}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos(n-2)\theta}{\cosh\xi+\cos\theta}\right.$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c_n-n\frac{c}{l}a_n\right)e^{-n(\xi-\xi_1)}\cos(n-1)\theta+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c_n+n\frac{c}{l}a_n\right)e^{-n(\xi-\xi_1)}\cos(n+1)\theta$$

$$-\sum_{n=1}^{\infty}n\left((\sinh\xi-n\cosh\xi)c_n-n\frac{c}{l}\sinh\xi a_n\right)e^{-n(\xi-\xi_1)}\cos n\theta-\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}e^{\xi}\frac{\cos(n-1)\theta}{\cosh\xi+\cos\theta}$$

$$-\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}e^{-\xi}\frac{\cos(n+1)\theta}{\cosh\xi+\cos\theta}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos(n+2)\theta}{\cosh\xi+\cos\theta}$$

$$-\frac{c}{l}\left(\sinh^2\xi+\frac{5}{2}\right)\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\cos n\theta}{\cosh\xi+\cos\theta}$$

$$+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)\left(\sinh\xi-(n+m)\cosh\xi\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta$$

$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n+m)^2+n-m\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m+1)\theta$$

$$\left.-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n+m)^2-(n-m)\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m-1)\theta\right\},, \qquad (88b)$$

and

$$\frac{\sigma_{\xi\theta}}{E}=-\frac{l}{h}\frac{l}{c}\left\{\frac{D_1}{32}\frac{c^4}{l^4}\frac{\sinh 2\xi\sin 2\theta}{(\cosh\xi+\cos\theta)^3}+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c_n-\frac{c}{l}na_n\right)e^{-n(\xi-\xi_1)}\sin(n-1)\theta\right.$$

$$-\sum_{n=1}^{\infty}n\left((\sinh\xi-n\cosh\xi)c_n-n\frac{c}{l}\sinh\xi a_n\right)e^{-n(\xi-\xi_1)}\sin n\theta$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c_n+\frac{c}{l}na_n\right)e^{-n(\xi-\xi_1)}\sin(n+1)\theta+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\sin(n-2)\theta}{\cosh\xi+\cos\theta}$$

$$-\frac{c}{l}\left(\sinh^2\xi+\frac{1}{2}\right)\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\sin n\theta}{\cosh\xi+\cos\theta}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na_ne^{-n(\xi-\xi_1)}\frac{\sin(n+2)\theta}{\cosh\xi+\cos\theta}$$

$$+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)\left(\sinh\xi-(n+m)\cosh\xi\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\sin(n-m)\theta$$

$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)(n-m+1)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\sin(n-m+1)\theta$$

$$\left.-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)(n-m-1)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\sin(n-m-1)\theta\right\}, \qquad (88c)$$

For D_2

$$\frac{\sigma'_{\xi\xi}}{E}=\frac{l}{h}\frac{l}{c}\left\{\frac{D'_1}{16}\frac{c^4}{l^4}\frac{\cosh\xi-\cos\theta}{\left(\cosh\xi+\cos\theta\right)^2}+\frac{D'_1}{32}\frac{c^4}{l^4}\frac{\cosh^2\xi\sin^2\theta}{\left(\cosh\xi+\cos\theta\right)^3}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos(n-2)\theta}{\cosh\xi+\cos\theta}\right.$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c'_n+\frac{c}{l}na'_n\right)e^{n(\xi-\xi_2)}\cos(n-1)\theta-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c'_n-\frac{c}{l}na'_n\right)e^{n(\xi-\xi_2)}\cos(n+1)\theta$$
$$-\sum_{n=1}^{\infty}n\left((\sinh\xi+n\cosh\xi)c'_n+n\frac{c}{l}\sinh\xi a'_n\right)e^{n(\xi-\xi_2)}\cos n\theta+\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}e^{-\xi}\frac{\cos(n-1)\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}e^{\xi}\frac{\cos(n+1)\theta}{\cosh\xi+\cos\theta}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos(n+2)\theta}{\cosh\xi+\cos\theta}$$
$$-\frac{c}{l}\left(\cosh^2\xi-\frac{5}{2}\right)\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos n\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)(n-m+1)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m+1)\theta$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)(n-m-1)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m-1)\theta$$
$$\left.+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n-m)^2\cosh\xi+(n+m)\sinh\xi\right)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta\right\}, \qquad (89a)$$

$$\frac{\sigma'_{\theta\theta}}{E}=\frac{l}{h}\frac{l}{c}\left\{\frac{D'_1}{16}\frac{c^4}{l^4}\frac{\cosh\xi-\cos\theta}{\left(\cosh\xi+\cos\theta\right)^2}+\frac{D'_1}{32}\frac{c^4}{l^4}\frac{\sinh^2\xi\cos^2\theta}{\left(\cosh\xi+\cos\theta\right)^3}-\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos(n-2)\theta}{\cosh\xi+\cos\theta}\right.$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c'_n+n\frac{c}{l}a'_n\right)e^{n(\xi-\xi_2)}\cos(n-1)\theta+\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c'_n-n\frac{c}{l}a'_n\right)e^{n(\xi-\xi_2)}\cos(n+1)\theta$$
$$+\sum_{n=1}^{\infty}n\left((\sinh\xi+n\cosh\xi)c'_n+n\frac{c}{l}\sinh\xi a'_n\right)e^{n(\xi-\xi_2)}\cos n\theta+\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}e^{-\xi}\frac{\cos(n-1)\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}e^{\xi}\frac{\cos(n+1)\theta}{\cosh\xi+\cos\theta}-\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos(n+2)\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{c}{l}\left(\sinh^2\xi+\frac{5}{2}\right)\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\cos n\theta}{\cosh\xi+\cos\theta}$$
$$-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)(\sinh\xi+(n+m)\cosh\xi)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n+m)^2+n-m\right)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m+1)\theta$$
$$\left.-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}\left((n+m)^2-(n-m)\right)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\cos(n-m-1)\theta\right\}, \qquad (89b)$$

and

$$\frac{\sigma'_{\xi\theta}}{E}=-\frac{l}{h}\frac{l}{c}\left\{\frac{D'_1}{32}\frac{c^4}{l^4}\frac{\sinh 2\xi\sin 2\theta}{(\cosh\xi+\cos\theta)^3}-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n-1)c'_n+\frac{c}{l}na'_n\right)e^{n(\xi-\xi_2)}\sin(n-1)\theta\right.$$
$$-\sum_{n=1}^{\infty}n\left((\sinh\xi+n\cosh\xi)c'_n+n\frac{c}{l}\sinh\xi a'_n\right)e^{n(\xi-\xi_2)}\sin n\theta$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}n\left((n+1)c'_n-\frac{c}{l}na'_n\right)e^{n(\xi-\xi_2)}\sin(n+1)\theta+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\sin(n-2)\theta}{\cosh\xi+\cos\theta}$$
$$-\frac{c}{l}\left(\sinh^2\xi+\frac{1}{2}\right)\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\sin n\theta}{\cosh\xi+\cos\theta}+\frac{1}{4}\frac{c}{l}\sum_{n=1}^{\infty}na'_ne^{n(\xi-\xi_2)}\frac{\sin(n+2)\theta}{\cosh\xi+\cos\theta}$$
$$+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n-m)(\sinh\xi+(n+m)\cosh\xi)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\sin(n-m)\theta$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)(n-m+1)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\sin(n-m+1)\theta$$
$$\left.+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\sum_{n,m=1}^{\infty}(n+m)(n-m-1)\alpha'_n\alpha'_me^{(n+m)(\xi-\xi_2)}\sin(n-m-1)\theta\right\},\qquad(89c)$$

19. THE MECHANICAL PROBLEM

We solve the mechanical problem for the two regions D_1 and D_2 separately to determine the constants of the harmonic functions by using the boundary conditions expressed in equations (85a, b). By use the orthogonality of trigonometric functions one get a system of linear equations in the constants of harmonic functions for D_1 and D_2 we solve them by any method to get these constants. They are some constants its value coming from applieg the additional simplifying conditions (41), (42) and (43) for two rigions.

20. THE DISPLACEMENT VECTOR COMPONENTS

From equations (37a, b) the displacement vector components in x and y directions for D_1 and D_2 are

$$\frac{1}{1+\nu}\frac{u_x}{l}=(3-4\nu)a_0-\frac{D_1}{16}\frac{c^3}{l^3}\frac{\cosh\xi-\cos\theta}{(\cosh\xi+\cos\theta)^2}\sinh\xi+4(1-\nu)\sum_{n=1}^{\infty}a_ne^{-n(\xi-\xi_1)}\cos n\theta$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\left[\frac{1}{2}e^{\xi}\cos(n-1)\theta+\cos n\theta+\frac{1}{2}e^{-\xi}\cos(n+1)\theta\right]nc_ne^{-n(\xi-\xi_1)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{x}{l}\left[\frac{1}{2}e^{\xi}\cos(n-1)\theta+\cos n\theta+\frac{1}{2}e^{-\xi}\cos(n+1)\theta\right]na_ne^{-n(\xi-\xi_1)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{y}{l}\left[\frac{1}{2}e^{\xi}\sin(n-1)\theta+\sin n\theta+\frac{1}{2}e^{-\xi}\sin(n+1)\theta\right]na_ne^{-n(\xi-\xi_1)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{x}{l}\right)\left[\frac{1}{2}e^{-\xi}\cos(n-1)\theta+\cos n\theta+\frac{1}{2}e^{\xi}\cos(n+1)\theta\right]a_ne^{-n(\xi-\xi_1)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{y}{l}\right)\left[\frac{1}{2}e^{-\xi}\sin(n-1)\theta+\sin n\theta+\frac{1}{2}e^{\xi}\sin(n+1)\theta\right]a_ne^{-n(\xi-\xi_1)}$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{-\xi}+me^{\xi}\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m+1)\theta$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{\xi}+me^{-\xi}\right)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m-1)\theta$$
$$\left.-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}(n+m)\alpha_n\alpha_me^{-(n+m)(\xi-\xi_1)}\cos(n-m)\theta+\frac{1}{1+\nu}\left(\frac{u_T}{l}+\frac{u_H}{l}\right)\right\},\qquad(90a)$$

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$$\frac{1}{1+\nu}\frac{v_y}{l} = -\frac{D_1}{16}\frac{c^3}{l^3}\frac{\cosh\xi - \cos\theta}{(\cosh\xi+\cos\theta)^2}\sin\theta - 4(1-\nu)\sum_{n=1}^{\infty} a_n e^{-n(\xi-\xi_1)}\sin n\theta$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\left[\frac{1}{2}e^{\xi}\sin(n-1)\theta + \sin n\theta + \frac{1}{2}e^{-\xi}\sin(n+1)\theta\right] nc_n e^{-n(\xi-\xi_1)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{x}{l}\left[\frac{1}{2}e^{\xi}\sin(n-1)\theta + \sin n\theta + \frac{1}{2}e^{-\xi}\sin(n+1)\theta\right] na_n e^{-n(\xi-\xi_1)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{y}{l}\left[\frac{1}{2}e^{\xi}\cos(n-1)\theta + \cos n\theta + \frac{1}{2}e^{-\xi}\cos(n+1)\theta\right] na_n e^{-n(\xi-\xi_1)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{x}{l}\right)\left[\frac{1}{2}e^{-\xi}\sin(n-1)\theta + \sin n\theta + \frac{1}{2}e^{\xi}\sin(n+1)\theta\right] a_n e^{-n(\xi-\xi_1)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\theta}\left(\frac{x}{l}\right)\left[\frac{1}{2}e^{-\xi}\cos(n-1)\theta + \cos n\theta + \frac{1}{2}e^{\xi}\cos(n+1)\theta\right] a_n e^{-n(\xi-\xi_1)}$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{-\xi} - me^{\xi}\right)\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\sin(n-m+1)\theta$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{\xi} - me^{-\xi}\right)\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\sin(n-m-1)\theta$$
$$\left.-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}(n-m)\alpha_n\alpha_m e^{-(n+m)(\xi-\xi_1)}\sin(n-m)\theta + \frac{1}{1+\nu}\left(\frac{v_T}{l}+\frac{v_H}{l}\right)\right\}, \qquad (90b)$$

and

$$\frac{1}{1+\nu}\frac{u_x^{'}}{l} = (3-4\nu)a_0^{'} - \frac{D_1^{'}}{16}\frac{c^3}{l^3}\frac{\cosh\xi-\cos\theta}{(\cosh\xi+\cos\theta)^2}\sinh\xi + 4(1-\nu)\sum_{n=1}^{\infty}a_n^{'}e^{n(\xi-\xi_2)}\cos n\theta$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\left[\frac{1}{2}e^{-\xi}\cos(n-1)\theta + \cos n\theta + \frac{1}{2}e^{\xi}\cos(n+1)\theta\right] nc_n^{'} e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{x}{l}\left[\frac{1}{2}e^{-\xi}\cos(n-1)\theta + \cos n\theta + \frac{1}{2}e^{\xi}\cos(n+1)\theta\right] na_n^{'} e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{y}{l}\left[\frac{1}{2}e^{-\xi}\sin(n-1)\theta + \sin n\theta + \frac{1}{2}e^{\xi}\sin(n+1)\theta\right] na_n^{'} e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{x}{l}\right)\left[\frac{1}{2}e^{\xi}\cos(n-1)\theta + \cos n\theta + \frac{1}{2}e^{-\xi}\cos(n+1)\theta\right] a_n^{'} e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{y}{l}\right)\left[\frac{1}{2}e^{\xi}\sin(n-1)\theta + \sin n\theta + \frac{1}{2}e^{-\xi}\sin(n+1)\theta\right] a_n^{'} e^{n(\xi-\xi_2)}$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{\xi} + me^{-\xi}\right)\alpha_n^{'}\alpha_m^{'} e^{(n+m)(\xi-\xi_2)}\cos(n-m+1)\theta$$
$$+\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{-\xi} + me^{\xi}\right)\alpha_n^{'}\alpha_m^{'} e^{(n+m)(\xi-\xi_2)}\cos(n-m-1)\theta$$
$$\left.+\frac{1}{4}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}(n+m)\alpha_n^{'}\alpha_m^{'} e^{(n+m)(\xi-\xi_2)}\cos(n-m)\theta + \frac{1}{1+\nu}\left(\frac{u_T^{'}}{l}+\frac{u_H^{'}}{l}\right)\right\}, \qquad (91a)$$

$$\frac{1}{1+\nu}\frac{v_y^{'}}{l}=-\frac{D_1^{'}}{16}\frac{c^3}{l^3}\frac{\cosh\xi-\cos\theta}{\left(\cosh\xi+\cos\theta\right)^2}\sin\theta+4\left(1-\nu\right)\sum_{n=1}^{\infty}a_n^{'}e^{n(\xi-\xi_2)}\sin n\theta$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\left[\frac{1}{2}e^{-\xi}\sin\left(n-1\right)\theta+\sin n\theta+\frac{1}{2}e^{\xi}\sin\left(n+1\right)\theta\right]nc_n^{'}e^{n(\xi-\xi_2)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{x}{l}\left[\frac{1}{2}e^{-\xi}\sin\left(n-1\right)\theta+\sin n\theta+\frac{1}{2}e^{\xi}\sin\left(n+1\right)\theta\right]na_n^{'}e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{y}{l}\left[\frac{1}{2}e^{-\xi}\cos\left(n-1\right)\theta+\cos n\theta+\frac{1}{2}e^{\xi}\cos\left(n+1\right)\theta\right]na_n^{'}e^{n(\xi-\xi_2)}$$
$$-\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{x}{l}\right)\left[\frac{1}{2}e^{\xi}\sin\left(n-1\right)\theta+\sin n\theta+\frac{1}{2}e^{-\xi}\sin\left(n+1\right)\theta\right]a_n^{'}e^{n(\xi-\xi_2)}$$
$$+\frac{l}{c}\sum_{n=1}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{y}{l}\right)\left[\frac{1}{2}e^{\xi}\cos\left(n-1\right)\theta+\cos n\theta+\frac{1}{2}e^{-\xi}\cos\left(n+1\right)\theta\right]a_n^{'}e^{n(\xi-\xi_2)}$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{\xi}+me^{-\xi}\right)\alpha_n^{'}\alpha_m^{'}e^{(n+m)(\xi-\xi_2)}\sin\left(n-m+1\right)\theta$$
$$-\frac{1}{8}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(ne^{-\xi}-me^{\xi}\right)\alpha_n^{'}\alpha_m^{'}e^{(n+m)(\xi-\xi_2)}\sin\left(n-m-1\right)\theta$$
$$-\frac{1}{4}\frac{\omega_1}{\mu_0^2}\frac{l}{c}\sum_{n,m=1}^{\infty}\left(n-m\right)\alpha_n^{'}\alpha_m^{'}e^{(n+m)(\xi-\xi_2)}\sin\left(n-m\right)\theta+\frac{1}{1+\nu}\left(\frac{v_T^{'}}{l}+\frac{v_H^{'}}{l}\right)\Bigg\}. \qquad (91b)$$

Where $\frac{u_T}{l}$, $\frac{v_T}{l}$, $\frac{u_H}{l}$, $\frac{v_H}{l}$ and $\frac{u_T^{'}}{l}$, $\frac{v_T^{'}}{l}$, $\frac{u_H^{'}}{l}$, $\frac{v_H^{'}}{l}$ are defined above.

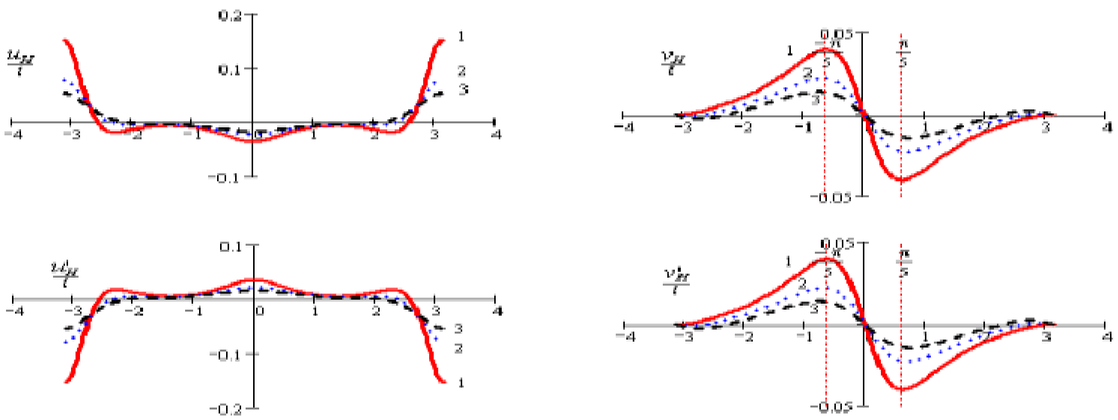
21. NUMERICAL RESULTS AND DISCUSSION

Consider the concrete case for which

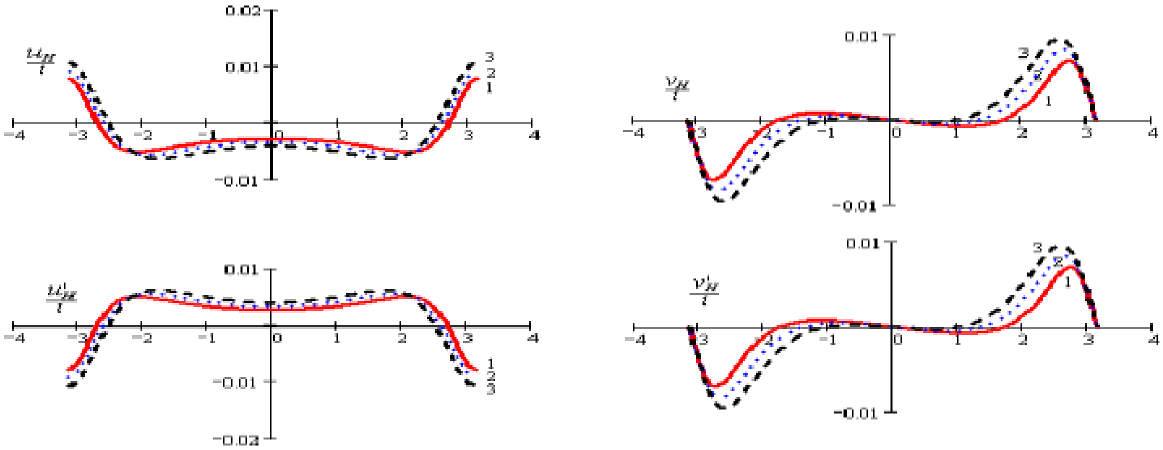
$a=b=3,\ d=11,\ B=1.5,\ \nu=0.25,\ \mu_0=0.9999,\ \mu_1=\mu_2=0.25,\ \gamma=0.25,$
$\beta_1=0.3,\ \beta_2=0.5,$

We have plotted the "magnetic" displacements $\frac{u_H}{l}$ and $\frac{v_H}{l}$ along the x- and y- directions respectively on the boundaries of the two cylinders as given by equations (75a, b) and (76a, b). In the following, we let the distance $d=8,9,10.$

Case (I): Let $J=1.$ The currents in the two similar cylinders are in the same direction and have equal intensities.



Case (II): Let $J = -1$. The currents in two similar cylinders are in opposite directions and have equal intensities.



We have displayed the magnetic displacement components $\frac{u_H}{l}$ and $\frac{v_H}{l}$ along the x - and y - directions respectively on the boundaries of the two cylinders for seven different cases, depending on the radii lengths of the two cylinders, the ratio between the two electric current densities and the relative directions of these currents. We have found that these displacements produce: a) an elongation of both cylinders along the x -direction in all cases; b) a compression of both cylinders along the y -direction when the currents have the same sense, in which as the cylinders acquire ellipse-like shapes; c) a compression of both cylinders along the y -direction on the nearer parts of their contours (|θ|≃0) and an elongation along the same axis on the farther parts of their contours (|θ|≃π) when the currents have opposite senses, in which case the cylinders acquire pear-like shapes with the narrow parts facing each other.

The bigger elongations take place at the far parts of the contours, while the relatively smaller elongations take place on the nearer parts of the contours. The larger compressions take place at the middle parts of the contours (|θ|≃π/2). When the currents are in opposite directions and the thinner cylinder is carrying the stronger current , the magnetic displacements become very small on most of the boundaries, with relatively larger values only around the far parts. All the magnetic displacements are relatively smaller when the currents have opposite senses, compared to the other case where both currents have the same sense, and become weaker as the distance between the centers increases.

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