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Asymptotic Properties of Third Order Nonlinear Difference Equations with Mixed Arguments

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Abstract

In this paper, we offer sufficient conditions for property (B) and/or the oscillation of third order nonlinear difference equation with mixed arguments of the form

$$\Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) = q_n f\left(x_{n-\ell}\right) + p_n h\left(x_{n+m}\right), \quad n \in N_0$$

where $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, α is a ratio of odd positive integer, and ℓ and m are positive integers. We deduce the properties of studied equation by establishing new comparison theorem, so that some asymptotic properties and oscillation are resulted from the oscillation of a set of first order difference equations. Both cases $\sum_{s=N_0}^{\infty} a_s^{-1/\alpha} = \infty$, and $\sum_{s=N_0}^{\infty} a_s^{-1/\alpha} < \infty$ are considered. Some examples are provided to illustrate the main results.

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1. Introduction

In this paper we are concerned with the following third order nonlinear difference equation with mixed arguments of the form

$$\Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) = q_n f\left(x_{n-\ell}\right) + p_n h\left(x_{n+m}\right), \quad n \in N_0 \tag{1.1}$$

Where $N_0 = \{n_0, n_0+1, n_0+2, ...\}, n_0$ is a nonnegative integer, ℓ and m are positive integers. $f, h : R \to R$ is continuous and nondecreasing and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, subject to the following conditions.

(H_1) α is a ratio of odd positive integers;

(H_2) $\{a_n\}$ is a positive real nondecreasing sequences;

(H_3) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences;

(H_4) $\frac{f(u)}{u^{\alpha}} > 0$, $uf(u) > 0$ and $uh(u) > 0$ for $u \neq 0$;

(H_5) $-f(-uv) \geq f(uv) \geq f(u)f(v)$ and $-h(-uv) \geq h(uv) \geq h(u)h(v)$ for $uv > 0$.

Let $\theta = \max\{\ell, m\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq n_0 - \theta$ and satisfies equation (1.1) for all $n \in N_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The oscillatory behavior of solutions of third order difference equations with or without delay have been investigated by several authors, see for example [3, 4, 5, 7, 13], and the references quoted therein.

In [6, 8, 12], the authors studied the oscillatory behavior of third order difference equation with mixed arguments.

In this paper based on the comparison results in which we compare studied equation with first order delay/ advanced difference equation yields property (B) or oscillation of equation (1.1). We say that equation (1.1) has property (B) if each of its nonoscillatory solution $\{x_n\}$ satisfies

$$\lim_{n\to\infty} x_n = \infty. \tag{1.2}$$

We will discuss both cases

$$\sum_{s=N_0}^{\infty} a_s^{-1/\alpha} < \infty \tag{1.3}$$

and

$$\sum_{s=N_0}^{\infty} a_s^{-1/\alpha} = \infty. \tag{1.4}$$

In Section 2, we obtain some important results on the nonoscillatory properties of equation (1.1) and in Section 3, we provide some examples to illustrate the main results.

2. Main Results

The following lemmas are elementary but useful to our main results.

Lemma 2.1. *Assume* $A \geq 0,\ B \geq 0,\ \gamma \geq 1$ *then*

$$\left(A+B\right)^{\gamma} \geq A^{\gamma} + B^{\gamma}. \tag{2.1}$$

Lemma 2.2. *Assume* $A \geq 0,\ B \geq 0,\ 0 < \gamma \leq 1$ *then*

$$\left(A+B\right)^{\gamma} \geq \frac{A^{\gamma}+B^{\gamma}}{2^{1-\gamma}}. \qquad (2.2)$$

The proof of Lemmas 2.3 and 2.4 can be found in [13].

Lemma 2.3. *Suppose* $\{p_n\}$ *and* $h(u)$ *satisfies* (H_3) *and* (H_4) *respectively. If the first order advanced difference inequality*

$$\Delta z_n - p_n h\left(z_{n+m}\right) \geq 0 \qquad (2.3)$$

has an eventually positive solution, then so does the advanced difference equation

$$\Delta z_n - p_n h\left(z_{n+m}\right) = 0. \qquad (2.4)$$

Proof: The proof can be found in [6].

Lemma 2.4. *Let* $\{x_n\}$ *be a nonoscillatory solution of equation (1.1). Then* $\{x_n\}$ *satisfies eventually one of the following conditions:*

(I) $x_n \Delta x_n > 0,\ x_n \Delta^2 x_n > 0$ *and* $x_n \Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) > 0;$

(II) $x_n \Delta x_n > 0,\ x_n \Delta^2 x_n < 0$ *and* $x_n \Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) > 0;$

and if (1.3) holds then also

(III) $x_n \Delta x_n < 0,\ x_n \Delta^2 x_n > 0$ *and* $x_n \Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) > 0.$

Proof: Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) and assume without loss of generality $x_n > 0$ for all $n \geq n_0$. It follows from (1.1),

$$\Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right) > 0.$$

Thus $x_n \Delta^2 x_n$ and $x_n \Delta x_n$ are fixed sign for $n \geq N_0$. We assume that $a_n \Delta^2 x_n < 0$. Then either $a_n \Delta x_n > 0$ and $a_n \Delta x_n < 0$ eventually. But $a_n \Delta^2 x_n < 0$ together with $a_n \Delta x_n < 0$ implies that $x_n < 0$, a contradiction.

Now suppose that $a_n \Delta^2 x_n > 0$, then either Case(I) or Case(II) holds.

On the otherhand, if (1.3) holds then the Case (III) implies that

$$a_n\left(\Delta^2 x_n\right)^{\alpha} \geq c > 0, \quad n \geq N. \qquad (2.5)$$

Summing from N to $n-1$ we have

$$\Delta x_n - \Delta x_N \geq c^{1/\alpha} \sum_{s=N}^{n-1} a_s^{-1/\alpha} \qquad (2.5)$$

which implies that $\lim_{n\to\infty} \Delta x_n = \infty$, and we deduce that Case(III) occur only if (1.3) is satisfied. The proof is complete.

Remark 2.1. It follows from Lemma 2.4 that if (1.4) holds, then only Case(I) and Case(II) occur.

The following results provide criteria for property (B) of equation (1.1). Define

$$P_n = \sum_{s=n}^{\infty} a_s^{-1/\alpha}\left[\sum_{j=s}^{\infty} p_j\right]^{1/\alpha}, \qquad (2.6)$$

$$Q_n = \sum_{s=n}^{\infty} a_s^{-1/\alpha} \left[\sum_{j=s}^{\infty} q_{j+\ell} \right]^{1/\alpha}, \tag{2.7}$$

and

$$E_n = \prod_{s=N_1}^{n} (1+Q_s). \tag{2.8}$$

Theorem 2.1. *Let $0<\alpha\le 1$. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). If the first order advanced difference equation*

$$\Delta w_n - P_n (E_n)^{-1} h^{1/\alpha}(E_{n+m-1}) h^{1/\alpha}(w_{n+m}) = 0 \tag{2.9}$$

Is oscillatory then Case(II) of Lemma 2.4 cannot hold.

Proof: Assume the contrary, let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) satisfying case (I) of Lemma 2.4.

We assume that $x_n > 0$ for $n \ge N$. Summing equation (1.1) from n to ∞ and using (2.9), we obtain

$$-a_n (\Delta^2 x_n)^{\alpha} \ge \sum_{s=n}^{\infty} q_s f(x_{s-\ell}) + \sum_{s=n}^{\infty} p_s h(x_{s+m}). \tag{2.10}$$

On the otherhand we substitute $s-\ell=u$, we have

$$\sum_{s=n}^{\infty} q_s f(x_{s-\ell}) = \sum_{u=n-\ell}^{\infty} q_{u+\ell} f(x_u) \ge \sum_{s=n}^{\infty} q_{s+\ell} f(x_s).$$

From (2.10) and the last inequality, we obtain

$$-\Delta^2 x_n \ge \frac{1}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} q_{s+\ell} f(x_s) + \sum_{s=n}^{\infty} p_s h(x_{s+m}) \right]^{1/\alpha}. \tag{2.11}$$

It follows from Lemma 2.1 that since $\Delta\left(a_n (\Delta x_n)^{\alpha}\right)$ is decreasing, we have

$$-\Delta^2 x_n \ge \frac{f^{1/\alpha}(x_n)}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} q_{s+\ell} \right]^{1/\alpha} + \frac{h^{1/\alpha}(x_{n+m})}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} p_s \right]^{1/\alpha}. \tag{2.12}$$

We have used and (H_3) and (H_4) summing from n to ∞ we obtain

$$\Delta x_n \ge \sum_{s=n}^{\infty} \frac{f^{1/\alpha}(x_s)}{a_s^{1/\alpha}} \left[\sum_{j=s}^{\infty} q_{j+\ell} \right]^{1/\alpha} + \sum_{s=n}^{\infty} \frac{h^{1/\alpha}(x_{s+m})}{a_s^{1/\alpha}} \left[\sum_{j=s}^{\infty} p_j \right]^{1/\alpha}$$

or

$$\Delta x_n \ge Q_n f^{1/\alpha}(x_n) + P_n h^{1/\alpha}(x_{n+m}). \tag{2.13}$$

It follows that $\{x_n\}$ is a positive solution of the difference inequality

$$\Delta x_n - Q_n x_n - P_n h^{1/\alpha}(x_{n+m}) \ge 0. \tag{2.14}$$

By setting $x_n = w_n \prod_{s=N_1}^{n-1} (1+Q_s)$, we can easily verify that $\{w_n\}$ is the positive solution of the advanced difference inequality

$$\Delta w_n - P_n\left(E_n\right)^{-1} h^{1/\alpha}\left(E_{n+m-1}\right) h^{1/\alpha}\left(w_{n+m}\right) \geq 0.$$

By Lemma 2.3, we deduce that the corresponding advanced difference equation (2.9) has a positive solution, which is a contradiction. This completes the proof.

Now we provide new criteria for property (B) of equation (1.1).

Theorem 2.2. *Let (1.4) holds and* $0 < \alpha \leq 1$ *. Assume that (2.9) is oscillatory, then equation (1.1) has property (B). Further each of its nonoscillatory solution holds the following*

$$|x_n| \geq c \sum_{s=n}^{n-1} a_s^{-1/\alpha}\left(n-s\right),\ c > 0. \tag{2.15}$$

Proof: Let $\{x_n\}$ is a positive solution of equation (1.1). It follows from Lemma 2.4 and Remark 2.1 that $\{x_n\}$ satisfy either Case(I) or Case(II) of Lemma 2.4. But by Theorem 2.1 implies that Case(II) cannot hold. Hence $\{x_n\}$ satisfies Case(I) of Lemma 2.4 which implies (1.2). Thus equation (1.1) has property (B).

On the otherhand there is a constant $c > 0$ such that

$$a_n\left(\Delta^2 x_n\right)^{\alpha} \geq c^{\alpha}. \tag{2.16}$$

Summing twice from N to $n-1$, we have

$$x_n \geq c \sum_{s=N}^{n-1} \sum_{j=N}^{s-1} a_j^{-1/\alpha} = c \sum_{s=N}^{n-1} a_s^{-1/\alpha}\left(n-s\right).$$

This completes the proof.

Corollary 2.1. Let $0 < \alpha \leq 1$ *and (1.4) hold. Assume that*

$$\frac{h^{1/\alpha}(u)}{u} \geq 1,\ |u| \geq 1, \tag{2.17}$$

and

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} P_s\left(E_s\right)^{-1} E_{s+m-1} > \left(\frac{m}{m+1}\right)^{m+1}, \tag{2.18}$$

then equation (1.1) has property (B).

Proof*:* We see that (2.18) implies

$$\sum_{s=n}^{\infty} P_s\left(E_s\right)^{-1} E_{s+m-1} = \infty. \tag{2.19}$$

By Theorem 2.1, it is sufficient to prove that equation (2.9) is oscillatory. Suppose, let (2.9) has a positive solution $\{w_n\}$. Then $\Delta w_n > 0$ and $w_{n+m} > c > 0.$

Summing (2.9) from N to $n-1$ we obtain

$$w_n \geq \sum_{s=N}^{n-1} P_s\left(E_s\right)^{-1} h^{1/\alpha}\left(E_{s+m-1}\right) h^{1/\alpha}\left(w_{s+m}\right) \geq h^{1/\alpha} c \sum_{s=N}^{n-1} P_s\left(E_s\right)^{-1} h^{1/\alpha}\left(E_{s+m-1}\right).$$

From (2.19) and the last inequality we obtain

$$\lim_{n\to\infty} w_n = \infty.$$

Using (2.17) in (2.8) we have $\{w_n\}$ is a positive solution of difference inequality

$$\Delta w_n - P_n\left(E_n\right)^{-1} E_{n+m-1} w_{n+m} \geq 0. \tag{2.20}$$

But by Corollary 2.2 in [7] and (2.18) ensure that (2.20) has no positive solution. This is a contradiction and we conclude that equation (1.1) has property (B).

Theorem 2.3. *Let* $\alpha \geq 1$. *Assume that* $\{x_n\}$ *is a nonoscillatory solution of equation (1.1). If the first order advanced difference equation*

$$\Delta w_n - 2^{(1-\alpha)/\alpha} P_n\left(2^{(1-\alpha)/\alpha} E_n\right)^{-1} h^{1/\alpha}\left(2^{(1-\alpha)/\alpha} E_{n+m-1}\right) h^{1/\alpha}\left(w_{n+m}\right) = 0 \tag{2.21}$$

is oscillatory then Case(II) of Lemma 2.4 cannot satisfied.

Proof: Suppose $\{x_n\}$ is a positive solution of equation (1.1) satisfies Case (II) of Lemma 2.4. Then from inequality (2.10) and Lemma 2.2, inview of $\{x_n\}$ and $\left(H_4\right)$ implies

$$-\Delta^2 x_n \geq \frac{f^{1/\alpha}\left(x_n\right)}{2^{(1-\alpha)/\alpha} a_n^{1/\alpha}}\left[\sum_{s=n}^{\infty} q_{s+\ell}\right]^{1/\alpha} + \frac{h^{1/\alpha}\left(x_{n+m}\right)}{2^{(1-\alpha)/\alpha} a_n^{1/\alpha}}\left[\sum_{s=n}^{\infty} p_s\right]^{1/\alpha}. \tag{2.22}$$

Summing (2.22) from n to ∞ and $\left(H_4\right)$, we obtain

$$\Delta x_n \geq 2^{(1-\alpha)/\alpha} Q_n x_n + 2^{(1-\alpha)/\alpha} P_n h^{1/\alpha}(x_{n+m}). \tag{2.23}$$

We see that $\{x_n\}$ is a positive solution of the first order difference inequality (2.23). By setting $x_n = w_n 2^{(1-\alpha)/\alpha} \prod_{s=N_1}^{n-1}(1+Q_s)$. It is easy to see that $\{w_n\}$ is the positive solution of the advanced difference inequality

$$\Delta w_n - 2^{(1-\alpha)/\alpha} P_n\left(2^{(1-\alpha)/\alpha} E_n\right)^{-1} h^{1/\alpha}\left(2^{(1-\alpha)/\alpha} E_{n+m-1}\right) h^{1/\alpha}\left(w_{n+m}\right) \geq 0.$$

By Lemma 2.2 we deduce that the corresponding difference equation (2.21) has a positive solution, which is a contradiction. Hence the proof is complete.

Theorem 2.4. *Let (1.4) hold and* $\alpha \geq 1$. *Assume that the equation (2.21) is oscillatory. Then equation (1.1) has property (B). Further each of its nonoscillatory solution satisfies (2.15).*

The proof of above theorem is obvious.

Corollary 2.2 *Let (1.4) and (2.17) hold and* $\alpha \geq 1$. *If*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} P_s\left(2^{(1-\alpha)/\alpha} E_s\right)^{-1} 2^{(1-\alpha)/\alpha} E_{s+m-1} > \left(\frac{m}{m+1}\right)^{m+1} \frac{1}{2^{(1-\alpha)/\alpha}}, \tag{2.24}$$

then equation (1.1) has property (B).

The proof is similar to that of Corollary 2.1, so it can be omitted.

It follows from Theorems 2.1 and 2.3 that, we have two criteria for the oscillation of equation (1.1).

Theorem 2.5. *Let* $\{x_n\}$ *be a nonoscillatory solution of equation (1.1). Assume that* $k > 0$ *and* $\eta_n = n - 2k + m$ *. If the first order advanced difference equation*

$$\Delta x_n - \sum_{s=n-k}^{n-1} a_s^{-1/\alpha}\left(\sum_{j=s-k}^{s-1} p_j\right) h^{1/\alpha}\left(x_{\eta(n)}\right) = 0 \tag{2.25}$$

is oscillatory then Case(I) 0f Lemma 2.4 cannot hold.

Proof: Let $\{x_n\}$ be an positive solution of equation (1.1) satisfying Case(I) of Lemma 2.4. It follows from equation (1.1) that

$$\Delta\left(a_n\left(\Delta x_n\right)^{\alpha}\right)\geq p_n h\left(x_{n+m}\right).$$

Summing above inequality from $n-k$ to $n-1$ we have

$$a_n\left(\Delta^2 x_n\right)^{\alpha}-a_{n-k}\left(\Delta^2 x_n\right)^{\alpha}\geq\sum_{s=n-k}^{n-1}p_s h\left(x_{s+m}\right)\geq h\left(x_{n-k+m}\right)\sum_{s=n-k}^{n-1}p_s$$

or

$$\Delta^2 x_n\geq h^{1/\alpha}\geq h^{1/\alpha}\left(x_{n-k+m}\right)a_n^{-1/\alpha}\left(\sum_{s=n-k}^{n-1}p_s\right)^{1/\alpha}.$$

Again summing from $n-k$ to $n-1,$ we get

$$\Delta x_n\geq\sum_{s=n-k}^{n-1}h^{1/\alpha}\left(x_{s-k+m}\right)a_s^{-1/\alpha}\left(\sum_{j=s-k}^{s-1}p_j\right)^{1/\alpha}\geq h^{1/\alpha}\left(x_{\eta(n)}\right)\sum_{s=n-k}^{n-1}a_s^{-1/\alpha}\left(\sum_{j=s-k}^{s-1}p_j\right)^{1/\alpha}.$$

Then we see that, $\{x_n\}$ is a positive solution of advanced difference inequality

$$\Delta x_n-\sum_{s=n-k}^{n-1}a_s^{-1/\alpha}\left(\sum_{j=s-k}^{s-1}p_j\right)h^{1/\alpha}\left(x_{\eta(n)}\right)\geq 0. \qquad (2.26)$$

But by Lemma 2.3, we conclude that the corresponding difference equation (2.25) has a positive solution, which is a contradiction. Hence the proof is complete.

Theorem 2.6. *Let (1.4) hold and* $0<\alpha\leq 1$. *Assume that first order advanced difference equation (2.9) is oscillatory then equation (1.1) is oscillatory.*

Proof: The proof of above theorem is obvious.

Theorem 2.8. *Let* $\{x_n\}$ *be a nonoscillatory solution of equation (1.1). Assume that (1.3) holds. If the first order advanced difference equation*

$$\Delta x_n+\left(\sum_{s=N}^{n-1}q_s\right)^{1/\alpha}\sum_{s=N}^{n-1}\left(a_s^{-1/\alpha}\right)f^{1/\alpha}\left(x_{n-\ell}\right)=0 \qquad (2.27)$$

is oscillatory then Case(III) 0f Lemma 2.4 cannot hold.

Proof: Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying case (III) of Lemma 2.4. We find that

$$-\Delta x_n\geq\sum_{s=n}^{\infty}\left(\Delta^2 x_s\right)=\sum_{s=n}^{\infty}\left(a_s^{1/\alpha}\Delta^2 x_s\right)a_s^{-1/\alpha}\geq a_n^{1/\alpha}\left(\Delta^2 x_n\right)\sum_{s=n}^{\infty}a_s^{-1/\alpha}. \qquad (2.28)$$

It follows from equation (1.1) we have

$$\Delta\left(a_n\left(\Delta^2 x_n\right)^{\alpha}\right)\geq q_n f\left(x_{n-\ell}\right).$$

Summing the last inequality from N to $n-1$ we obtain

$$a_n\left(\Delta^2 x_n\right)^{\alpha}\geq\sum_{s=N}^{n-1}q_s f\left(x_{s-\ell}\right)\geq f\left(x_{n-\ell}\right)\sum_{s=N}^{n-1}q_s.$$

Thus,

$$a_n^{1/\alpha}\left(\Delta^2 x_n\right)^{\alpha}\geq f^{1/\alpha}\left(x_{n-\ell}\right)\left(\sum_{s=N}^{n-1}q_s\right)^{1/\alpha}. \qquad (2.29)$$

Combining inequalities (2.28) and (2.29), we obtain

$$\Delta x_n + \left(\sum_{s=N}^{n-1} q_s \right)^{1/\alpha} \sum_{s=N}^{n-1} \left(a_s^{-1/\alpha} \right) f^{1/\alpha} \left(x_{n-\ell} \right) \le 0.$$

It follows from [11] that the corresponding difference equation (2.27) has a positive solution. We get a contradiction and we conclude that $\{x_n\}$ cannot satisfies Case(III) of Lemma 2.4. This completes the proof.

Theorem 2.9. *Let (1.3) hold and* $0 < \alpha \le 1$. *Assume that first order advanced difference equations (2.9) and (2.25) are oscillatory then equation (1.1) has property (B).*

Proof: It follows from Theorems 2.1 and 2.5 hence the details are omitted.

Theorem 2.10. *Let (1.3) hold and* $\alpha \ge 1$. *Assume that first order difference equations (2.21) and (2.27) are oscillatory then equation (1.1) has property (B).*

Proof: It follows from Theorems 2.3 and 2.8 hence the details are omitted.

Theorem 2.11. *Let (1.3) hold and* $0 < \alpha \le 1$. *Assume that first order difference equations (2.9) and (2.25) are oscillatory then equation (1.1) is oscillatory.*

Proof: It follows from Theorems 2.1 and 2.5 hence the details are omitted.

Theorem 2.12. *Let (1.3) hold and* $\alpha \ge 1$. *Assume that first order difference equations (2.21) and (2.27) are oscillatory then equation (1.1) is oscillatory.*

Proof: It follows from Theorems 2.3 and 2.8 hence the details are omitted.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the third order nonlinear difference equation with mixed arguments

$$\Delta\left(n\Delta^2 x_n\right) = \frac{1}{(n-2)^2} x_{n-2} + \frac{1}{(n+3)^2} x_{n+3}, \;\; n \ge 3. \tag{3.1}$$

Here $a_n = n,\ \alpha = 1,\ \ell = 2,\ m = 3,\ q_n = \dfrac{1}{(n-2)^2}$ and $p_n = \dfrac{1}{(n+3)^2}$. Then, it is easy to see that all conditions of Corollary 2.1 are satisfied and hence equation (3.1) has property (B). In fact $\{x_n\} = \left\{n^2\right\}$ is one such solution of equation (3.1) having property (B).

Example 3.2. Consider the third order nonlinear difference equation with mixed arguments

$$\Delta\left(2^n \left(\Delta^2 x_n\right)^3\right) = 2^{3n+5} x_{n-2} + 2^{3n-3} x_{n+3}, \;\; n \ge 3 \tag{3.2}$$

with $a_n = 2^n,\ \alpha = 3,\ \ell = 2,\ m = 3,\ q_n = 2^{3n+5}$ and $p_n = 2^{3n-3}$. Then, it is easy to see that all conditions of Corollary 2.2 are satisfied and hence equation (3.2) has property (B). In fact $\{x_n\} = \left\{2^n\right\}$ is one such solution of equation (3.2) having property (B).

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