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Degree of approximation of Conjugate Series of a Fourier Series by $(E,r)(N,p,q)$ Means

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Abstract: In this paper a theorem on degree of approximation of a function $f \in Lip\,\alpha$ by product Summability $(E,r)(N,p,q)$ of conjugate series of Fourier series associated with f, has been established.

Keywords: Degree of Approximation; $Lip\,\alpha$ class of function; (E,r) mean; (N,p,q) mean; $(E,r)(N,p,q)$ product mean; conjugate Fourier series; Lebesgue integral.

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1. Introduction:

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{t_n\}$ denote the sequence of (N, p, q) mean of the sequence $\{s_n\}$. Then $\{t_n\}$ is defined as follows:

(1.1) $$t_n = \frac{1}{r_n}\sum_{\upsilon=0}^{n} p_{n-\upsilon} q_\upsilon s_\upsilon ,$$

where $$r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 (\neq 0)$$

$$p_{-1} = q_{-1} = r_{-1} = 0$$

If

(1.2) $$t_n \to s \quad \text{, as} \quad n \to \infty ,$$

then the series $\sum a_n$ is said to be (N, p, q) summable to s .

The necessary and sufficient conditions for the regularity of (N, p, q) method are:

(1.3) (i) $$\frac{p_{n-\nu} q_\nu}{r_n} \to 0 \text{ , as } n \to \infty \text{ for each integer } \nu \geq 0$$

and

(1.4) (ii) $$\sum_{\nu=0}^{n} |p_{n-\nu} q_\nu| < H |r_n| ,$$

where H is a positive number independent of n .The sequence –to-sequence transformation [1],

(1.5) $$T_n = \frac{1}{(1+r)^n}\sum_{\nu=0}^{n}\binom{n}{\nu} r^{n-k} s_\nu \quad ,$$

defines the sequence $\{T_n\}$ of the (E, r) mean of the sequence $\{s_n\}$. If

(1.6) $$T_n \to s \text{ , as } n \to \infty ,$$

then the series $\sum a_n$ is said to be (E, r) summable to s .Clearly (E, r) method is regular[1].

Further, the (E, r) transform of the (N, p, q) transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+r)^n}\sum_{k=0}^{n}\binom{n}{k} r^{n-k} T_k$$

(1.7) $$= \frac{1}{(1+r)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon s_\upsilon\right\}$$

If

(1.8) $$\tau_n \to s \text{ , as } n \to \infty ,$$

then $\sum a_n$ is said to be $(E, r)(N, p, q)$-summable to s .

Let $f(t)$ be a periodic function with period 2π , L-integrable over $(-\pi,\pi)$, The Fourier series associated with f at any point x is defined by

(1.9) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)$$

and its conjugate series is

(1.10) $$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

Let $\bar{s}_n(f;x)$ be the n-th partial sum of the series given by (1.10).The L_∞-norm of a function $f: R \to R$ is defined by

(1.11) $$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and the L_υ-norm is defined by

(1.12) $$\|f\|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \upsilon \geq 1.$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\| \,.\, \|_\infty$ is defined by

(1.13) $$\|P_n - f\|_\infty = \sup\{|p_n(x) - f(x)| : x \in R\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by

(1.14) $$E_n(f) = \min_{P_n} \|P_n - f\|_\nu .$$

This method of approximation is called Trigonometric Fourier approximation.

A function $f \in Lip\,\alpha$ if

(1.15) $$|f(x+t) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1.$$

We use the following notation throughout this paper:

(1.16) $$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},$$

and $$\overline{K_n}(t) = \frac{1}{2\pi(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \left\{ \frac{1}{r_n} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\nu \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}.$$

Further, the method $(E,r)(N,p,q)$ is assumed to be regular and this case is supposed throughout the paper.

2. Known Theorems:

Dealing with the degree of approximation by the product $(E,q)(C,1)$-mean of Fourier series, Nigam et al [3] proved the following theorem.

Theorem 2.1:

If a function f is 2π - periodic and of class $Lip\ \alpha$, then its degree of approximation by $(E,q)(C,1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by $\left\|E_n^q C_n^1 - f\right\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$ $,0<\alpha<1$,where $E_n^q C_n^1$ represents the (E,q) transform of $(C,1)$ transform of $\overline{s_n}(f;x)$.

Subsequently Misra et al [2] have proved the following theorem on degree of approximation by the product mean $(E,q)(\overline{N},p_n)$ of the conjugate series (1.10) of the Fourier series (1.9).

Theorem 2.2:

If f is a $2\pi-$ Periodic function of class $Lip\ \alpha$, then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means on the conjugate series of its Fourier series (defined above) is given by $\left\|\tau_n - f\right\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$ $,0<\alpha<1$, where τ_n as defined in (1.7) .

3. Main theorem:

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,r)(N,p,q)$ of the Fourier series of a function of class $Lip\alpha$. We prove:

Theorem -3.1:

If f is a $2\pi-$ Periodic function of the class $Lip(\alpha,r)$, then degree of approximation by the product $(E,r)(N,p,q)$ summability means on its Fourier series (1.9) is given by, $\left\|\tau_n - f(x)\right\|_\infty == O\left(\frac{1}{(n+1)^\alpha}\right)$ $,0<\alpha<1$, where τ_n is as defined in (1.7).

4. Required Lemmas:

We require the following Lemma for the proof the theorem.

Lemma -4.1:

$$\left|K_n(t)\right| = O(n) \quad ,0 \le t \le \frac{1}{n+1}.$$

Proof of Lemma-4.1:

For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$ then

$$\left|K_n(t)\right| = \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\left(O(\upsilon)+O(\upsilon)\right)\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\frac{O(k)}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\right|$$

$$= O(n).$$

This proves the lemma.

Lemma-4.2:

$$\left|\overline{K}_n(t)\right| = O\left(\frac{1}{t}\right),\ for\ \ \frac{1}{n+1}\le t\le\pi .$$

Proof of Lemma-4.2:

For $\frac{1}{n+1}\le t\le\pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right)\ge\frac{t}{\pi}$, $\sin nt\le 1$.

Then

$$\left|\overline{K}_n(t)\right| = \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}\frac{\pi\ p_{k-\upsilon}q_\upsilon}{2t}\right\}\right|$$

$$= \frac{1}{4(1+r)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_\upsilon\right\}\right|.$$

$$= \frac{1}{4(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

5. Proof of Theorem 3.1:

Using Riemann –Lebesgue theorem, for the n-th partial sum $s_n(f;x)$ of the Fourier series (1.9) of $f(x)$ and following Titchmarch [4], we have

$$\overline{s_n}(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\ \overline{K_n}\ dt \quad .$$

Using (1.1), the (N,p,q) transform of $\overline{s_n}(f;x)$ is given by

$$t_n - f(x) = \frac{1}{2\pi\, r_n}\int_0^\pi \psi(t)\sum_{k=0}^{n} p_{n-k}q_k\ \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\ dt\,.$$

Denoting the $(E,r)(N,p,q)$ transform of $\overline{s_n}(f;x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{1}{2\pi(1+r)^n}\int_0^\pi \psi(t)\sum_{k=0}^{n}\binom{n}{k} r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_\upsilon\ \frac{\cos\frac{t}{2} - \cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt$$

$$= \int_0^\pi \psi(t)\ \overline{K_n}(t)\, dt$$

$$= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\psi(t)\ \overline{K_n}(t)\ dt$$

(5.1) $\qquad = I_1 + I_2\,,\ say$

Now

$$|I_1| = \frac{1}{2\pi(1+r)^n}\left|\int_0^{1/n+1}\psi(t)\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt\right|$$

$$\leq O(n)\int_0^{\frac{1}{n+1}}\left|\psi(t)\right|dt, \qquad \text{using lemma-4.1}$$

$$= O(n)\int_0^{\frac{1}{n+1}}\left|t^\alpha\right|dt \ ,$$

$$= O(n)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$= O(n)\left[\frac{1}{(\alpha+1)(n+1)}\right]$$

(5.2) $$= O\left(\frac{1}{(n+1)^{\alpha+1}}\right)$$

Next

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi}|\psi(t)|\left|\overline{K_n}(t)\right|dt$$

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi}|\psi(t)|O\left(\frac{1}{t}\right)dt, \quad \text{using lemma-4.2}$$

,

$$= \int_{\frac{1}{n+1}}^{\pi}\left|t^\alpha\right|O\left(\frac{1}{t}\right)dt$$

$$= \int_{\frac{1}{n+1}}^{\pi}t^{\alpha-1}dt$$

(5.3) $$= O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Then from (5.2) and (5.3) , we have

$$\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha}}\right), 0 < \alpha < 1$$

$$\left\|\tau_n - f(x)\right\|_{\infty} = \sup_{-\pi<x<\pi}\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha}}\right), 0 < \alpha < 1$$

This completes the proof of the theorem.

References

[1] **G.H. Hardy**: Divergent series, First edition, Oxford University press 70(19).

[2] **Misra, U.K. Misra, M. Misra, B.P. Padhy and S.K. Buxi:** On degree of approximation by product mean $(E,q)(\bar{N},p_n)$ of the conjugate series of the Fourier series**",** International Journal of Math. and Sciences and Engineering Applications, ISSN 0973 – 9424, Vol.6, No.122 (2012), pp 363-370.

[3] **H.K. Nigam and Ajay Sharma:** On degree of Approximation by product means, Ultra Scientist of Physical Sciences, Vol.22 (3) M, 889-894, (2010).

[4] **E.C. Titchmarch**: The theory of functions, oxford university press, p.p402-403(1939).

[5] **A . Zygmund** : Trigonometric Series , second Edition ,Vol.I , Cambridge University press , Cambridge , (1959).