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Skew Injective Modules Relative to Torsion Theories

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ABSTRACT

The purpose of this paper is to extend results about skew injective modules to a torsion theoretic setting. Given a hereditary torsion theory τ, a module M is called τ-skew injective if all endomorphisms of τ-dense submodules of M can be extended to endomorphismsof M. A characterization of τ-skew injectivity using split short exact sequences is given.

Indexing terms/Keywords

Torsion theory;τ-torsion module;τ-dense submodule;skew injective module.

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1 Preliminaries

All modules considered will be right unital R-modules, where R is some associative ring with a nonzero identity. By $\tau = (\mathcal{T}, \mathcal{F})$ we denote a hereditary torsion theory on the category mod-R of R-modules, where $\mathcal{T}$ (resp. $\mathcal{F}$) denotes the class of all τ-torsion (resp. τ-torsion free) R-modules.

A submodule N of a module M is said to be *τ-dense* in M (denoted $N \leq^{\tau d} M$) if M/N is τ-torsion, and M is τ-torsion if and only if all its elements are annihilated by τ-dense right ideals of R. A submodule N of M is called *τ-essential* in M (denoted $N \leq^{\tau e} M$) if N is both τ-dense and essential in M. In this case, M is called a *τ-essential extension* of N. The intersection of any finite number of τ-dense (resp. τ-essential) submodules is again τ-dense (resp. τ-essential). If N and K are submodules of a module M such that $N \leq^{\tau e} M$ then $N \cap K \leq^{\tau e} K$. Any submodule that contains a τ-dense (resp. τ-essential) submodule is itself τ-dense (resp. τ-essential). An R-module is called *τ-injective* if it is injective with respect to every short exact sequence having a τ-torsion cokernel. Every R-module M admits a *τ-injective envelope* $E = E_\tau(M)$, i.e. a τ-injective R-module E containing M as a τ-essential submodule. A module M is called *τ-quasi injective* if homomorphisms from τ-dense submodules of M into M are extendable to endomorphisms of M. For preliminaries about torsion theories, we refer to [2].

Charalambides [3] introduced the concept of *τ-essentially closed* submodules. A submodule N of a module M is called τ-essentially closed in M (denoted $N \leq^{\tau c} M$) if N has no proper τ-essential extensions in M.

A module M is called *skew injective* [4] if whenever N is a submodule of M, any f in $\mathrm{End}(N)$ can be extended to $g \in \mathrm{End}(M)$. Note that in [5] skew injective modules are called semiinjective. In this paper, we generalize this concept to torsion theoretic setting.

2 τ-Skew Injective Modules

Definition. A module M is called *τ-skew injective* if whenever N is a τ-dense submodule of M, any f in $\mathrm{End}(N)$ can be extended to $g \in \mathrm{End}(M)$.

Remarks.

a) Every skew injective module is τ-skew injective.
b) Every τ-quasi injective module (and hence every τ-injective) module is τ-skew injective.
c) If M is a τ-torsion τ-skew injective module, then it is skew injective.
d) If τ is the torsion theory in which every R-module is τ-torsion, then a module is τ-skew injective if and only if it is skew injective.

Proposition 1: A module M is τ-skew injective if and only if for every τ-essential submodule N of M, any endomorphism of N can be extended to an endomorphism of M.

*Proof***.** Let N be a τ-dense submodule of M and $f \in \mathrm{End}(N)$. Let N' be a relative complement of N in M. Then $N \oplus N'$ is a τ-essential submodule of M. Moreover, f can be extended to an R-endomorphism g of $N \oplus N'$ by putting $g(N') = 0$. By the given condition, there is an R-homomorphism h of M which extends g hence f. The other direction is trivial. □

Given a submodule M of a module E and an endomorphism f of E, we call f an *M-τ-essential* endomorphism if $f(N) \subseteq N$ for some τ-essential submodule N of M.

Theorem 2: If E is the τ-injective envelope of a module M, then the following statements are equivalent:

(a) M is τ-skew injective and $f(M) \subseteq M$ for any endomorphism f of E having a τ-essential kernel.
(b) $g(M) \subseteq M$ for any M-τ-essential endomorphism g of E.

*Proof***.** (a) ⇒ (b) Let g be an M-τ-essential endomorphism of E. Then there is a τ-essential submodule N of M such that $g(N) \subseteq N$. By τ-skew injectivity of M, there exists $h \in \mathrm{End}(M)$ that extends g. Again τ-injectivity of E gives existence of a k in $\mathrm{End}(E)$ such that $k|_M = h$. So $(g-k)(N) = 0$. Hence $N \subseteq \ker(g-k)$. So $\ker(g-k) \leq^{\tau e} E$. Then by hypothesis $(g-k)(M) \subseteq M$. Therefore, for any x in M we have $(g-k)(x) = m \in M$, hence $g(x) = m + k(x) \in M$, i.e. $g(M) \subseteq M$.

(b) ⇒ (a) Since any endomorphism of E having a τ-essential kernel is necessarily an M-τ-essential homomorphism, we need only prove that M is τ-skew injective. By Proposition 1, let N be a τ-essential submodule of M and $f \in \mathrm{End}(N)$. By τ-injectivity of E we have a $g \in \mathrm{End}(E)$ such that $g(N) = f(N) \subseteq N$. Hence, g is an M-τ-essential homomorphism, so $g(M) \subseteq M$. Then, $g|_M \in \mathrm{End}(M)$ is an extension of f. □

The following Theorem generalizes Lemma 6 of [5].

Theorem 3: Let M be a τ-skew injective module, E a τ-essential extension of M and f an M-τ-essential endomorphism of E. If for each $x \in M$ there exists a positive integer n such that $f^{n+1}(x) = f^n(x)$, then $f(M) \subseteq M$.

*Proof***.** Let N be the sum of all τ-dense submodules N' of M such that $f(N') \subseteq N'$. Therefore $f(N) \subseteq N$ and by hypothesis, N is a τ-essential submodule of M. We see that $f^n(N) \subseteq N$ for all $n \geq 1$, hence by τ-skew injectivity of M, there exist endomorphisms $g_1, g_2, \ldots$ of M such that $(f^n - g_n)(N) = 0$ for all $n \geq 1$. So we have well-defined homomorphisms h_n from M/N into E: $h_n(m+N) = (f^n - g_n)(m)$ for all $m \in M$, $n \geq 1$. Let $\bar{A}_n = h_n^{-1}(N \cap \mathrm{Im}\, h_n)$ for all n. Hence $\bar{A}_n$ are τ-essential submodules of $\bar{M} = M/N$, for $\bar{A}_n$ is the inverse image under the homomorphism h_n of the essential submodule $N \cap \mathrm{Im}\, h_n$ of

$\mathrm{Im}\, h_n$, this gives essentiality of $\bar{A}_n$ in $\bar{M}$. Moreover,$\bar{M} = M/N$ is a τ-torsion module. This means that $\bar{A}_n$ is τ-dense in$\bar{M}$ for all n. If $\bar{M} = \bar{0}$, then $f(M) \subseteq M$ and everything is proved. Assume $b \in M \setminus N$ so that $\overline{N + bR}$ is a non-zero submodule of $\bar{M}$ and choose a natural number n such that $f^{n+1}(b) = f^n(b)$. Put $\bar{A} = \bar{A}_1 \cap \cdots \cap \bar{A}_n$, hence $\bar{A}$ is a τ-essential submodule of M since $\bar{A}$ is the intersection of a finite number of τ-essential submodules of $\bar{M}$. Now $\bar{A} \cap \overline{N + bR} \neq 0$. Therefore, there exists an element $r \in R$ such that $br \in (N + bR) \setminus N$and $br + N \in \bar{A}_1 \cap \cdots \cap \bar{A}_n$. It follows from the definition of the modules $\bar{A}_i$ that $h_i(br) \in M$for $i = 1, \cdots, n$. If $b_1 = br$, then $g_m(b_1) \in M$ for all m. From the definition of the homomorphisms$h_i$we see that $f^i(b_1) \in M$for $i = 1, \cdots, n$. But then $f^m(b_1) \in M$ for all m, since $f^{m+1}(b_1) = \left(f^{m+1}(b)\right)r = f^m(b_1)$. Now put $N_1 = N + \sum_{i=0}^{\infty} f^i(b_1 R)$. Hence N_1 is a τ-densesubmodule of M with$f(N_1) \subseteq N_1$and$N_1 \nsubseteq N$, in contradiction with the choice of the module N. □

Charalambides [3] defines a module M to be τ-*quasi continuous* if it is invariant under idempotentsof $\mathrm{End}(E_\tau(M))$. From the above Theorem,we see that if f is an idempotent in $\mathrm{End}(E_\tau(M))$ and M is τ-skew injective, then $f^2(x) = f(x)$ for all $x \in M$, hence $f(M) \subseteq M$. So we have the following corollary.

Corollary 1: Every τ-skew injective module is τ-quasi continuous. □

In [3], a module M is defined to be τ-*CS* if every (τ-essentially) closed τ-dense submodule of M is a direct summand. There, it is proved that τ-quasi continuous modules are τ-CS. Hence we have:

Corollary 2: Any τ-skew injective module is τ-CS. □

τ-skew injectivity is preserved by taking direct summands:

Proposition 4: A direct summand of a τ-skew injective module is τ-skew injective.

Proof. Let M be τ-skew injective such that $M = N \oplus N'$. Let K be a τ-dense submodule of N. Then $N/K \cong (N \oplus N')/(K \oplus N')$ is τ-torsion, which means that $K \oplus N'$ is τ-dense in M. Any homomorphism $f: K \to K$ can be extended to a homomorphism $f': K \oplus N' \to K \oplus N'$ by putting $f'(k + n') = f(k)$ for all $k + n' \in K \oplus N'$. Now τ-skew injectivity of M gives a homomorphism$g \in \mathrm{End}(M)$ that extends f'. Hence $h = pgi_N$extends f, where p is the projection map of M onto N. □

We end this section with a characterization of τ-skew injective modules using split short exact sequences:

Theorem 5: For a module A, the following statements are equivalent:

(1) Aisτ-skew injective.

(2) Any short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ splits whenever there exists $\beta \in \mathrm{Hom}\,(A, B)$ such that:
  (a) $\alpha(A) + \beta(A) = B$,
  (b) $\alpha(A) \cap \beta(A) \subseteq \alpha\left(\beta^{-1}(\alpha(A))\right)$ and
  (c) $\beta^{-1}(\alpha(A)) \leq^{\tau-d} A$.

*Proof***.** (1) ⇒ (2) Let $N = \beta^{-1}(\alpha(A))$. By hypothesis, $\alpha(A) \cap \beta(A) = \beta\left(\beta^{-1}(\alpha(A))\right) = \beta(N) \subseteq N$. Hence by τ-skew injectivityof A, the homomorphism $\beta: N \to A$ extends to a homomorphism $\gamma: A \to A$. By assumption, $B = \alpha(A) + \beta(A)$, i.e. for each $b \in B$ there exist a, a' in A such that $b = \alpha(a) + \beta(a')$. Define $\delta: B \to A$by $\delta(b) = \alpha(a) + \gamma(a')$. It is an easy matter to verify that δ is an R-homomorphism. Moreover, for $a \in A$, $\delta(\alpha(a)) = \delta(\alpha(a) + \beta(0)) = \alpha(a) + \gamma(0) = \alpha(a)$ so that $\alpha(A)$ is a direct summand of B.

(2) ⇒ (1) Let N be a τ-dense submodule of A and $g \in \mathrm{End}(N)$. Form the pushout diagram:

$$\begin{array}{ccc} N & \xrightarrow{i} & A \\ g\downarrow & & \downarrow\beta \\ 0 \to A & \xrightarrow{\alpha} & B \end{array}$$

where $B = (A \oplus A)/W$, with $W = \{(n, -g(n)), n \in N\}$. If the second row splits then we get a homomorphism $\alpha': B \to A$ such that $\alpha'\alpha = 1_A$. Hence $\alpha'\beta i$ is an extension of g. So we need to show that conditions in (2) hold. But it is easy to see that conditions (a) and (b) hold. Moreover, $\beta^{-1}(\alpha(A)) = N$ which is τ-dense in A. Hence the lower sequence splits by (2). □

3 Direct sum of τ-skew injective modules

In Theorem 2 of the last section, we proved that a module M is (a) τ-skew injective and (b) $f(M) \subseteq M$ for any $f \in \mathrm{End}(\mathrm{E}_\tau(M))$ having a τ-essential kernel if and only if $g(M) \subseteq M$ for any M-τ-essential endomorphism $g$of E. In this section, we seek conditions on the module M and/or the ring R so that condition (b) above is already satisfied.

Let us see first what happens if we take direct sum or summands of modules satisfying condition (b):

Proposition 6.A module M satisfies condition (b) if and only if any direct summand of M satisfies condition (b).

*Proof***.** Let $M = M_1 \oplus M_2$ so that $\mathrm{E}_\tau(M) = \mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$. Let f be an endomorphism of $\mathrm{E}_\tau(M_1)$ having a τ-essential kernel. Now f can be easily extended to an endomorphism of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$ by $(x, y) \mapsto (f(x), 0)$ whose kernel is now equal to $\ker f \oplus \mathrm{E}_\tau(M_2)$ which is clearly a τ-essential submodule of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$. By assumption, the image of Munder this map is contained in M. So $f(M_1) \subseteq M_1$. Conversely, Let f be an endomorphism of $\mathrm{E}_\tau(M)$ having a τ-essential kernel.

Now for all i, $M_i \cap \ker f$ is τ-essential in M_i. But $M_i \cap \ker f$ is the kernel of $p_i \circ (f|M_i)$, where p_i is the projection map of M onto M_i. Hence $p_i \circ (f|_{M_i})(M_i) \subseteq M_i$ for all i. So $f(M) \subseteq M$. □

Recall that a module M is called *τ-nonsingular* [1] if $Z_\tau(M) = 0$where $Z_\tau(M) = \{m \in M | \operatorname{ann}_R(m) \leq^{\tau e} R\}$. The following proposition shows that if we assume τ-nonsingularity of the module M, then we can remove condition (b) above from Theorem 2:

Proposition 7. Let M be a τ-nonsingular module. Then M is τ-skew injective if and only if M is invariant over endomorphisms $g \in \operatorname{End}(\mathrm{E}_\tau(M))$ having M-τ-essential kernels.

Proof. We will show that the only endomorphism of M that has a τ-essential kernel is the zero homomorphism. But this implies that M is invariant under such homomorphisms and hence by Theorem 2, the result follows. To this end, let $f \in \operatorname{End}(M)$with $\ker f \leq^{\tau - e} M$, and let $g = 1_M - f \in \operatorname{End}(M)$. We will show that $g = 1_M$ and hence $f = 0$. For each $x \in M$, there exists a non-zero element $r \in R$ such that $0 \neq xr \in \ker f$, so $f(xr) = 0$ hence $g(x)r = g(xr) = xr - f(xr) = xr$, then $(g(x) - x)r = 0$ and $\operatorname{ann}_R(g(x) - x)$ is a non-zero ideal of R. But $(\ker f : x) \leq^{\tau e} R$ and hence $\operatorname{ann}_R(g(x) - x) \leq^{\tau e} R$, i.e. $g(x) - x \in Z_\tau(M) = 0$ and therefore$f(x) = 0$. □

In the next result, if the τ-injective envelope of M satisfies some ascending chain condition, then we can get rid of condition (b).

Proposition 8. Let M be a τ-skew injective module. If $E = \mathrm{E}_\tau(M)$ satisfies the ascending chain condition on τ-essential submodules, then $f(M) \subseteq M$ for every M-τ-essential $f \in \operatorname{End}(E)$.

Proof. Consider the ascending chain $M \cap \ker f \subseteq M \cap \ker f^2 \subseteq \cdots \subseteq M$. It is clear that $M \cap \ker f^k \leq^{\tau e} M$ for each $k \geq 1$, so by assumption there is a positive integer $n_\circ$ such that $M \cap \ker f^n = M \cap \ker f^{n+1}$ for all $n \geq n_\circ$. We claim that $\operatorname{Im}(f^n) \cap \ker fn \cap M=0$. To see this, let $x \in \operatorname{Im} fn \cap \ker fn \cap M$. So there is $y \in E$ such that $x=fny$and $0=fnx=fnfny=f2ny$. Hence $y \in \ker f^{2n} = \ker f^n$. So $x = f^n(y) = 0$. But $\ker f^n \leq^{\tau - e} E$ implies that $\operatorname{Im} f^n = 0$. Now for $x \in M$, $x - f(x) \in E$ and $0 = f^n\big(x - f(x)\big) = f^n(x) - f^{n+1}(x)$ or $f^n(x) = f^{n+1}(x)$ for $n \geq n_\circ$. So by Theorem 3, we have $f(M) \subseteq M$. □

Now, combining the above propositions with Theorem 2, we get:

Corollary. If M is a τ-nonsingular module or $\mathrm{E}_\tau(M)$ satisfies the ascending chain condition on τ-essential submodules, then M is τ-skew injectiveif and only if it is invariant under all M-τ-essential endomorphisms of $\mathrm{E}_\tau(M)$. □

Now, we put conditions on the ring R to help us remove condition (b). For this we give a concept that generalizes both noetherian and weakly noetherian modules in [5].

Definition. A module M is said to be *τ-weakly noetherian* if for every ascending chain $L_1 \subseteq L_2 \subseteq \cdots$ of submodules of M with $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for all i, there is a positive integer k such that $L_{n+1} = L_n$ for all $n \geq k$. A ring R is called τ-weakly noetherian if it is τ-weakly noetherian as an R-module.

Remarks.

(1) Every module with ascending chain condition on τ-essential submodules is τ-weakly noetherian.
(2) If M is τ-weakly noetherian then so is any homomorohic image of M.
(3) Every cyclic module over a τ-weakly noetherian ring is τ-weakly noetherian.

Proof.(1) Let $L_1 \subseteq L_2 \subseteq \cdots$ be an ascending chain of submodules of a τ-weakly noetherian module M with $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for all i. For each i, under the natural map $M \to M/L_i$, we have L_{i+1} is the preimage of L_{i+1}/L_i. So it must be essential in in M. Moreover, $M/L_{i+1} \cong (M/L_i)/(L_{i+1}/L_i)$ is τ-torsion, hence $L_{i+1} \leq^{\tau e} M$. Thus the ascending chain $L_2 \subseteq L_3 \subseteq \cdots$ (and hence the ascending chain $L_1 \subseteq L_2 \subseteq \cdots$) terminates.

(2) Let N be a submodule of M. We want to show that M/N is τ-weakly noetherian. Let $L_1/N \subseteq L_2/N \subseteq \cdots$ be an ascending chain of submodules of M/N such that $(L_{i+1}/N)/(L_i/N) \leq^{\tau e} (M/N)/(L_i/N)$ for each i. Hence $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for each i. Now for every i, $(M/L_i)/(L_{i+1}/L_i) \cong \frac{(M/N)/(L_i/N)}{(L_{i+1}/N)/(L_i/N)}$, so that $L_{i+1}/L_i \leq^{\tau e} M/L_i$. So by assumption there is a positive integer k such that $L_{n+1} = L_n$ for all $n \geq k$or $L_{n+1}/N = L_n/N$.

(3) Let M be a cyclic module over a τ-weakly noetherian ring R. This means that $M \cong R/\operatorname{ann}_R(m)$ for some $m \in M$. By (2) it follows that M is τ-weakly noetherian. □

Theorem 9. Let M be a module over a τ-weakly noetherian ring, then for each endomorphism f of $\operatorname{End}(\mathrm{E}_\tau(M))$ that has a τ-essential kernel, there is a positive integer n such that $f^n(x) = 0$ for every $x \in \mathrm{E}_\tau(M)$.

Proof. Put $E = \mathrm{E}_\tau(M)$ and let $K_0 = 0$, $K_1 = \ker f$, … , $K_{n+1} = f^{-1}\big(K_n \cap f(E)\big)$. Hence $K_1 \subseteq K_2 \subseteq \cdots$ is an ascending chain of submodules of E. Now $\ker f = K_1 \leq^{\tau d} E$. This implies that $K_n \leq^{\tau d} E$ for each n and hence $E/K_{n+1} \cong (E/K_n)/(K_{n+1}/K_n)$ which is τ-torsion, gives that $K_{n+1}/K_n \leq^{\tau d} E/K_n$. So $K_{n+1}/K_n \leq^{\tau e} E/K_n$ for each n since $K_{n+1}/K_n \leq^{e} E/K_n$ for each n. For each $x \in E$, let $A = xR$ which by remark (3) is τ-weakly noetherian. Put $A_0 = 0$, $A_1 = A \cap \ker f$, … , $A_n = A \cap K_n$, … which gives an ascending chain $A_0 \subseteq A_1 \subseteq \cdots$ of submodules of A. Since each K_n is τ-essentialin E, $A_n = A \cap K_n$ is τ-essential in A [3], and hence $A_{n+1}/A_n \leq^{\tau d} A/A_n$ for all n. But A is τ-weakly noetherian. So there is a positive integer k such that $A_{n+1} = A_n$for $n \geq k$. But A_{n+1}/A_n is an essential submodule of E/A_n for all $n \geq k$. This is equivalent to saying that $A_n = A$

for all $n \geq k$. Hence $A = A_n = A \cap K_n$. So $A \subseteq K_n$, but $x \in A$ which implies that $f(x) \in K_{n-1} = f^{-1}(K_{n-2} \cap f(E))$. Thus $f^2(x) \in K_{n-2}$ and so on, we have $f^n(x) = 0$. □

Now we can remove condition (b) provided R is τ-weakly noetherian:

Theorem 10. Let M be a module over a τ-weakly noetherian ring. Then M is τ-skew injective if and only if it is invariant under M-τ-essential endomorophisms of $\mathrm{E}_\tau(M)$.

Proof. By Theorem 2 it is enough to show that M is invariant under all endomorphisms of $E = \mathrm{E}_\tau(M)$ that have τ-essential kernels. Let f be such an endomorphism, thus by Theorem 9, there is a positive integer n such that $f^n(x) = 0$ for all $x \in E$. In particular, for every $m \in M$ we have $f^n\big(m - f(m)\big) = 0$. Thus $f^{n+1}(m) = f^n(m)$. Using Theorem 3, we have $f(M) \subseteq M$. □

So far, examples of modules satisfying condition (b) are:

1. τ-nonsingular modules,
2. Modules whose τ-injective envelopes satisfy the ascending chain condition on τ-essential submodules, and
3. Modules over τ-weakly noetherian rings.

Now we are ready to study direct sums of τ-skew injective modules. Here we give necessary and sufficient conditions for a direct sum of τ-skew injective modules to be τ-skew injective.

Theorem 11. Let $M = M_1 \oplus \cdots \oplus M_n$ be an R-module satisfying condition (b). Then M is τ-skew injectiveif and only if $K_{ij} M_i \subseteq M_j$ for each $i, j = 1,2, \dots, n$,

where $K_{ij} = \{f \in \mathrm{Hom}_R\big(\mathrm{E}_\tau(M_i), \mathrm{E}_\tau(M_j)\big) \mid f(N_i) \subseteq N_j$ for some τ-essential submodules N_i of M_i and N_j of $M_j\}$.

Proof. Put $E = \mathrm{E}_\tau(M)$and $E_i = \mathrm{E}_\tau(M_i)$, $i = 1,2, \dots, n$. Then $E = E_1 \oplus \cdots \oplus E_n$. Suppose that M is τ-skew injective and let $f_{ij} \in K_{ij}$, i.e $f_{ij}: E_i \to E_j$ is a map with $f_{ij}(N_i) \subseteq N_j$ for some τ-essential submodules N_i of M_i and N_j of M_j. Consider the direct sum $N = \oplus N_k'$, where $N_k' = N_k$ if $k = i$ or $k = j$ and otherwise $N_k' = E_k$. Now N is clearly τ-essential in E and hence $N \cap M$ is τ-essential in M. But f_{ij} can easily be extended to a map $f: E \to E_j$, $(x_1, \dots, x_n) \mapsto f_{ij}(x_i)$. So that $f(N) \subseteq N_j$. Hence $f(N \cap M) \subseteq N_j = N_j \cap M \subseteq N \cap M$. This means that f is an M-τ-essential endomorphism of E. Hence $f(M) \subseteq M$ since M is τ-skew injective satisfying condition (b). So $f_{ij}(M_i) \subseteq M_j$. Conversely, let f be an M-τ-essential endomorphism of E, so that $f(N) \subseteq N$ for some τ-essential submodule N of M. Now for each $i$and j, we have $N \cap M_i \leq^{\tau-e} M_i$ and $p_j(N \cap M_i) \subseteq N \cap M_j$, where p_j is the projection of M onto M_j. So if we compose p_j with the restriction of f on M_i we get a map $f_{ij} \in K_{ij}$, i.e. $f_{ij}(M_i) \subseteq M_j$ for all i and j. Now it easy to see that $f(M) \subseteq M$. □

Corollary. If M is a τ-skew injective$R$-module satisfying condition (b) then M^n is also τ-skew injective. □

Proposition 12. Let $M = M_1 \oplus M_2$ be a τ-skew injective$R$-module satisfying condition (b). Then $\mathrm{E}_\tau(M_1) \cong \mathrm{E}_\tau(M_2)$ if and only if $M_1 \cong M_2$.

Proof. Let $f: \mathrm{E}_\tau(M_1) \to \mathrm{E}_\tau(M_2)$ be an isomorphism. Then f extends to an endomorphism F of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$by $(x, y) \mapsto \big(0, f(x)\big)$. If we prove that F is M-τ-essential then, by Theorem 11 we must have $F(M_1) \subseteq M_2$. Now since M_2 is essential in $\mathrm{E}_\tau(M_2)$ we have $f^{-1}(M_2)$ is essential in $\mathrm{E}_\tau(M_1)$, and since M_1 is τ-essential in $\mathrm{E}_\tau(M_1)$ we must have $M_1 \cap f^{-1}(M_2)$ is τ-essential in M_1 and hence $\big(M_1 \cap f^{-1}(M_2)\big) \oplus M_2$ is τ-essential in $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$. Now $F\Big(\big(M_1 \cap f^{-1}(M_2)\big) \oplus M_2\Big) = f\big(M_1 \cap f^{-1}(M_2)\big) \subseteq f(M_1) \cap M_2 \subseteq M_2 \subseteq \big(M_1 \cap f^{-1}(M_2)\big) \oplus M_2$. So F is M-τ-essential and $F(M_1) \subseteq M_2$. Similarly we get an M-τ-essential endomorphism G of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$ so that $G(M_2) \subseteq M_1$. Now for every $m_1 \in M_1$ we have $G \circ F(m_1) = G\big(F(m_1)\big) = G\big(f(m_1)\big) = m_1$. So $G \circ F = 1_{M_1}$. And similarly we have $F \circ G = 1_{M_2}$. The other direction is obvious. □

Corollary. Let M be a τ-skew injective module satisfying condition (b) and $E = \mathrm{E}_\tau(M)$. Then $M \oplus E$ is τ-skew injective if and only if $M = E$.

*Proof.*It is obvious that E satisfies condition (b), and by Proposition 6 so does $M \oplus E$. So we can apply Theorem 11 on $M \oplus E$. Clearly$1_{E \oplus E}$ is an $M \oplus M$-τ-essential endomorphism of $E \oplus E$. So if $M \oplus E$ is τ-skew injective then $1_E(E) \subseteq M$ by Theorem 11. But this means that $M = E$. The other direction is trivial. □

Proposition 13. The following statements are equivalent for any ring R:

(1) The direct sum of any two τ-skew injective R-modules satisfying condition (b) is τ-skew injective.
(2) Every τ-skew injective$R$-module satisfying condition (b) is τ-injective.

Proof.(1) ⇒ (2) Let M be τ-skew injective satisfying condition (b). Then $M \oplus E$ is τ-skew injective by 1. So by the above corollary, $M = E$. (2) ⇒ (1) is trivial. □

Corollary. The following statements are equivalent for a τ-weakly noetherian ring:

(1) The direct sum of any two τ-skew injective R-modules is τ-skew injective.
(2) Every τ-skew injective R-module is τ-injective. □

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