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On the solution of Fredholm-Volterra integral equation with discontinuous kernel in time

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ABSTRACT

The existence of a unique solution of Fredholm-Volterra integral equation (F-VIE) under certain conditions, are discussed and proved. The Fredholm integral term (FIT) is considered in position with continuous kernel, while the Volterra integral term (VIT) in time with singular kernel. Using a numerical method, the F-VIE is transformed to a linear system of Volterra integral equations (LSVIEs). Then after using Toeplitz matrix method (TMM), we have a linear algebraic system (LAS). Finally, two applications are given, numerical results are obtained, and the error, in each case, is calculated .

Keywords

Fredholm–Volterra integral equation (F–VIE); Linear system of Volterra integral equations (LSVIEs); Toeplitz matrix method (TMM); Linear algebraic system (LAS) .

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1. INTRODUCTION

Singular integral equations arise in many problems of mathematical models of physical phenomena, specifically in various kinds of mixed boundary value problem of mathematical physics and engineering problems see [1,2,3]. Since closed form solutions to these integral equations are generally not available, much attention has been focused on numerical methods. In the work of Abdou and Nasr [4], the solution of Fredholm – Volterra integral equation is discussed analytically and numerically, when the Fredholm integral term is considered in position with discontinuous kernel in $L_2(\Omega)$, where Ω is the domain of integration. While the Volterra integral term is considered in time t , $t \in [0,T]$, $T < \infty$, with continuous kernel. In Linz [5], the solution of VIE was obtained analytically and numerically when the kernel has a singularity. Abdou, in [6], discussed numerically the solution of F-VIE in the Banach space $L_2[-1,1] \times C[0,T]$, $T < \infty$, when the kernel of FIT has a singularity. Also, in the work of Abdou and Raad [7], the solution of F-VIE was obtained in the Banach space $L_2(\Omega) \times C[0,T]$, $T < \infty$, when the FIT has a logarithmic kernel.

Consider the linear integral equation

$$\mu\, \phi(x,t) = f(x,t) + \lambda \int_a^b F(x,y)\, \phi(y,t)dy + \lambda \int_0^t K(|t-\tau|)\, \phi(x,\tau)d\tau. \tag{1.1}$$

This formula is called Fredholm-Volterra integral equation (F-VIE), and it contains all previous linear integral terms in the Banach space $C([a,b] \times [0,T]), T < \infty$, where $[a,b]$ is the domain of integration with respect to position and the time $t \in [0,T]$. Here, in equation (1.1) the Fredholm integral term is considered in position with continuous kernel $F(x,y)$, while the Volterra integral term in time and it's kernel $K(|t-\tau|)$ is singular, and$0 \le \tau \le t \le T < \infty$. The coefficient λ is a constant, may be complex, and has a physical meaning in applied science, while the constant μ defines the kind of the integral equation. The given function $f(x,t)$is called the free term and $\phi(x,t)$ is the unknown function.

2. Existence of a unique solution of F-VIE

In this section, the existence of a unique solution of equation (1.1) will be discussed and proved by virtue of Banach fixed point theorem which can be applied to IEs of the first and second kinds. In this aim, we write equation (1.1) in the integral operator form :

$$\overline{W}\phi(x,t) = \frac{1}{\mu} f(x,t) + W\phi(x,t) \quad ; \quad \mu \neq 0 \ , \tag{2.1}$$

where

$$W\phi(x,t) = \frac{\lambda}{\mu}\int_a^b F(x,y)\phi(y,t)dy + \frac{\lambda}{\mu}\int_0^t K(|t-\tau|)\phi(x,\tau)d\tau. \tag{2.2}$$

Also , we assume the following conditions :

(i) The kernel of position $F(x,y) \in C([a,b] \times [a,b])$ and satisfies

$$|F(x,y)| \le c^* \quad , \quad (c^* \text{ is a constant}).$$

(ii) The kernel of time $K(|t-\tau|)$ is a discontinuous function and satisfies :

(a) For each continuous function $\phi(x,t)$and $0 \le t_1 \le t_2 \le t$, the integrals

$$\int_{t_1}^{t_2} K(|t-\tau|)\phi(x,\tau)d\tau \text{ , and } \int_0^t K(|t-\tau|)\phi(x,\tau)d\tau \text{ ,}$$

are continuous in$([a,b] \times [0,T])$.

(b) $K(|t-\tau|)$ is absolutely integrable with respect to τ for all $0 \le t \le T, i.e.$

$$\int_0^t |K(|t-\tau|)|d\tau \le M \quad , \quad (M \text{ is a constant}).$$

(iii) The given function $f(x,t)$ with its partial derivatives with respect to position and time belong to $C\ ([a,b] \times [0,T])$ and its norm is defined by :

$$\|f(x,t)\|_{C\ ([a,b]\times[0,T])} = \max_{x,t} |f(x,t)| \le L\ .$$

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Theorem 1 :

Equation (1.1) has a unique solution in the space $C([a,b]\times[0,T])$, under the condition

$$|\,\mu\,| > |\,\lambda\,|(c^{*}B + M)\,. \tag{2.3}$$

To prove this theorem , we must consider the following lemmas :

Lemma 1 :

The integral operator $\overline{W}$ maps the space$C([a,b]\times[0,T])$ into itself .

Proof :

From equations (2.1) and (2.2), we have

$$\left|\overline{W}\phi(x,t)\right| \le \frac{1}{|\mu|}|f\,(x,t)| + \left|\frac{\lambda}{\mu}\right|\int_a^b |F\,(x,y)||\,\phi\,(y,t)|\,dy + \left|\frac{\lambda}{\mu}\right|\int_0^t |K(|t-\tau|)||\,\phi\,(x,\tau)|d\tau.$$

In view of the conditions (i) - (iii) , we obtain

$$\left\|\overline{W}\phi(x,t)\right\| \le \frac{L}{|\mu|} + \sigma\|\phi\,(x,t)\|\,,\left(\sigma = \left|\frac{\lambda}{\mu}\right|(c^{*}B + M)\ ,\ \ B = (b-a)\right). \tag{2.4}$$

The previous inequality (2.4) shows that , the operator$\overline{W}$ maps the ball $S_\rho$into itself, where

$$\rho = \frac{L}{[|\mu| - |\lambda|(c^{*}B + M)]}\quad . \tag{2.5}$$

Since $\rho > 0$ and $L > 0$, therefore we have $\sigma < 1$. Also, the inequality (2.4) involves the boundedness of the operator W, where

$$\|W\phi(x,t)\| \le \sigma\,\|\,\phi\,(x,t)\|\,. \tag{2.6}$$

Moreover, the inequalities (2.4) and (2.6) define the boundness of the operator $\overline{W}$.

Lemma 2 :

The integral operator $\overline{W}$ is continuous and contraction in the Banach space $C([a,b]\times[0,T])$.

Proof :

For the two functions $\phi_1\,(x,t)$ and $\phi_2(x,t)$ in $C([a,b]\times[0,T])$, the formulas (2.1) and (2.2) yield

$$\left|\overline{W}\phi_1(x,t) - \overline{W}\phi_2(x,t)\right| \le \left|\frac{\lambda}{\mu}\right|\int_a^b |F\,(x,y)||\phi_1(y,t) - \phi_2(y,t)|\,dy + \left|\frac{\lambda}{\mu}\right|\int_0^t |K\,(|t-\tau|)||\phi_1(x,\tau) - \phi_2\,(x,\tau)|d\tau.$$

In view of the conditions (i) and (ii), we get

$$\left\|\overline{W}\phi_1(x,t) - \overline{W}\phi_2(x,t)\right\| \le \sigma\|\phi_1(x,t) - \phi_2(x,t)\|\quad . \tag{2.7}$$

From inequality (2.7), we see that the operator $\overline{W}$ is continuous in the space $C([a,b]\times[0,T])$. Moreover, $\overline{W}$ is a contraction operator under the condition $\sigma < 1$.

Proof of theorem 1 :

Lemmas (1) and (2) show that, the operator $\overline{W}$ of equation (2.1) is contractive in the space $C([a,b]\times[0,T])$. So, from Banach fixed point theorem, $\overline{W}$ has a unique fixed point which is, of course, the unique solution of equation (1.1).

3. The linear system of VIEs

In this section, a numerical method is used in the mixed integral equation (1.1) to obtain a linear system of VIEs in time, see (Atkinson [8,9], Delves and Mohamed [10]) .

If we divide the interval$[a,b]$ into p intervals , by means of the points : $a = x_0 < x_1 < x_2 < \cdots < x_l < \cdots < x_p = b$, where $x = x_l\ \ ,y = x_j,\ l,j = 0,1,2,\dots,p$, then use the quadrature formula , the Fredholm integral term in (1.1) becomes

$$\int_a^b F(x_l,y)\,\phi(y,t)\,dy\ = \sum_{j=0}^{p} u_j F\big(x_l,x_j\big)\phi\big(x_j,t\big) + E_{p,l}(t)\quad . \tag{3.1}$$

u_j are the weights, such that

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$$u_j = \begin{cases} h/2 & j = 0\ , \quad j = p \\ h & 0 < j < p \end{cases}$$

where h is the stepsize of integration, and the error $E_{p,l}(t)$ is determined by the following relation :

$$E_{p,l}(t) = \left| \int_a^b F(x,y)\,\phi(y,t)dy - \sum_{j=0}^{p} u_j F_{l,j}\,\phi_j(t) \right| ,$$

and satisfies $\lim_{p\to\infty} E_{p,l}(t) = 0$.

Using (3.1) in (1.1) and neglecting $E_{p,l}(t)$, we have

$$\mu\phi_l(t) = \lambda \sum_{j=0}^{p} u_j\, F_{l,j}\,\phi_j(t) + \lambda \int_0^t K(|t-\tau|)\phi_l(\tau)d\tau + f_l(t)\ . \quad (3.2)$$

The formula (3.2) can be adapted in the form

$$\mu_l\phi_l(t) = \psi_l(t) + \lambda \int_0^t K(|t-\tau|)\phi_l(\tau)\,d\tau \quad ,(l = 0,1,2,\dots,p)\,, \quad (3.3)$$

where we used the following notations :

$$\psi_l(t) = \lambda \sum_{j=0}^{l-1} u_j F_{l,j}\,\phi_j(t) + \lambda \sum_{j=l+1}^{p} u_j F_{l,j}\,\phi_j(t) + f_l(t)\,, \mu_l = \mu - \lambda u_l F_{l,l} \ ,\ \phi_l(t) = \phi(x_l,t)\,,\ F_{l,j} = F(x_l,x_j) \ , f_l(t) = f(x_l,t)$$

The formula (3.3) represents a linear system of Volterra integral equations of the second kind (LSVIEs) .

Remark :

Let $\mathbb{E}$ be the set of all continuous functions $\Phi(t) = \{\phi_0(t)\,,\phi_1(t)\,,\dots,\phi_l(t)\,,\dots\}$, where $\phi_l(t) \in C[0,T]$, and define on $\mathbb{E}$, the norm:

$$\|\Phi(t)\|_{\mathbb{E}} = \operatorname{Sup}_l \max_{0\le t\le T} |\phi_l(t)| = \operatorname{Sup}_l \|\phi_l(t)\|_{C[0,T]} \quad , \quad \forall\, l\,,$$

then $\mathbb{E}$ is a Banach space .

4. The existence of a unique solution of LSVIEs

In order to guarantee the existence of a unique solution of the LSVIEs (3.3) in the space $\mathbb{E}$, we write this system in the following integral operator form :

$$\bar{U}\phi_l(t) = \frac{1}{\mu_l}\psi_l(t) + U\,\phi_l(t) \quad ; \quad \mu_l \neq 0\ ,\forall\, l \quad (4.1)$$

where

$$U\,\phi_l(t) = \frac{\lambda}{\mu_l}\int_0^t K(|t-\tau|)\phi_l(\tau)\,d\tau \quad , \quad (4.2)$$

then we assume in addition to condition (ii) of theorem (1), the following conditions :

$$(1)\ \operatorname{Sup}_l \max_{0\le t\le T} |f_l(t)| = \|f(t)\|_{\mathbb{E}} \le L^* \,, \qquad (2) \sum_{j=0}^{p} \operatorname{Sup}_j |u_j F_{l,j}| \le A \quad , \quad \forall \quad 0 \le l \le p \ , \quad (L^* \text{and } A \text{ are constants}).$$

Hence, the formula (4.1) has a unique solution in the space $\mathbb{E}$ under the following condition :

$$\delta = \left|\frac{\lambda}{\mu^*}\right| (\,A + M\,) < 1 \quad , \quad \left(\mu^* = \min_{0\le l\le p} \mu_l\right) \ . \quad (4.3)$$

Also, for $p \to \infty$, the sum

$$\sum_{j=0}^{p} u_j F_{l,j}\,\phi_j(t) \text{ tends to } \int_a^b F(x,y)\,\phi(y,t)dy\,,$$

thus the solution of the LSVIEs (4.1) becomes the solution of equation (1.1) .

5. The Toeplitz matrix method and linear algebraic system

Here , we present the Toeplitz matrix method to obtain the numerical solution of a linear VIE of the second kind with singular kernel. For this, we assume the LVIE :

$$\mu\,\phi(t) = \psi(t) + \lambda \int_0^t K(|t-\tau|)\,\phi(\tau)d\tau \quad . \tag{5.1}$$

Following the same way of Abdou et.al. [11,12], we can apply the Toeplitz matrix method for Volterra term to obtain the following principal equation

$$\mu\,\phi(t) - \lambda \sum_{n=0}^{N} D_n(t)\,\phi(nh_1) = \psi(t)\,. \tag{5.2}$$

Putting $t = mh_1$, $h_1 = T/N$, in (5.1) and using the following notations :

$$\phi(mh_1) = \phi_m \quad , \qquad D_n(mh_1) = D_{m\,n} \quad , \quad \psi(mh_1) = \psi_m \;, \tag{5.3}$$

we get the following linear algebraic system (LAS) :

$$\mu\,\phi_m - \lambda \sum_{n=0}^{N} D_{m\,n}\phi_n = \psi_m \quad ; \quad 0 \le m \le N \quad , \tag{5.4}$$

where

$$D_{m\,n} = \begin{cases} A_0(mh_1) & , \quad n = 0 \\ A_n(mh_1) + B_{n-1}(mh_1) & , \quad 0 < n < N \\ B_{N-1}(mh_1) & , \quad n = N \end{cases} \tag{5.5}$$

$$A_n(t) = \frac{[(nh_1 + h_1)U(t) - V(t)]}{h_1} \quad , \tag{5.6}$$

$$B_n(t) = \frac{[\,V(t) - (nh_1)U(t)]}{h_1} \quad , \tag{5.7}$$

and

$$U(t) = \int_{nh_1}^{nh_1+h_1} K(|t-\tau|)\;d\tau \;\;,\;\; V(t) = \int_{nh_1}^{nh_1+h_1} \tau\,K(|t-\tau|)\;d\tau \;\;. \tag{5.8}$$

The matrix D_{mn} can be written in the Toeplitz matrix form :

$$D_{mn} = G_{n-m} - P_{mn} \quad ,$$

here, the matrix

$$G_{n-m} = A_n(mh_1) + B_{n-1}(mh_1) \quad , \quad 0 \le m \,, \qquad n \le N \;, \tag{5.9}$$

is called a Toeplitz matrix of order $(N+1)$ and

$$P_{m\,n} = \begin{cases} B_{-1}(mh_1) & , \; n = 0 \\ 0 & , \; 0 < n < N \\ A_N(mh_1) & , \; n = N \end{cases} \tag{5.10}$$

represents a matrix of order $(N+1)$ whose elements are zeros except the first and the last rows (columns) .

Finally, the solution of the LAS (5.1) takes the form

$$\phi_m = [\,\mu\, I - \lambda\, D_{m\,n}]^{-1}\psi_m \quad , \; |\,\mu\, I - \lambda\, D_{m\,n}| \ne 0 \,, \; (I \text{ is the identity matrix}) \;. \tag{5.11}$$

Definition 1 :

The Toeplitz matrix method is said to be convergent of order r in the interval $[0,T]$, if and only if for sufficiently large N, there exists a constant $d > 0$ independent on N such that

$$\|\phi(t) - \phi_N(t)\| \le d\,N^{-r} \quad . \tag{5.12}$$

Definition 2 :

The estimate local error $E_{N,n}$ takes the form

$$E_{N,n} = \left| \int_0^t K(|t-\tau|)\, \phi(\tau) d\tau - \sum_{n=0}^{N} D_{mn}\, \phi_n \right| . \tag{5.13}$$

Lemma 3 :

If the kernel $K(|t-\tau|)$ of equation (1.1) satisfies condition (ii) of theorem (1) and the following condition

$$\lim_{\acute{t}\to t} \int_{nh}^{nh+h} |K(|\acute{t}-\tau|) - K(|t-\tau|)|\, d\tau = 0 \ , \tag{5.14}$$

then

$$\sup_N \sum_{n=0}^{N} |D_{mn}| \ \ exists \quad , \quad and \quad \lim_{\acute{m}\to m} \sup_N \sum_{n=0}^{N} |D_{\acute{m}n} - D_{mn}| = 0 \,.$$

Proof :

From the formulas (5.6) and (5.8), we have

$$|A_n(t)| \le \frac{(nh_1+h_1)}{h_1} \int_{nh_1}^{nh_1+h_1} |K(|t-\tau|)| d\tau + \frac{1}{h_1} \int_{nh_1}^{nh_1+h_1} \tau |K(|t-\tau|)| d\tau \ ,$$

in view of condition (ii), we get

$$|A_n(t)| \le (n+1)M + \frac{1}{h_1} M = \frac{M}{h_1}(nh_1 + h_1 + 1) \ , \quad (0 < \tau < t < T < \infty).$$

Summing from $n = 0$ to $n = N$, we obtain

$$\sum_{n=0}^{N} |A_n(t)| \le \frac{M}{h_1} \sum_{n=0}^{N} (nh_1 + h_1 + 1) \,,$$

hence, there exists a small constant E_1 , such that

$$\sum_{n=0}^{N} |A_n(t)| \le E_1 \,, \qquad \forall\, N \ .$$

Since , each term of $\sum_{n=0}^{N} |A_n(t)|$ is bounded above , so that for $t = mh_1$, wededuce

$$\sup_N \sum_{n=0}^{N} |A_{mn}| \le E_1 \ . \tag{5.15}$$

Similarly from the formulas (5.7) and (5.8),we can find a small constant E_2 , such that

$$\sup_N \sum_{n=0}^{N} |B_{mn}| \le E_2 \ . \tag{5.16}$$

In the light of (5.5), and the help of (5.15) and (5.16), there exists a small constantE^*, such that

$$\sup_N \sum_{n=0}^{N} |D_{mn}| \le \sup_N \sum_{n=0}^{N} |A_{mn}| + \sup_N \sum_{n=0}^{N} |B_{mn}| \le E^* \ ; (E^* = E_1 + E_2) \ ,$$

hence , $\sup_N \sum_{n=0}^{N} |D_{mn}|$ exists .

By virtue of the formulas (5.6) and (5.8), we have for $t, \acute{t} \in [0, T]$

$$|A_n(\acute{t}) - A_n(t)| \le \frac{(nh_1+h_1)}{h_1} \int_{nh_1}^{nh_1+h_1} |K(|\acute{t}-\tau|) - K(|t-\tau|)| d\tau + \frac{1}{h_1} \int_{nh_1}^{nh_1+h_1} \tau\, |K(|\acute{t}-\tau|) - K(|t-\tau|)| d\tau \ ,$$

thus, for $0 < \tau < T < \infty$, the above inequality after summing from $n = 0$ to $n = N$, can be adapted in the form

$$\operatorname{Sup}_{N}\sum_{n=0}^{N}|A_n(\acute{t})-A_n(t)| \leq \operatorname{Sup}_{N}\sum_{n=0}^{N}\frac{(nh_1+h_1+1)}{h_1}\int_{nh}^{nh+h}|K(|\acute{t}-\tau|)-K(|t-\tau|)|d\tau\,,$$

putting $t = mh_1$, $\acute{t} = \acute{m}h_1$, then using the condition (5.14) , we obtain as $t \to \acute{t}$,

$$\lim_{\acute{m}\to m}\operatorname{Sup}_{N}\sum_{n=0}^{N}|A_{\acute{m}n}-A_{mn}| = 0\,. \tag{5.17}$$

Similarly, in view of the formulas (5.7) and (5.8), we can prove

$$\lim_{\acute{m}\to m}\operatorname{Sup}_{N}\sum_{n=0}^{N}|B_{\acute{m}n}-B_{mn}| = 0\,. \tag{5.18}$$

Finally, from (5.5), (5.17) and (5.18) , we have

$$\lim_{\acute{m}\to m}\operatorname{Sup}_{N}\sum_{n=0}^{N}|D_{\acute{m}n}-B_{mn}| = 0\,.$$

5.1. The existence of a unique solution of the LAS

The LSVIEs (3.3) after using Toeplitz matrix method becomes

$$\mu_l\phi_{\,l,m} = \psi_{l,m} + \lambda\sum_{n=0}^{N}D_{mn}^{[l]}\phi_{\,l,n}\qquad , \tag{5.19}$$

where

$$\psi_{l,m} = \lambda\sum_{j=0}^{l-1}u_j\,F_{l,j}\,\phi_{j,m} + \lambda\sum_{j=l+1}^{p}u_j\,F_{l,j}\,\phi_{j,m} + f_{l,m}\quad . \tag{5.20}$$

Lemma 4 : (without proof)

For the LAS (5.19) if the kernel $K(|t-\tau|)$ of equation (1.1) satisfies the condition (ii) of theorem (1) and the condition (5.14), then

$$\operatorname{Sup}_{l,N}\sum_{n=0}^{N}\left|D_{mn}^{[l]}\right| \text{exists,}\qquad \text{and}\ \lim_{\acute{m}\to m}\ \sum_{n=0}^{N}\left|D_{\acute{m}n}^{[l]}-D_{mn}^{[l]}\right| = 0\,.$$

In order to guarantee the existence of a unique solution of the LAS (5.19), in the Banach space ℓ^∞, we write this system in the following operator form :

$$\bar{\mathcal{H}}\phi_{\,l,m} = \frac{1}{\mu_l}\psi_{l,m} + \mathcal{H}\,\phi_{\,l,m}\qquad ;\quad \mu_l \neq 0\quad \forall\ l\,, \tag{5.21}$$

where

$$\mathcal{H}\,\phi_{\,l,m} = \frac{\lambda}{\mu_l}\sum_{n=0}^{N}D_{mn}^{[l]}\phi_{\,l,n}\quad . \tag{5.22}$$

Then we assume in addition to condition (2), the following conditions :

$(\acute{1})\operatorname{Sup}_{l,m}\left|f_{l,m}\right| \leq H$, (H *is a constant*) .

$(\acute{2})\operatorname{Sup}_{l,N}\sum_{n=0}^{N}\left|D_{mn}^{[l]}\right| \leq e$, (e is a constant) .

Theorem 3: (without proof)

The formula (5.19) has a unique solution in the Banach space ℓ^∞ under the following condition

$$\alpha = \left|\frac{\lambda}{\mu^*}\right|(A+e) < 1\qquad ,\quad \left(\mu^* = \min_{0\leq l\leq p}\mu_l\right)\,. \tag{5.28}$$

Definition 3 :

The estimate total error R_q is determined by the following relation :

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$$R_q = \left| \int_a^b F(x,y)\phi(y,t)dy + \int_0^t K(|t-\tau|)\,\phi\,(x,\tau)d\tau - \sum_{j=0}^{p} u_j\, F_{l,j}\, \phi_{j,m} - \sum_{n=0}^{N} D_{mn}^{[l]}\,\phi_{l,n} \right|.$$

When $q = \max\{N,p\} \to \infty$, the sums $\sum_{j=0}^{p} u_j\, F_{l,j}\, \phi_{j,m} + \sum_{n=0}^{N} D_{mn}^{[l]}\,\phi_{l,n}$ tend to $\int_a^b F(x,y)\phi(y,t)dy + \int_0^t K(|t-\tau|)\,\phi\,(x,\tau)d\tau$

and the solution of the LAS (5.19) becomes the solution of equation (1.1).

Theorem 4 :

If the sequence of continuous functions $\{f_q(x,t)\}$ converges uniformly to the function $f(x,t)$ in the Banach space $C([a,b] \times$ *0,T,* then under the conditions of theorem (1), the sequence *ϕq(x,t)* converges uniformly to the exact solution of equation (1.1) in$C([a,b] \times [0,T])$.

Proof :

The formula (1.1) with its approximate solution give

$$|\phi(x,t) - \phi_q(x,t)| \le \frac{1}{|\mu|}|f(x,t) - f_q(x,t)| + \left|\frac{\lambda}{\mu}\right| \int_a^b |F(x,y)|\,|\phi(y,t) - \phi_q(y,t)|dy + \left|\frac{\lambda}{\mu}\right| \int_o^t |K(|t-\tau|)|\,|\phi(x,\tau) - \phi_q(x,\tau)|d\tau$$

In view of the conditions of theorem (1), we get

$$\|\phi(x,t) - \phi_q(x,t)\| \le \frac{1}{[|\mu| - |\lambda|(c^*\beta + M)]}\|f(x,t) - f_q(x,t)\| .$$

Hence $\|\phi(x,t) - \phi_q(x,t)\| \to 0$, since$\|f(x,t) - f_q(x,t)\| \to 0$ as $q \to \infty$.

Corollary 1 :

The total error satisfies $\lim_{q\to\infty} R_q = 0$.

Proof :

From the definition of R_q ,we have

$$|R_q| \le \operatorname{Sup}_{l,m}\left|\phi_l(mh) - (\phi_l(mh))_q\right| + \sum_{j=0}^{p} \operatorname{Sup}_{j}|u_j F_{i,j}| \operatorname{Sup}_{l,m}\left|\phi_l(mh) - (\phi_l(mh))_q\right| + \operatorname{Sup}_{l,N}\sum_{n=0}^{N}\left|D_{mn}^{[l]}\right| \operatorname{Sup}_{l,n}\left|\phi_l(nh) - (\phi_l(nh))_q\right| .$$

Using the conditions (2) and $(2')$, we get

$$|R_q| \le (1 + A + e)\left\|\phi_l(mh) - (\phi_l(mh))_q\right\|_{\ell^\infty} \quad , \quad \forall q .$$

Since each term R_q is bounded above, hence for $t = mh$, we deduce

$$\operatorname{Sup}_q|R_q| \le (1 + A + e)\operatorname{Sup}_l\max_t\left|\phi_l(t) - (\phi_l(t))_q\right| = (1 + A + e)\left\|\phi_l(t) - (\phi_l(t))_q\right\|_E .$$

Thus, the above inequality can be adapted in the form

$$\|R_q\|_{\ell^\infty} \le (1 + A + e)\left\|\phi(x,t) - (\phi(x,t))_q\right\|_{C([a,b]\times[0,T])} .$$

Since$\left\|\phi(x,t) - (\phi(x,t))_q\right\|_{C([a,b]\times[0,T])} \to 0asq \to \infty$, then$\|R_q\|_{\ell^\infty} \to 0$and consequently$R_q \to 0$.

6. Applications :

Here, we will consider the linear integral equation (1.1) when the Volterra kernel takes a logarithmic and Carleman forms .

Application 1 :

In equation (1.1), let the Volterra kernel takes the Carleman form$K(|t-\tau|) = |t-\tau|^{-\upsilon}$, $0 < \upsilon < 1/2$,whereυ is called Poisson ratio, and the relation between υ, λ, μ is given by

$$\lambda = \upsilon\,\mu/[(1+\upsilon)(1-2\upsilon)] .$$

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The results are obtained numerically, for $x \in [-1,1]$ at the times $T = 0.004$, $T = 0.02$and $T = 0.6$, with $\lambda = 0.16279$, 0.3158 *and* 0.51515, and the parameter $\mu = 1$.The position interval $[-1,1]$ is divided into 30 units, while the time interval $[0,T]$is divided into $n = 20$ units .

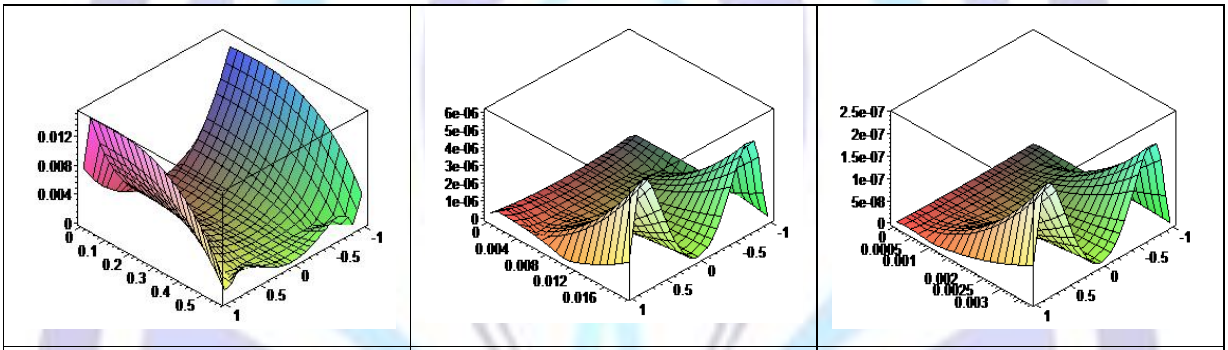
Application 2 :

In equation(1.1) , let the Volterra kernel takes the logarithmic form$K(|t-\tau|) = \ln|t-\tau|$, the results are obtained numerically for $x \in [-1,1]$, at the times $T = 0.004$, $0.02$and 0.6, with $\lambda = 0.0684$, 0.25and0.4368, and the parameter $\mu = 1$.The position interval $[-1,1]$ is divided into 30 units, while the time interval $[0,T]$ is divided into $n = 20$ units.

The following tables and diagrams are selected among a large amount of data to compare between the exact solution of equation (1.1) (Exact) and its numerical solution (Approx.sol.) in case of Carleman kernel when $\lambda = 0.16279$, $v = 0.07$ (see Table 1 and diagram 1), and in case of logarithmic kernel when $\lambda = 0.0684$ (see Table 2 and diagram 2) , where the exact solution $\phi(x,t) = x^2t^2$.

X	$T = 0.6$			$T = 0.02$			$T = 0.004$		
	Exact	**Approx. sol.**	**Error**	**Exact**	**Approx.**	**Error**	**Exact**	**Approx.**	**Error**
-1	0.36	3.54090E-01	5.90953E-03	4.00E-	3.99696E-	3.04038E-	1.600E-	1.59973E-	2.72953E-
-0.8	0.2304	2.21795E-01	8.60540E-03	2.56E-	2.50390E-	5.61012E-	1.024E-	1.00215E-	2.18463E-
-0.6	0.1296	1.23128E-01	6.47179E-03	1.44E-	1.39013E-	4.9866E-	5.760E-	5.56385E-	1.96154E-
-0.4	0.0576	5.44454E-02	3.15464E-03	6.40E-	6.14695E-	2.53045E-	2.560E-	2.46025E-	9.97543E-
-0.2	0.0144	1.35938E-02	8.06181E-04	1.60E-	1.53474E-	6.52611E-	6.400E-	6.14261E-	2.57392E-
0	0	0	0	0	0	0	0	0	0
0.2	0.0144	1.35928E-02	8.07203E-04	1.60E-	1.53461E-	6.53937E-	6.400E-	6.14208E-	2.57925E-
0.4	0.0576	5.44126E-02	3.18742E-03	6.40E-	6.14270E-	2.57296E-	2.560E-	2.45854E-	1.01462E-
0.6	0.1296	1.22878E-01	6.72241E-03	1.44E-	1.38688E-	5.3116E-	5.760E-	5.55079E-	2.09208E-
0.8	0.2304	2.20719E-01	9.68096E-03	2.56E-	2.48995E-	7.0046E-	1.024E-	9.96552E-	2.74478E-
1	0.36	3.53672E-01	6.32805E-03	4.00E-	3.99471E-	5.29069E-	1.600E-	1.59975E-	2.47808E-

Table 1



$T = 0.6$	$T = 0.02$	$T = 0.004$

diagram (1)

X	$T = 0.6$			$T = 0.02$			$T = 0.004$		
	Exact	**Approx. sol.**	**Error**	**Exact**	**Approx.**	**Error**	**Exact**	**Approx.**	**Error**
-1	0.36	0.364949919	4.94992E-03	4.00E-	4.00477E-	4.77386E-	1.600E-	1.60049E-	4.93495E-
-0.8	0.2304	0.231451349	1.05135E-03	2.56E-	2.54014E-	1.98599E-	1.024E-	1.01517E-	8.83379E-
-0.6	0.1296	0.129468449	1.31551E-04	1.44E-	1.42100E-	1.90024E-	5.760E-	5.67904E-	8.09600E-
-0.4	0.0576	5.74E-02	1.84521E-04	6.40E-	6.30201E-	9.79893E-	2.560E-	2.51862E-	4.13831E-
-0.2	0.0144	1.43E-02	5.43191E-05	1.60E-	1.57464E-	2.53623E-	6.400E-	6.29309E-	1.06906E-
0	0	0	0	0	0	0	0	0	0
0.2	0.0144	1.43E-02	5.49688E-05	1.60E-	1.57458E-	2.54209E-	6.400E-	6.29286E-	1.07136E-
0.4	0.0576	5.74E-02	2.05324E-04	6.40E-	6.30013E-	9.98654E-	2.560E-	2.51788E-	4.21220E-
0.6	0.1296	0.12931004	2.89960E-04	1.44E-	1.41957E-	2.04311E-	5.760E-	5.67341E-	8.65867E-
0.8	0.2304	0.230778854	3.78854E-04	2.56E-	2.53407E-	2.59262E-	1.024E-	1.01578E-	8.22295E-
1	0.36	0.363686973	3.68697E-03	4.00E-	3.99512E-	4.87603E-	1.600E-	1.59975E-	2.50046E-

Table 2

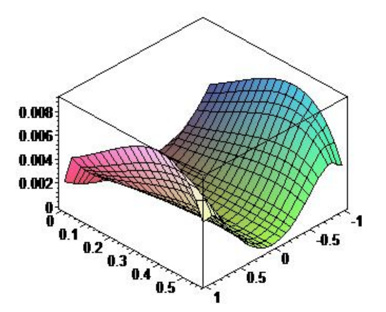
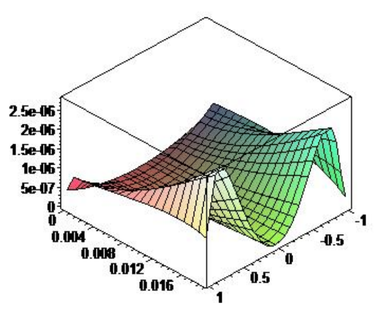
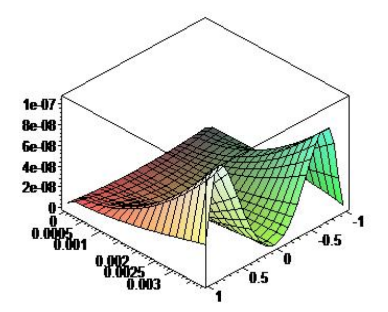
		
$T = 0.6$	$T = 0.02$	$T = 0.004$

diagram (2)

7. Conclusion:

From the numerical results, we notice that :

1. For fixed values of λ , the error values are increasing with the increase of time .
2. For fixed values of T , the error values are gradually increasing with the increase values of λ . So , the change in the values of λ is lightly effective in the numerical calculations.
3. The increasing of the error values at the end points of the x – axis is more than at the middle points.
4. The error values are symmetric with the x – axis's values .
5. The error is 0 at $x = 0$ for all cases we have studied .
6. The maximum value of the error for the Carleman case is 0.012729884 , at $\lambda = 0.51515$, $T = 0.6$ and $x = -1$.
7. The maximum value of the error for the logarithmic case is 0.028835575 , at $\lambda = 0.4368$, $T = 0.6$ and $x = 1$.
8. In general , the error values of the logarithmic kernel are higher than the error values of the Carleman kernel .

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