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(ψ,α,β)-weak contraction in fuzzy metric spaces

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Abstract.

The aim of this paper is to establish some new common fixed point theorems for mappings employing (ψ,α,β)-weak contraction in fuzzy metric spaces. Our results generalize and improve various well known comparable results.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [22]. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [12]. Grabiec [8] proved the contraction principle in the setting of fuzzy metric space. Also, George and Veermani [7] modified the notion of fuzzy metric space with the help of continuous t-norm. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc.

Boyd and Wong [3] introduced the notion of Φ-contractions. In 1997, Alber and Guerre- Delabriere [2] defined the ϕ-weak contraction which is a generalization of Φ-contractions. Many researchers [6, 9, 19] studied the notion of weak contractions on different settings which generalize the Banach Contraction Mapping Principle. Another interesting and significant fixed point results as a generalization of Banach Contraction Principle have been established by using the notion of alerting distance function, a new notion propounded by Khan et al. [11]. Altering Distance Function are control functions which alter the distance between two points in a metric space. Altering distances have been generalized to a two variable function and in [4] a generalization to a three-variable function has been introduced and applied for obtaining fixed point results in metric spaces.

In 2011, Abbas et. al [1] introduced the notion of ψ-weak contraction and proved fixed point results for a pair of self maps in fuzzy metric space. Further, Vetro et. al [20] introduced $(\phi-\psi)$-weak contraction in fuzzy metric space and established a common fixed point theorem for the sequence of self maps. Following this trend, we explore another extension of $(\phi-\psi)$ -weak contractions under the name of (ψ,α,β)-weak contraction and propose a generalization of altering distances to a three-variable function in fuzzy metric spaces. Our results improve and generalize the results of [5, 10, 13-16, 18, 19, 21].

2. reliminaries

To set up our results in the next section, we recall some basic definitions.

Definition 2.1 [17] A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 2.2 [12] A binary operation * : [0,1] × [0,1]→ [0,1] is a continuous t-norm if ([0,1], *) is a topological abelian monoid with unit 1 s.t. a * b ≤ c * d whenever a ≤ c and

b ≤ d , ∀ a, b, c, d ∈ [0,1]. Some examples are below:

(i) *(a, b) = a b, (ii) *(a, b) = min {a, b}.

Definition 2.3 [7] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(FM-1) M(x, y, t) > 0 and M(x, y, 0) = 0

(FM-2) M(x, y, t) = 1 iff x = y,

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),

(FM-5) M(x, y, t) : (0,∞) → [0,1] is continuous, for all x, y, z ∈ X and s, t > 0.

We note that M(x, y, .) is non-decreasing for all x, y ∈ X.

Definition 2.4 Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ is said to be

(i) G-Cauchy (i.e , Cauchy sequence in sense of Grabiec [8] sequence if $\lim_{n\to\infty} M\left(x_{n+p},x_n,t\right)=1$ for all t > 0 and each p > 0.

(ii) convergent to a point x ∈ X if $\lim_{n\to\infty} M\left(x_n,x,t\right)=1$ for all t > 0.

Definition 2.5[13] A pair of self mappings (f, g) on fuzzy metric space $(X,M,*)$ is said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz, \lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=z$ for some *z* in *X*.

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Definition 2.6[15] A pair of self mappings (f,g) on fuzzy metric space $(X,M,*)$ is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X*.

Thus the mappings *f* and *g* will be non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X* but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist.

Definition 2.7[15] A pair of self mappings (f, g) on fuzzy metric space $(X,M,*)$ is said to be sub-compatible iff there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X* and $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$.

Definition 2.8 [15] A pair of self mappings (f, g) on fuzzy metric space $(X,M,*)$ is said to be sub-sequentially continuous iff there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X* and $\lim_{n\to\infty} fgx_n = fz, \lim_{n\to\infty} gfx_n = gz$.

Definition 2.9 Let $(X,M,*)$ be a fuzzy metric space. f and g be self mappings on *X*. A point *x* in *X* is called a coincidence point of f and g iff fx = gx. In this case, w = fx = gx is called a point of coincidence of f and g.

Definition 2.10 A pair of self mappings (f, g) on a fuzzy metric space is said to be weakly compatible if they commute at the coincidence points i.e. fu = gu for some *u* in *X* , then

fgu = gfu.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11 An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which is

(i) monotone increasing and continuous and

(ii) $\psi(t) = 0$ iff $t = 0$.

Example 2.1 Let X=R equipped with $a * b = a \cdot b$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ then $(X,M,*)$ be a fuzzy metric space . Define the maps $f,g : X \to X$ as follows

$$f(x) = \begin{cases} x, x \le 1 \\ x+1, x > 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2x-1, x \le 1 \\ 2x, x > 1 \end{cases}$$

Here the mappings f and g are both discontinuous. Consider the sequence $x_n = 1 - \frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 1 \text{ and } \lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore the mappings are sub-compatible but not compatible as if we consider the sequence $x_n = 1 + \frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 2 \text{ and } \lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1.$$

Moreover, it is easy to see that f and g are reciprocally continuous maps with a coincidence point x=1, which is also a fixed point of the maps f and g.

Example 2.2 Let $X = [0,\infty)$ equipped with $a * b = a \cdot b$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ then $(X,M,*)$ be a fuzzy metric space . Define the maps $f,g : X \to X$ as follows

$$f(x)=\begin{Bmatrix}1+x, x<1\\ x, x\geq 1, x=0\end{Bmatrix} \text{ and } g(x)=\begin{Bmatrix}1-x, x\leq 1\\ 2x-1, x>1\end{Bmatrix}$$

Here the mappings f and g are both discontinuous. Consider the sequence $x_n = 1+\frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 1 \text{ and } \lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore, the mappings are sub-compatible but not compatible as if we consider the sequence $x_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 1 \text{ and } \lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1.$$

Moreover, it is easy to see that f and g are reciprocally continuous mappings without a coincidence point.

Example 2.3 Let $X=[0,\infty)$ equipped with $a*b=a\cdot b$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ then $(X,M,*)$ be a fuzzy metric space. Define the mappings $f,g: X\to X$ as follows

$$f(x)=\begin{Bmatrix}2, x<3\\ x, x\geq 3\end{Bmatrix} \text{ and } g(x)=\begin{Bmatrix}2x-4, x\leq 3\\ 3, x>3\end{Bmatrix}$$

Here the maps f and g are both discontinuous. Consider the sequence $x_n = 3+\frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 3 \text{ and}$$

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} f(3) = 3 = f(3), \quad \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g(3+\frac{1}{n}) = 3 \neq g(3) = 2$$

Notice that the mappings are not reciprocal continuous but are sub-sequentially continuous as if we consider the sequence $x_n = 3-\frac{1}{n}$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \to 2 \text{ and}$$

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} f(2-\frac{2}{n}) = 2 = f(2), \quad \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g(2) = 0 = g(2)$$

Moreover, it is easy to see that f and g don't have a coincidence point. Also the mappings are sub-compatible.

Hence it is worth mentioning that sub-compatible and sub-sequentially continuity are more general than compatibility and reciprocally continuity.

3. Main results

Theorem 3.1 Let $(X,M,*)$ be a fuzzy metric space and $A,B,S,T: X\to X$ be the mappings such that

(i) $A(X)\subseteq T(X), B(X)\subseteq S(X)$

(ii) One of $A(X), B(X), S(X)$ or $T(X)$ is G- complete subspace of X.

(iii) $$\psi\left(\frac{1}{M(Ax,By,t)}-1\right)\leq \alpha\left(\frac{1}{M(x,y,t)}-1\right)-\beta\left(\frac{1}{M(x,y,t)}-1\right)$$

With $M(x,y,t)=\min\{M(Ax,Sx,t),M(Sx,Ty,t),M(Ax,Sy,t),M(Ax,Ty,t)\}$

where ψ and α are altering distance functions and $\beta:[0,\infty)\to[0,\infty)$ is continuous with $\beta(t)>0$ for $t>0$ and $\beta(0)=0$ and $\psi(t)-\alpha(t)+\beta(t)>0$ for $t>0$.Then the mappings

A, B, S and T have a common fixed point provided the pairs (A, S) and (B, T) are weak compatible.

Proof: Let x_0 be any point in X. Then using condition (i), we can define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n}=Ax_{2n}=Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}$ for $n\geq 1$. First we claim $\{y_n\}$ is a G-Cauchy sequence.

If $y_{2n}=y_{2n+1}$ for some n. Then, using condition (iii), we get $y_{2n+1}=y_{2n+2}$ and $y_m=y_{2n}$ for each m > 2n. Thus the sequence $\{y_n\}$ is a G-Cauchy sequence. So without loss of generality, we may assume $y_n\neq y_{n+1}$ for all $n\in N$. In this case, first we prove $M(y_n,y_{n+1},t)\geq M(y_{n-1},y_n,t)$. Let if possible,

$$M(y_n,y_{n+1},t)<M(y_{n-1},y_n,t)$$

$$\Rightarrow\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)>\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)$$ as ψ is increasing so we have

$$\Rightarrow\psi\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)>\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)$$ and hence

$$\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)<\psi\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)$$

$$=\psi\left(\frac{1}{M(Ax_n,Bx_{n+1},t)}-1\right)\leq\alpha\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)$$

$$-\beta\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)$$

$$M(x_n,x_{n+1},t)=\min\{M(Ax_n,Sx_n,t),M(Sx_n,Tx_{n+1},t),M(Ax_n,Sx_{n+1},t),M(Ax_n,Tx_{n+1},t)\}$$

$$=\min\{M(y_n,y_{n-1},t),M(y_{n-1},y_n,t),M(y_n,y_n,t),M(y_n,y_n,t)\}=M(y_n,y_{n-1},t).$$

And hence

i.e $$\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)\leq\alpha\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)-\beta\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right) \qquad (1)$$

it implies $M(y_{n-1},y_n,t)=1$, a contradiction. Thus $M(y_n,y_{n+1},t)\geq M(y_{n-1},y_n,t)$ for all $n\in N$ and hence $M(y_{n-1},y_n,t)$ is an increasing sequence of positive real numbers in $(0,1]$.

Let $\gamma(t)=\lim_{n\to\infty}M(y_n,y_{n+1},t)$, then we show that $\gamma(t)=1$ for all $t>0$. If not, then there corresponds some $t>0$ such that $\gamma(t)<1$. Taking $n\to\infty$ in (1) , we get

$\psi\left(\frac{1}{\gamma(t)}-1\right)\leq\alpha\left(\frac{1}{\gamma(t)}-1\right)-\beta\left(\frac{1}{\gamma(t)}-1\right)\Rightarrow\gamma(t)=1$. Therefore $M(y_n,y_{n+1},t)\to 1$ as $n\to\infty$. Note that, for each positive integer p

$$M(y_n,y_{n+p},t)\geq M\left(y_n,y_{n+1},\frac{t}{p}\right)*M\left(y_{n+1},y_{n+2},\frac{t}{p}\right)*\ldots\ldots\ldots\ldots*M\left(y_{n+p-1},y_{n+p},\frac{t}{p}\right)$$

$$M(y_n,y_{n+p},t)\geq 1*1*\ldots\ldots\ldots\ldots*1=1$$

Therefore, $\{y_n\}$ is a G-Cauchy sequence. Now, we suppose S(X) is G-complete, there exists $z\in S(X)$ such that $y_n\to z$. Consequently, we can take v ∊ X such that Sv = z. Now we claim Av = z. Using (iii) for x= v and y = x_{n+1}

$$M(v,x_{n+1},t)==\min\{M(Av,Sv,t),M(Sv,Tx_{n+1},t),M(Av,Sx_{n+1},t),M(Av,Tx_{n+1},t)\}$$

and so ,

$$\psi\left(\frac{1}{M(Av,Bx_{n+1},t)}-1\right)\leq\alpha\left(\frac{1}{M(v,x_{n+1},t)}-1\right)-\beta\left(\frac{1}{M(v,x_{n+1},t)}-1\right)$$

Taking $n\to\infty$, we get

$$\psi\left(\frac{1}{M(Av,z,t)}-1\right)\leq\alpha\left(\frac{1}{M(Av,z,t)}-1\right)-\beta\left(\frac{1}{M(Av,z,t)}-1\right)$$

Therefore, $Av=Sv=z$. Since the pair (A, S) is weakly compatible so $Sz=SAv=ASv=Az$. Now, we prove $Az=z$. Using (iii) for x= z and y = x_{n+1}, we get

$M(z,x_{n+1},t)==\min\{M(Az,Sz,t),M(Sz,Tx_{n+1},t),M(Az,Sx_{n+1},t),M(Az,Tx_{n+1},t)\}$ and so

$$\psi\left(\frac{1}{M(Az,Bx_{n+1},t)}-1\right)\leq\alpha\left(\frac{1}{M(z,x_{n+1},t)}-1\right)-\beta\left(\frac{1}{M(z,x_{n+1},t)}-1\right)$$

Taking $n\to\infty$, we get

$$\psi\left(\frac{1}{M(Az,z,t)}-1\right)\leq\alpha\left(\frac{1}{M(Az,z,t)}-1\right)-\beta\left(\frac{1}{M(Az,z,t)}-1\right)$$

Thus $Az=z=Sz$.

Again $A(X)\subseteq T(X)$, so there exist some $u\in X$ such that $Az=Tu$ and hence $z=Sv=Av=Tu$. We claim $Bu=z$. Using (iii) for x= z and y = u , we get

$$M(z,u,t)=\min\{M(Az,Sz,t),M(Sz,Tu,t),M(Az,Su,t),M(Az,Tu,t)\}=1$$

and so

$$\psi\left(\frac{1}{M(Az,Bu,t)}-1\right)\leq\alpha\left(\frac{1}{M(z,u,t)}-1\right)-\beta\left(\frac{1}{M(z,u,t)}-1\right)$$

$$=\alpha(0)-\beta(0)=0,$$

A contradiction which gives $z=Sz=Az=Tu=Bu$.

Now, weak compatibility of the pair (B, S) implies $Tz = TBu = BTu = Bz$. Using (iii) for x = z and y = x_{n+1}, we get $z = Sz = Az = Tz = Bz$ and hence z is a fixed point of the mappings A, B, S and T .

Theorem 3.2 Let $(X, M, *)$ be a fuzzy metric space and $A, B, S, T : X \to X$ be the mappings such that

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(ii) One of $A(X), B(X), S(X)$ or $T(X)$ is G- complete subspace of X.

(iii) $$\psi\left(\frac{1}{M(Ax, By, t)} - 1\right) \le \alpha\left(\frac{1}{M(x, y, t)} - 1\right) - \beta\left(\frac{1}{M(x, y, t)} - 1\right)$$

With $M(x, y, t) = \min\{M(Ax, Sx, t), M(Sx, Ty, t), M(Ax, Sy, t), M(Ax, Ty, t)\}$

where ψ and α are altering distance functions and $\beta : [0, \infty) \to [0, \infty)$ is continuous with $\beta(t) > 0$ for $t > 0$ and $\beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$.Then the mappings A, B, S and T have a common fixed point provided the pairs (A, S) and (B, T) are sub-compatible and reciprocally continuous.

Proof : Following the Theorem 3.1, we can have $\{y_n\}$ a G-Cauchy sequence. Now, we suppose S(X) is G-complete , there exist $z \in S(X)$ such that $y_n \to z$. Clearly

$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and

$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$

Since the pair (A,S) is reciprocally continuous so $\lim_{n\to\infty} ASx_{2n} \to Az$ and $\lim_{n\to\infty} SAx_{2n} \to Sz$, but by sub-compatibility of (A, S), we have $\lim_{n\to\infty} M(ASx_{2n}, ASx_{2n}, t) = 1 \Rightarrow Az = Sz$ and rest of the proof follows on the lines of theorem 3.1.

Theorem 3.3 Let $(X, M, *)$ be a fuzzy metric space and $A, B, S, T : X \to X$ be the maps such that

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(ii) One of $A(X), B(X), S(X)$ or $T(X)$ is G- complete subspace of X.

(iii) $$\psi\left(\frac{1}{M(Ax, By, t)} - 1\right) \le \alpha\left(\frac{1}{M(x, y, t)} - 1\right) - \beta\left(\frac{1}{M(x, y, t)} - 1\right)$$

With $M(x, y, t) = \min\{M(Ax, Sx, t), M(Sx, Ty, t), M(Ax, Sy, t), M(Ax, Ty, t)\}$

where ψ and α are altering distance functions and $\beta : [0, \infty) \to [0, \infty)$ is continuous with $\beta(t) > 0$ for $t > 0$ and $\beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$.Then the maps A, B, S and T have a common fixed point provided the pairs (A, S) and (B, T) are compatible and sub-sequentially continuous.

Proof : Proof follows directly from theorem 3.1 and 3.2.

Theorem 3.4 Let $(X, M, *)$ be a fuzzy metric space and $A, B, S, T : X \to X$ be the mappings such that

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(ii) One of $A(X), B(X), S(X)$ or $T(X)$ is G- complete subspace of X.

(iii) $$\psi\left(\frac{1}{M(Ax,By,t)}-1\right)\le\alpha\left(\frac{1}{M(x,y,t)}-1\right)-\beta\left(\frac{1}{M(x,y,t)}-1\right)$$

With $M(x,y,t)=\min\{M(Ax,Sx,t),M(Sx,Ty,t),M(Ax,Sy,t),M(Ax,Ty,t)\}$

(iv) The pair (A, S) is sub-compatible and (B, T) is weak compatible.

(v) The pairs (A, S) and (B, T) reciprocally continuous.

where ψ and α are altering distance functions and $\beta:[0,\infty)\to[0,\infty)$ is continuous with $\beta(t)>0$ for $t>0$ and $\beta(0)=0$ and $\psi(t)-\alpha(t)+\beta(t)>0$ for $t>0$.Then the mappings A, B, S and T have a common fixed point.

If we take A = B and S = T, then we have

Theorem 3.5 Let $(X,M,*)$ be a fuzzy metric space and $A,S:X\to X$ be the mappings such that

(i) $A(X)\subseteq S(X)$

(ii) One of $A(X)$ or $S(X)$ is G- complete subspace of X.

(iii) $$\psi\left(\frac{1}{M(Ax,Ay,t)}-1\right)\le\alpha\left(\frac{1}{M(x,y,t)}-1\right)-\beta\left(\frac{1}{M(x,y,t)}-1\right)$$

With $M(x,y,t)=\min\{M(Ax,Sx,t),M(Sx,Sy,t),M(Ax,Sy,t),M(Ay,Sx,t)\}$

where ψ and α are altering distance functions and $\beta:[0,\infty)\to[0,\infty)$ is continuous with $\beta(t)>0$ for $t>0$ and $\beta(0)=0$ and $\psi(t)-\alpha(t)+\beta(t)>0$ for $t>0$.Then the mappings A,B,S and T have a common fixed point provided the pairs (A, S) and (B, T) are weak compatible.

Now, we furnish our theorem 3.5 with an example.

Example 3.2 Let $X=[-1,1)$ equipped with $a*b=ab$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ for all $x,y\in X$ and $t>0$. Define the mappings $f,g:X\to X$ as follows

$$A(x)=\begin{cases}0, \text{if } x\le 0;\\ \frac{1}{2}, \text{if } x>0\end{cases}\qquad S(x)=\begin{cases}0, \text{if } x\le 0;\\ 1, \text{if } x\ge\frac{1}{2}:\\ \frac{1}{2}, if\ 0<x<\frac{1}{2}\end{cases}$$

Define $\psi,\alpha,\beta:[0,\infty)\to[0,\infty)$ as $\psi(t)=t$, $\alpha(t)=2t$ and $\beta(t)=\frac{3t}{2}$, then we can see that $\psi(t)-\alpha(t)+\beta(t)=\frac{t}{2}>0$. One can easily verified condition (iii). Discuss the following subcases.

Case 1: when $x,y\le 0$ or $x,y>0$ then we have $M(Ax,Ay,t)=1=$ and condition (iii) is trivial.

Case 2: If $x\le 0, y>\frac{1}{2}$ then $M(fx,fy,t)=\frac{t}{t+\frac{1}{2}}\Rightarrow\left(\frac{1}{M(fx,fy,t)}-1\right)=\frac{1}{2t}$, $M(x,y,t)=\frac{t}{t+1}\Rightarrow\left(\frac{1}{M(x,y,t)}-1\right)=\frac{1}{t}$ and hence $\psi\left(\frac{1}{2t}\right)-\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)$ that is $\frac{1}{2t}\le\frac{2}{t}-\frac{3}{2t}\Rightarrow\frac{1}{2t}\le\frac{1}{2t}$ which is true.

Case 3: If $x>0, y\le 0$, then case is similar to previous one and other subcases are also true. Hence all the conditions of Theorem 3.5 are satisfied and 0 is a common fixed point of the mappings f and g.

4. Conclusion :

Some authors [10,14,15,16,21] proved the results by considering the pairs of maps sub-comatible as well as sub-sequentially continuous which are unfortunately not true .The results can be recovered either by replacing subcompatible pairs with compatible pairs or by replacing sub sequential continuity of pairs with reciprocal continuity of pairs. To substantiate this viewpoint , we furnish the following example

Example 4.1 Let $X = R$ equipped with $a * b = ab$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Define the mappings $A, S: X \to X$ as follows

$$A(x) = \begin{cases} x+1, x \in (-\infty, 1) \\ 2x-1, x \in [1, \infty) \end{cases} \text{ and } S(x) = \begin{cases} \frac{x}{2}, x \in (-\infty, 1) \\ 3x-2, x \in [1, \infty) \end{cases}$$

Consider a sequence $\langle x_n \rangle = 1 + \frac{1}{n}$, then we see

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$$

$$\lim_{n\to\infty} ASx_n = 1 = A(1), \lim_{n\to\infty} SAx_n = 1 = S(1)$$

And $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$

Next if we consider a sequence $\langle x_n \rangle = \frac{1}{n} - 2$, then we see

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = -1$$

$$\lim_{n\to\infty} ASx_n = 0 = A(-1), \lim_{n\to\infty} SAx_n = \frac{-1}{2} = S(-1)$$

And $\lim_{n\to\infty} M(ASx_n, SAx_n, t) \neq 1$

Thus the pair (A, S) is reciprocal continuous as well as sub-compatible but not compatible .Note that this example cannot be covered by those fixed point results which involve compatibility and reciprocal continuity.

Consequently, our results generalize and improve the results and follow –up corollaries of [10,14,15,16,21]

Also, our results generalize, improve and extend the results of [2,3,9,11,17-18,20] in the sense that the compatibility and commutativity of maps has been replaced by weak compatibility which is more general. Besides this , the condition of completeness of the space and continuity of maps has been completely relaxed.

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