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A Class Of Diameter Six Graceful Trees

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Abstract

In this paper we give graceful labelings to diameter six trees $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$ satisfying the following property:

$m+n$ is odd, degree of each neighbour of a_0 is even, and the centers of the branches incident on the center a_i of diameter four trees are either all odd branches or all even branches.

Keywords: graceful labeling; n distant tree; component moving transformation; transfer of the first type; BD8TF

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1 Introduction

Definition 1.1

A *diameter six tree* is a tree which has a representation of the form $(a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; $a_i, i = 1,2,\ldots,m$, $b_j, j = 1,2,\ldots,n$, and $c_k, k = 1,2,\ldots,r$ are the vertices of the tree adjacent to a_0; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. We observe that in a diameter six tree with above representation $m \geq 2$, i.e. there should be at least two (vertices) a_i s adjacent to c which are the centers of diameter four trees. In this we use the notation D_6 to denote a diameter six tree.

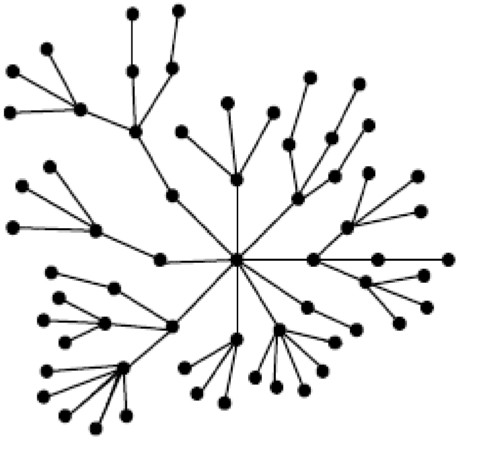

Figure1:-A diameter six tree.

In the year 1964 the famous " *Graceful Tree Conjecture*" of Ringel (1964) got published. The conjecture is yet to be resolved. Some specific type of trees are known to be graceful. One may refer to Gallian (2012), Robeva (2011), Edward and Howard (2006) to have an idea on the progress made so far in resolving the graceful tree conjecture. In 1966 Rosa proved that if a tree with n vertices is graceful then K_{2n+1} decomposes into $2n+1$ isomorphic copies of that tree. From the surveys of Gallian (2012), Robeva (2011), Edward and Howard (2006) and the work of Hrnčiar and Havier (2001), we know that all trees up to diameter five are graceful. As far as diameter six trees are concerned only *banana trees* are graceful. A *banana tree* is a tree obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). Sethuraman and Jesintha (2009a, 2009b) and Jesintha (2005) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful. Figure 1 is an example of a banana tree.

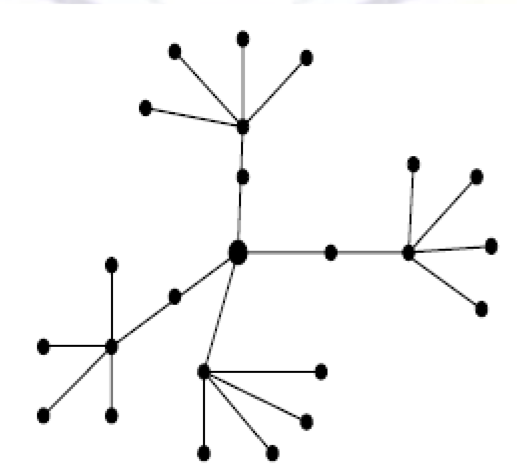

Figure2:-A banana tree.

Here we give graceful labelings to diameter six trees $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$ satisfying the following property:

$m+n$ is odd, degree of each neighbour of a_0 is even, and the centers of the branches incident on the center a_i of diameter four trees are either all odd branches or all even branches.

2 Preliminaries

In order to prove our results we require some existing terminologies and results which are given below.

Lemma 2.1

If g is a graceful labeling of a tree T with n edges then the labeling g_n defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the *inverse transformation* of g is also a graceful labeling of T.

Definition 2.2

For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a *component incident on* a and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called the *component moving transformation*. Here we say the component $u(T)$ has been transferred or moved from a to b. Throughout the paper we write "the component u" instead of writing "the component $u(T)$". Whenever we wish to refer u as a vertex, we write "the vertex u". By the label of the component "$u(T)$" we mean the label of the vertex u.

Notation 2.3

For any two vertices a and b of a tree T, the notation $a \to b$ *transfer* means that we move some components incident on the vertex a to the vertex b. If we consider successive transfers $a \to b$, $b \to c$, $c \to d$, $\ldots$, we simply write $a \to b \to c \to d \ldots$ transfer. In a transfer $a_1 \to a_2 \to a_3 \to \ldots \to a_n$, we call each vertex except a_n a vertex of the transfer.

Lemma 2.4

[Hrnčiar and Havier (2001)] Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a, where $b \notin u(T) \cup v(T)$. Then the following hold:

(i) if $f(u) + f(v) = f(a) + f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(ii) if $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Definition 2.5

Let T be a labelled tree and a and b be two vertices of T, and a be attached to some components. The $a \to b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers. The $a \to b$ transfer is called *a transfer of the second type* if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers.

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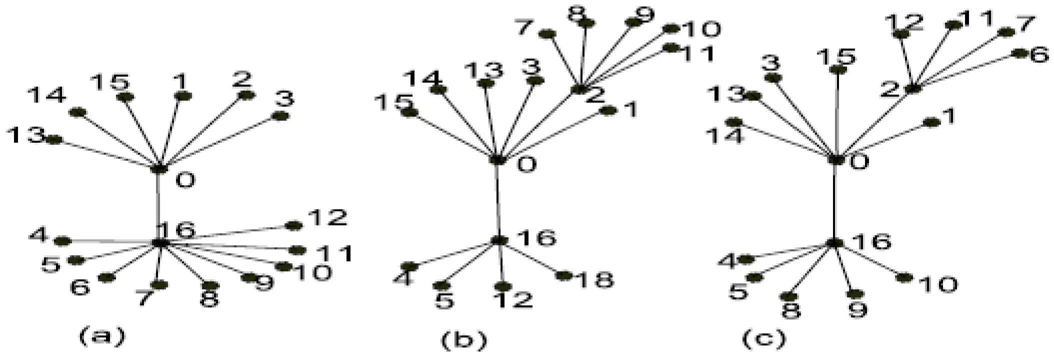

Figure3 :-The tree in (a) is a tree with a graceful labeling. The trees in (b) and (c) are obtained from (a) by applying a transfer of the first type $16 \to 2$ and a transfer of the second type $16 \to 2$, respectively.

In the following result i.e. Lemma 2.6 we state the conditions under which we can carry out a sequence of transfers of the first and second type so as form new graceful trees from given one.

Lemma 2.6 [Mishra and Panigrahi (2007)] In a graceful labeling f of a tree T, let $a, a-1, a-2, \ldots, a-p_1$, $b, b+1, b+2, \ldots, b+p_2$ (respectively, $a, a+1, a+2, \ldots, a+r_1$, $b, b-1, b-2, \ldots, b-r_2$) be some vertex labels. Let the vertex a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels) in f and satisfying either $(n+i+1)+(n+p-i)=a+b$ or $(n+i)+(n+p-1-i)=a+b$, $0 \le i \le [\frac{p+1}{2}]$, then the following hold.

1.By making a sequence of transfers of the first type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots \to x$ (respectively, $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots \to x$), where $z = a-p_1$ or $b+p_2$ (respectively, $z = a+r_1$ or $b-r_2$), an odd number of elements from A can be kept at each vertex of the transfer and the resultant tree thus formed will be graceful.

2. If A contains an even number of elements, then by making a sequence of transfers of second type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots \to z$ (respectively, $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots \to z$),where $z = a - p_1 or b + p_2$ (respectively, $z = a + r_1 or b - r_2$), an even number of elements from A can be kept at each vertex of the trasfer,such that the resultant tree thus formed will be graceful.

3. Let A contain an odd number of elements.By making a transfer $a \to b$ of the first type followed by a transfer $b \to a-1$ (respectively, $b \to a+1$) of the second type,we can keep from A an odd number of elements at a and an even number of elements at b and move the rest to $a-1$ (respectively, $a+1$), that the resultant tree thus formed will be graceful.

3 Results

Theorem 3.1

$D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ with $m+n$ odd and degree of a_i and b_j are even, for $i = 1,2,3,\ldots,m;\ j = 1,2,3,\ldots,n$. If the branches incident on the center a_i of the diameter four tree are either all odd branches or all even branches, then D_6 has a graceful labeling.

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Proof: Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Consider the graceful tree G as represented in Figure4.

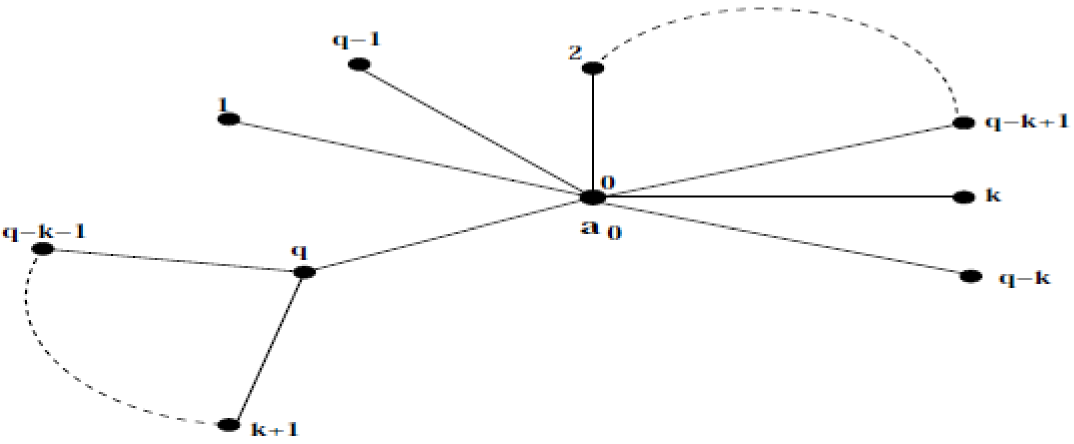


Figure4: Starting graceful tree for giving graceful labeling to diameter six trees in Theorem 3.1.

Let $A = \{k+1, k+2, \ldots, q-k-1\}$. Observe that $(k+i)+(q-k-i) = q$. Consider the sequence of transfer $T_1 : q \to 1 \to q-1 \to 2 \to q-2 \to \ldots \to k \to q-k \to k+1$ of the first type of the vertex levels in the set A. Observe that the transfer T_1 and the set A satisfy the properties of Lemma 2.6. We execute the transfer T_1 by keeping an odd number of elements of A at each vertex of the transfer. In the transfer T_1, the first m vertices are designated as the vertices $a_1, a_2, \ldots, a_m$, respectively, and the remaining n vertices are designated as the vertices b_1, b_2, …, b_n. Observe that

$$a_i = \begin{cases} q - \frac{i-1}{2} & \text{if } i \text{ is odd} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases} \quad \text{and } b_j = \begin{cases} \begin{cases} q - \frac{m+j-1}{2} & \text{if } j \text{ is odd} \\ \frac{m+j}{2} & \text{if } i \text{ is even} \end{cases} & \text{if } m \text{ is even} \\ \begin{cases} \frac{m+j}{2} & \text{if } j \text{ is odd} \\ q - \frac{m+j-1}{2} & \text{if } j \text{ is even} \end{cases} & \text{if } m \text{ is odd} \end{cases}$$

Let A_1 be the set of vertex labels of A which have come to the vertex $k+1$ after the transfer T_1. Since each transfer in T_1 is a transfer of 1st type, the elements of A are the consecutive integers. Next consider the transfer $T_2 : k+1 \to q-k-1 \to k+2 \to q-k-2 \to k+3 \to q-k-3 \to \ldots, \to r$, where

$$r = \begin{cases} k + k_1 + 1; & \text{if } m \text{ is odd} \\ q - k - k_1; & \text{if } m \text{ is even} \end{cases} \quad , k_1 = \sum_{i=1}^{m} deg(a_i)$$

Observe that the vertices of transfer T_2 and the elements of A_1 satisfy the properties of the transfer and the set A of vertex label in Lemma 2.6. If the branches incident on each a_i are all even branches then each transfer in T_2 is a transfer of second type, else each transfer in T_2 is a transfer of 1st type and keep the required number of vertices of A_1 at each vertex of T_2 so that we form the tree D_6. By virtue of Lemma 2.6, the resultant tree thus formed has a graceful labeling.

Example 3.2

Figure 5 represents graceful lableing of a diameter six tree of the type in Theorem 3.1. Here $q = 52$, $m = 4, n = 3, deg(a_0) = 7$. So $k = 3$ $A = \{k+1, k+2, \ldots, q-k-1\} = \{4,5,\ldots,48\}$. The vertices 52, 1, 51, and 2 are the centers of diameter four trees and the vertices 50, 3, and 49 are the centers of stars. T_1 is the transfer $52 \to 1 \to 51 \to 2 \to 50 \to 3 \to 49 \to 4$. Here the branches incident on the centers of diameter four trees are even branches. T_2 is the transfer $4 \to 48 \to 5 \to 47 \to 6 \to 46 \to 7 \to 45 \to 8 \to 44$, where each transfer is a transfer of the second type.

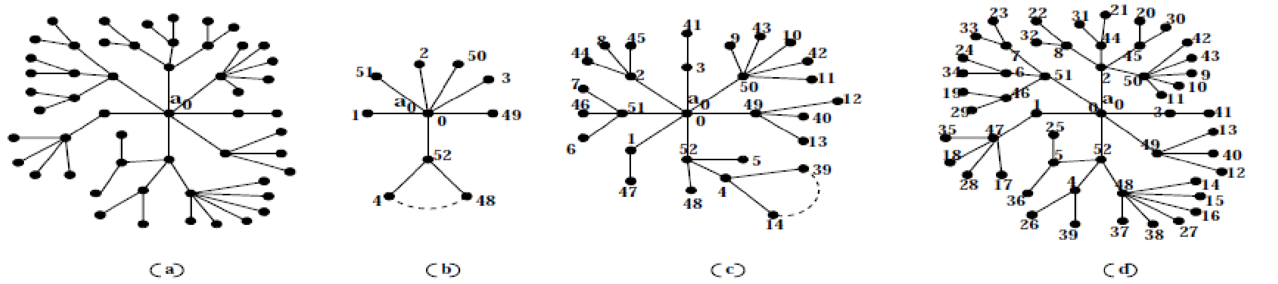


Figure 5:- A graceful diameter six tree.

As an immediate consequence of Theorem 3.1. We get the following result.

Corollary 3.3

If m is odd, degree of each $a_i, i = 1,2,\ldots,m$, is even and the branches incident on the centers $a_i, i = 1,2,\ldots,m$, of the diameter four trees are either all odd branches or all even branches. Then

(a) $D_6 = \{a_0; a_1, a_2 \ldots, a_m\}$ has a graceful labeling.

(b) $D_6 = \{a_0; a_1, \ldots, a_m; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

Proof: (a) The proof follows immediately from the same involving Theorem 3.1 by setting $n = 0$.

(b) Let us construct a tree G_6 from D_6 by removing the pendant vertices $c_1, c_2, ,\ldots, c_r$. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof involving part (a) of this corollary by replacing q with $q-r$ and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach the pendant vertices $c_1, c_2, ,\ldots, c_r$ and assign the labels $q-r+1$, $q-r+2$, …, q to them. Obviously, the tree $G_6 \cup \{c_1, c_2, ,\ldots, c_r\}$ is graceful and is seen to be the tree D_6.

Theorem 3.4

(a) $D_6 = \{a_0; a_1, \ldots, a_m\}$ with m even and degree of a_i are even , $i = 1,2,3,\ldots,m$. If the branches incident on the center a_i of the diameter four tree are either all odd branches or all even branches then D_6 has a graceful labeling.

(b) $D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ with $m+n$ even and degree of a_i and b_j are even,

$i=1,2,3,\ldots,m;\ j=1,2,3,\ldots,n$. If the branches incident on the center a_i of the diameter four tree are either all odd branches or all even branches then D_6 has a graceful labeling.

(c) $D_6=\{a_0;a_1,\ldots,a_m;b_1,b_2,\ldots,b_n;c_1,c_2,\ldots,c_r\}$ degree of a_i and b_j are even, $i=1,2,3,\ldots,m;\ j=1,2,3,\ldots,n$. If the branches incident on the center a_i of the diameter four tree are either all odd branches or all even branches then D_6 has a graceful labeling.

(d) $D_6=\{a_0;a_1,\ldots,a_m;c_1,c_2,\ldots,c_r\}$ with m even, degree of a_i are even $i=1,2,3,\ldots,m-1$ and that of a_m is odd. If the branches incident on the center a_i of the diameter four trees are either all odd branches or all even branches, then D_6 has a graceful labeling.

Proof: (a) Let us designate the vertices a_1, a_2, $\ldots$, a_m such that $deg(a_1)\le deg(a_2)\ge deg(a_3)\ge\ldots\ge deg(a_m)$, i.e. the degree a_m is minimum among all the neighbours of a_0. Excluding a_0 let there be $2p_i+1$ neighbours of $a_i, i=1,2,\ldots,m$ in D_6. Remove a_m and all the components incident on it, i.e. construct the tree $D_6/\{a_m\}$. Make any $2p_m$ neighbours of a_m adjacent to the vertex a_2. The resultant tree thus formed from D_6 is obviously a diameter six tree and let it be denoted by G_6. Let $|G_6|=q_1$. Repeat the procedure in the proof involving Corollary 3.3 (Theorem 3.1 with $n=0$) by replacing m with $m-1$ and q with q_1 and give a graceful labeling to G_6. We observe that the vertex a_2 gets label 1, and the $2(p_2+p_m)+1$ neighbours of a_2 get the labels q_1-x, $x+1+i, q_1-x-i$, $x=k+p_1+1$, $i=1,2,\ldots,p_2+p_m$. While labeling G_6 we allot labels $x+i+2, q_1-x-i, i=1,2,\ldots,p_m$ to $2p_m$ neighbours of a_m that were shifted to a_2 while constructing G_6. Next we attach the vertex a_m to a_0 and assign label q_1+1 to a_m. Now we move the vertices $x+i+2, q_1-x-i, i=1,2,\ldots,p_m$, to a_m. Since $(x+i+2)+(q_1-x-i)=q_1+2=1+(q_1+1)$, for $i=1,2,\ldots,p_m$, by Lemma 2.4 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to G_1 so that the label of the vertex a_m becomes 0. By Lemma 2.1, g_{q_1+1} is a graceful labeling of G_1. Now attach one remaining vertex to a_m and assign the label q_1+2 to it. Let this graceful labeling of the new tree, say G_2 thus formed be g_1. Let there be p neighbours of q_1+2 in D_6. Apply inverse transformation g_{1q_1+2} to G_2 so that the label of the vertex q_1+2 of G_2 becomes 0. By Lemma 2.1, g_{q_1+2} is a graceful labeling of G_2. Now attach the p pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1+3, q_1+4,\ldots,q_1+p+2$. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

(b) Let us construct a tree G_6 from D_6 by removing the vertex b_n and pendant vertices incident on it. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6|=q_1$. Repeat the procedure in the proof involving Theorem 3.1 by replacing n with $n-1$ and q with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach b_n to a_0 and assign the label q_1+1 to it. Obviously, the tree $G_6\cup\{b_n\}$ is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to $G_6\cup\{b_n\}$ so that the label of the vertex b_n becomes 0. By Lemma 2.1, g_{q_1+1} is a graceful labeling of $G_6\cup\{b_n\}$. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign labels $q_1+2, q_1+3,\ldots,q_1+p+1$ to them. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

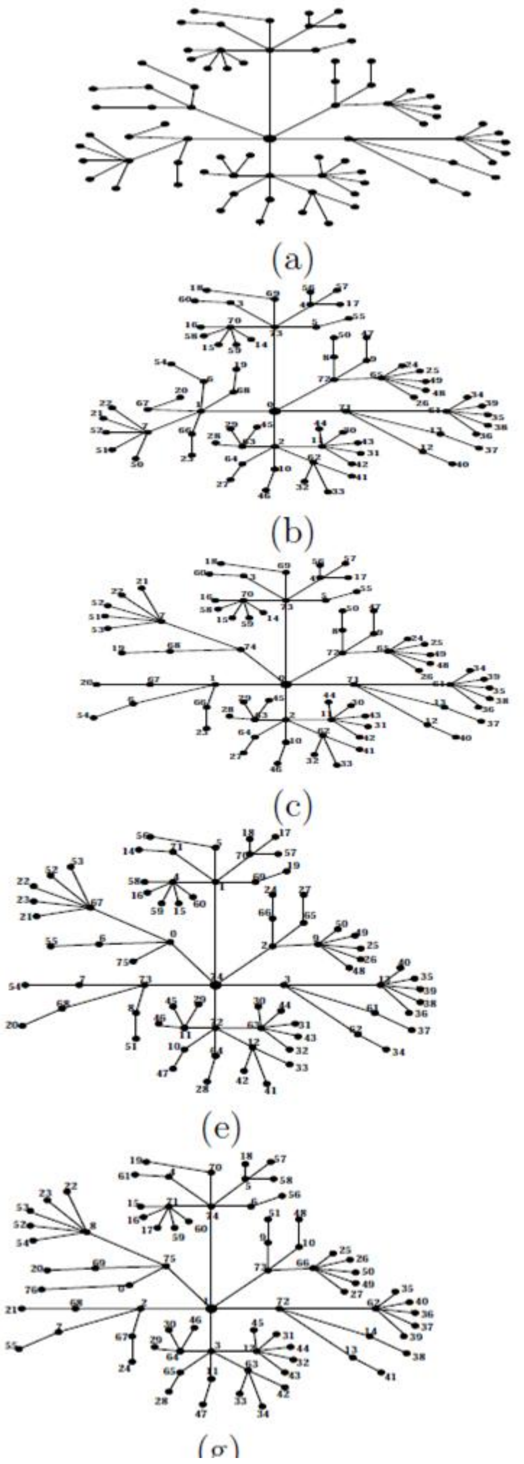
(c) Case - I Let $m+n$ be even. Let us construct a tree G_6 from D_6 by removing the vertices b_n, c_1, c_2, $\ldots$, c_r incident on a_0. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6|=q_1$. Repeat the procedure in the proof involving Theorem 3.1 by replacing n with $n-1$ and q with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach c_1, c_2, $\ldots$, c_r and b_n to a_0 and assign the labels q_1+1, q_1+2, $\ldots$, q_1+r, and q_1+r+1, respectively. Obviously, the tree $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$ is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+r+1} to $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$ so that the label of the vertex b_n becomes 0. By Lemma 2.1, g_{q_1+r+1} is a graceful labeling of $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign labels $q_1+r+2, q_1+r+3, \ldots, q_1+r+p+1$ to them. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

(c) Case - II Let $m+n$ be odd. Let us construct a tree G_6 from D_6 by removing the pendant vertices c_1, c_2, $\ldots$, c_r incident on a_0. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6|=q_1$. Repeat the procedure in the proof involving Theorem 3.1 by replacing q with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach c_1, c_2, $\ldots$, and c_r to a_0 and assign the labels q_1+1, q_1+2, $\ldots$, and q_1+r, respectively. Obviously, the tree $G_6 \cup \{c_1, c_2, \ldots, c_r\}$ is graceful.

(d) Let us designate the vertices a_1, a_2, $\ldots$, a_m such that $deg(a_1) \le deg(a_2) \ge deg(a_3) \ge \ldots \ge deg(a_m)$, i.e. the degree a_m is minimum among all the non pendant vertices adjacent to a_0. Excluding a_0 let there be $2p_i+1$ neighbours of $a_i, i=1,2,\ldots,m-1$ and $2p_m$ neighbours of a_m in D_6. Remove the vertices $c_1, c_2, \ldots, c_r, a_m$ and all the components incident on a_m, i.e. construct the tree $D_6 \{c_1, c_2, \ldots, c_r, a_m\}$. Make $2p_m$ neighbours of a_m adjacent to the vertex a_2. The resultant tree thus formed from D_6 is obviously a diameter six tree and let it be denoted by G_6. Let $|G_6|=q_1$. Repeat the procedure in the proof involving Corollary 3.3 (Theorem 3.1 with $n=0$) by replacing m with $m-1$ and q with q_1 and give a graceful labeling to G_6. We observe that the vertex a_2 gets label 1, and the $2(p_2+p_m)+1$ neighbours of a_2 get the labels $q-x$, $x+1+i, q_1-x-i$, $x=k+p_1+1$, $i=1,2,\ldots,p_2+p_m$. While labeling G_6 we allot labels $x+i+2, q_1-x-i, i=1,2,\ldots,p_m$ to $2p_m$ neighbours of a_m that were shifted to a_2 while constructing G_6. Next we attach the vertices $a_m, c_1, c_2, \ldots, c_r$ to a_0 and assign labels $q_1+1, q_1+2, \ldots, q_1+r, q_1+r+1$, respectively. Now we move the vertices $x+i+2, q_1-x-i, i=1,2,\ldots,p_m$, to a_m. Since $(x+i+2)+(q_1-x-i)=q_1+2=1+(q_1+1)$, for $i=1,2,\ldots,p_m$, by Lemma 2.6 the resultant tree, say G_1 thus formed is graceful.

Example 3.5

The diameter six tree in Figure 3 (a) is a diameter six of the type in Theorem 3.4(a). Here $q=76$ and $m=6$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing the vertex adjacent to a_0 with minimum degree and attaching two of its three components to another neighbour of a_0. In the graceful labeling of G_6 the neighbour of a_0 to which additional components have been added gets the label 1. Figure (c) represents the tree obtained from the graceful tree in (b) by attaching a vertex to a_0, assigning to it the label 74 and shifting two components incident on the vertices 68 and 7 from the vertex with label 1. The graceful tree in Figure (d) is obtained by applying inverse transformation to the graceful tree in Figure (c). The graceful tree in Figure (e) is obtained when we attach a vertex with label 75 to the vertex labelled 0 of the graceful tree in Figure (d).The graceful tree in Figure (f) is obtained by applying inverse transformation to the graceful tree in Figure (e). Finally, he graceful tree D_6 in Figure (g) is obtained when we attach a vertex with label 76 to the vertex labelled 0 of the graceful tree in Figure (f).



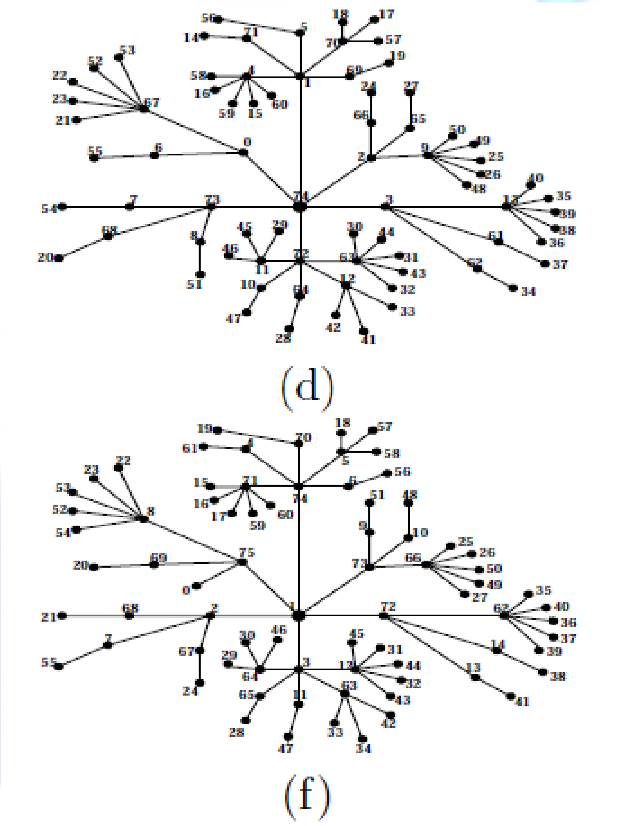


Figure 6:- A diameter six tree of the type in Theorem 3.4(a) with a graceful labeling.

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