ISSN 2347-1921

On rank one λ*-commuting operators

Nuha H. Hamada

Al Ain University of Science and Technology, UAE

Nuha.Hamada@aau.ac.ae

Abstract

Let λ be a non zero complex number. An operator A is a rank one λ *-commutes with B if $AB-\lambda BA^{*}$ has rank one. If, moreover, B is compact operator then A is called to belong to $\Delta_{\lambda}^{*}(\mathrm{H})$. In other words, $\Delta_{\lambda}^{*}(\mathrm{H})=\{A\in \mathbf{B}(\mathrm{H})\,|\,AB-\lambda BA^{*}$ has rank one for some compact operator $B\}$. We study the basic properties of $\Delta_{\lambda}^{*}(\mathrm{H})$. We prove that if $A\in \mathbf{B}(\mathrm{H})$ has an eigenvalue different than λ, and A has a fixed point then $A\in \Delta_{\lambda}^{*}(\mathrm{H})$.

Key words and phrases

Hilbert space; rank one operator; λ *-commutator.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol. 9 , No. 5

www.cirjam.com , editorjam@gmail.com

Introduction

Let H be a complex, separable, infinite dimensional Hilbert space, and let $\mathbf{B}(\mathsf{H})$ denote the algebra of all linear bounded operators on H. The "λ-commuting" property has its history related to the Invariant Subspace Problem of operators on Hilbert space. It was proved by V. Lomonosov [9] that any nonscalar operator $A \in \mathbf{B}(\mathsf{H})$ that commutes with a nonzero operator $B \in \mathbf{B}(\mathsf{H})$ has a nontrivial hyperinvariant subspace. Subsequently, the result was improved by Lomonosov and many other authors to operators that λ-commute, that is $AB = \lambda BA$, see for instance [3], [4], [10]. In [7] it was proved that if $A \in \mathbf{B}(\mathsf{H})$ is a normal operator with empty point spectrum and B is a compact operator such that $AB = \lambda BA$ for some complex number λ, then $B = 0$. This result was extended to hyponormal operators. For more details see [8].

On the other hand, It was proved that all operators A in $\mathbf{B}(\mathsf{H})$ with the property that there exists a compact operator B such that $AB - BA$ has rank one, have a nontrivial hyperinvariant subspace [5]. This class of operators was extensively studied in [6]. In [2], Hamada proved that every nonscalar operator with the property that there exists a finite rank operator B such that $AB - BA^*$ has rank one, has a nontrivial hyperinvariant subspace.

This paper is devoted to study a new class of operators, named $\Delta^*_\lambda(\mathsf{H})$. Let λ be a compex number. An operator $A \in \Delta^*_\lambda(\mathsf{H})$ if $AB - \lambda BA^*$ has rank one for some compact operator B. In other word, A is rank one λ *-commutes with a compact operator B. We construct some basic properties of $\Delta^*_\lambda(\mathsf{H})$, giving different examples of operators. However, we couldn't prove that the nonscalar elements in $\Delta^*_\lambda(\mathsf{H})$ have nontrivial hyperinvariant subspaces.

Remember that for $f, g \in \mathsf{H}$, $f \otimes g$ is the rank one operator defined by,

$(f \otimes g)x = \langle x, g \rangle f$ for each $x \in \mathsf{H}$

1 Main Results

Definition 1.1

Let $0 \neq \lambda \in \mathsf{C}$. An operator A is a rank one λ *-commutes with B if $AB - \lambda BA^*$ has rank one. By $\Delta^*_\lambda(\mathsf{H})$ we mean the set of all operators A in $\mathbf{B}(\mathsf{H})$ with the property that there exists a compact operator B such that $AB - \lambda BA^*$ has rank one. Note that the existence of B depends on A.

Remark 1.2

One can easily prove that $\Delta^*_\lambda(\mathsf{H})$ is not empty. In fact $iI \in \Delta^*_\lambda(\mathsf{H})$, for let $B = x \otimes y$ be the rank one operator that sends y to x provided that x and y are nonzero unit vectors, then clearly B is compact. For $A = iI$, $AB - \lambda BA^* = Ax \otimes y - \lambda(x \otimes Ay) = ix \otimes y + i\lambda x \otimes y = i(1+\lambda)x \otimes y$ which is of rank one.

We start with some basic properties of the class of operators $\Delta^*_\lambda(\mathsf{H})$.

Proposition 1.3

1. *For any non zero scalar operator* λI , $\lambda I \in \Delta^*_\lambda(\mathsf{H})$ iff $\lambda \neq 1$
2. $A \in \Delta^*_\lambda(\mathsf{H})$ iff $\alpha A \in \Delta^*_\lambda(\mathsf{H})$ *for eachnon zero real number* α .
3. If $A \in \Delta^*_\lambda(\mathsf{H})$, then $A \in \Delta^*_{1/\bar{\lambda}}(\mathsf{H})$

Proof.

1. Let $A=\lambda I$ be a non zero scalar operator. If B is an operator of rank one, then

$$AB-\lambda BA^{*}=(\lambda I)B-\lambda B(\overline{\lambda}I)=(\lambda-|\lambda|^{2})B$$

has rank one iff $\lambda\neq 1$.

2. Suppose that $A\in\Delta_{\lambda}^{*}(\mathrm{H})$, then there exists a compact operator B such that $AB-\lambda BA^{*}$ has rank one. Let α ba a non zero real number and $\beta\in\mathrm{C}$ s.t. $\beta=\lambda\overline{\beta}$, then

$$(\alpha A)B-\lambda B(\alpha A)^{*}=\alpha AB-\overline{\alpha}\lambda BA^{*}$$

$$=\alpha AB-\overline{\alpha}\lambda BA^{*}$$

$$=\alpha(AB-\lambda BA^{*})$$

which has rank one. Thus, $\alpha A\in\Delta_{\lambda}^{*}(\mathrm{H})$. The converse is trivial, just put $\alpha=1$.

3. If $A\in\Delta_{\lambda}^{*}(\mathrm{H})$ then there exists a compact operator B such that $AB-\lambda BA^{*}=f\otimes g$ for some nonzero vectors $f,g\in\mathrm{H}$. Thus $(AB-\lambda BA^{*})^{*}=(f\otimes g)^{*}$ but $(f\otimes g)^{*}=g\otimes f$. Therefore

$$(AB-\lambda BA^{*})^{*}=B^{*}A^{*}-\overline{\lambda}AB^{*}=-\overline{\lambda}(AB^{*}-(1/\overline{\lambda})B^{*}A^{*})$$

has rank one, and $AB^{*}-(1/\overline{\lambda})B^{*}A^{*}$ is so. Note that B^{*} is also compact. Thus $A\in\Delta_{1/\overline{\lambda}}^{*}(\mathrm{H})$

Now, we go through some examples of operators that belong to $\Delta_{\lambda}^{*}(\mathrm{H})$.

Proposition 1.4

Let A be any nonzero operator which is not 1-1, then $A\in\Delta_{\lambda}^{*}(\mathrm{H})$.

Proof.

By assumption, there exist nonzero vectors $x,y\in\mathrm{H}$ such that $Ax\neq 0$ and $Ay=0$. Let $B=x\otimes y$, then B is of rank one hence compact and

$$AB-\lambda BA^{*}=A(x\otimes y)-\lambda(x\otimes y)A^{*}=Ax\otimes y-\lambda x\otimes Ay=Ax\otimes y$$

which has rank one. Thus $A\in\Delta_{\lambda}^{*}(\mathrm{H})$.

Corollary 1.5

Any nonzero nilpotent operator belongs to $\Delta_{\lambda}^{*}(\mathrm{H})$

Proof.

Let A be a nonzero nilpotent operator, then there exists $n\in\mathrm{N}$ such that $A^{n}=0$ and $A^{n-1}\neq 0$. Let $f\in\mathrm{H}$ be such that $A^{n-1}f\neq 0$ then $A(A^{n-1}f)=A^{n}f=0$ so A is not 1-1. Hence by (1.4), $A\in\Delta_{\lambda}^{*}(\mathrm{H})$.

One can easily prove that:

Corollary 1.6

Any nonzero finite rank operator belongs to $\Delta^*_\lambda(\mathrm{H})$.

We turn now to results related to invertible operators.

Proposition 1.7

Let A be an invertible operator. If $A \in \Delta^*_\lambda(\mathrm{H})$ then $A^{-1} \in \Delta^*_{1/\lambda}(\mathrm{H})$.

Proof.

Let B be a compact operator such that $AB - \lambda BA^*$ has rank one. Now,

$$AB - \lambda BA^* = (ABA^*)(A^*)^{-1} - \lambda A^{-1}(ABA^*) = A^{-1}(ABA^*) - (1/\lambda)(ABA^*)(A^*)^{-1}.$$

Since (ABA^*) is compact, the result follows.

Recall that an operator A is called algebraic if $p(A) = 0$ for some non zero polynomial p. The following theorem gives a condition on an algebraic operator to be rank one λ *-commutes with a compact operator.

Theorem 1.8

Let A be a nonzero algebraic operator. If A is non invertible then $A \in \Delta^*_\lambda(\mathrm{H})$.

Proof.

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o$ be a polynomial such that $p(A) = 0$. If $a_o \neq 0$ then $a_n A^n + a_{n-1}A^{n-1} + \ldots + a_1 A = -a_o I$, So $A(a_n A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1) = -a_o I$ which contradicts the non invertiblity of A.

Let $q(A) = a_n A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1$, then $Aq(A) = 0$. One can assume that the degree of p is the smallest one among those polynomials that annihilate A. Thus, $q(A) \neq 0$ and there exists non zero vectors f and g such that $q(A)f \neq 0$ and $Ag \neq 0$ (since $A \neq 0$). Let $B = q(A)f \otimes g$ then

$$AB - \lambda BA^* = Aq(A)f \otimes g - \lambda q(A)f \otimes Ag = -\lambda q(A)f \otimes Ag$$

which is of rank one. Thus $A \in \Delta^*_\lambda(\mathrm{H})$.

The following theorem find a relation between operators that have eigenvalues and the $\Delta^*_\lambda(\mathrm{H})$.

Theorem 1.9

Let $A \in \mathbf{B}(\mathrm{H})$ satisfies the following conditions:

1. A *has an eigenvalue different than* λ, and
2. A *has a fixed point*,

then $A \in \Delta^*_\lambda(\mathrm{H})$

Proof. Let $\lambda \neq \mu \in \mathrm{C}$ such that $Af = \mu f$ for some $0 \neq f \in \mathrm{H}$, μ is the eigenvalue of A. Let $0 \neq g \in \mathrm{H}$ be the fixed point of A, hence $Ag = g$. Define $B = f \otimes g$, then

$$AB - \lambda BA^* = A(f \otimes g) - \lambda(f \otimes g)A^*$$
$$= Af \otimes g - \lambda f \otimes Ag$$
$$= \mu \otimes g - \lambda f \otimes g$$
$$= (\mu - \lambda) f \otimes g$$

has rank one as $\mu \neq \lambda$. Consequently, $A \in \Delta^*_\lambda(\mathrm{H})$

Corollary 1.10 Let $0 \neq \lambda \in \mathrm{C}$. If $A \in \mathbf{B}(\mathrm{H})$ has an eigenvalue 1, then there exists a compact operator B such that $AB - \lambda BA^*$ has rank at most 1.

Proof. If $\lambda \neq 1$ then A has an eigenvalue and fixed point of 1. Hence by theorem (1.9), $A \in \mathbf{B}(\mathrm{H})$. If $\lambda = 1$ then there exists a non zero vector g in H such that $Ag = g$. Define $B = g \otimes g$, then

$$AB - \lambda BA^* = A(g \otimes g) - (g \otimes g)A^*$$
$$= Ag \otimes g - g \otimes Ag$$
$$= g \otimes g - g \otimes g$$
$$= 0$$

Consequently, $AB = \lambda BA^*$.

Open Question: Does an operator in $\Delta^*_\lambda(\mathrm{H})$ have a nontrivial hyperinvariant subspace?

References

[1] Cho, L., Ohwada, T. and Zhang, L. 2011. On λ-commuting operators, International Math. Forum 6 , no. 34, 1685-1690.

[2] Hamada, N. 2002.*Jordan*-derivations on B(H),* PhD. Thesis, University of Baghdad.

[3] Kim, H., Moore, R. and Pearcy, C. 1979. A variation of Lomonosovs theorem, J. Operator Theory 2, 131-140.

[4] Kim, H., Moore, R. and Pearcy, C. 1981. A variation of Lomonosovs theorem, II, J. Operator Theory 5, 283-287.

[5] Kim, H., Pearcy, C. and Shields, A. 1975.*Rank-one commutators and hyperinvariant subspaces,* Michigan Math. J., 22, 193-194.

[6] Kim, H., Pearcy, C. and Shields, A. 1976.*Sufficient conditions for rank-one commutators and hyperinvariant subspaces,* Michigan Math. J., 23, 235-243.

[7] Lauric, V. 1997. Operators α-commuting with a compact operator, Proc. Amer. Math. Soc. 125, 2379-2384.

[8] Lauric, V. 2013. *On λ-Commuting Hyponormal Operators,* Int. Journal of Math. Analysis, Vol. 7, no. 49, 2441 - 2444.

[9] Lomonosov, V. 1973. Invariant subspaces for operators commuting with compact operators, Funkcional Anal. i Prilozen 7, 55-56 (Russian).

[10] Pearcy, C. and Shields, A. 1974. A survey of the Lomonosov technique in the theory of invariant subspaces, Mathematical Surveys, Amer. Math. Soc. 13, 219-229.