ISSN 2347-1921

COMMON FIXED POINT FOR TANGENTIAL MAPS OF GREGUS TYPE ON FUZZY METRIC SPACES

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Abstract: In this paper, we define extend the results of many others. We prove common fixed point theorems on tangential property for a Gregus type on pair of fuzzy metric spaces. We also deal on some coupled coincidence and common fixed point theorems.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol. 9 , No. 5
www.cirjam.com , editorjam@gmail.com

1 INTRODUCTION:

The notion of fuzzy sets introduced by Zadeh (1965, [4]) proved a turning point in the development of Mathematics. The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Markin (1973, [2]) and Nadler (1972, [3]).

Bhaskar and Lakshmikantham (2006, [1]) introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem.

2 Definitions and Preliminaries:

To set up our results in the next section we recall some definitions and facts.

2.1 Definition: A fuzzy set A in X is a function with domain X and values in [0, 1].

2.2 Definition: A binary operation *: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if ([0, 1], *) is a topological abelian monoid with unit 1 s.t. a * b ≤ c * d whenever a ≤ c and b ≤ d, ∀ a, b, c, d ∈ [0, 1].

Some examples are below:

(i) *(a, b) = ab,

(ii) *(a, b) = min{a, b}.

2.3 Definition: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X^2 x [0, ∞) satisfying the following conditions:

(FM-1) M(x, y, t) > 0 and M(x, y, 0) = 0

(FM-2) M(x, y, t) = 1 iff x = y,

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),

(FM-5) M(x, y, .) : (0, ∞) → [0, 1) is continuous, for all x, y, z ∈ X and s, t > 0.

therefore, M(x, y, .) is non-decreasing for all x, y ∈ X

2.4 Definition: Let (X, M,) be a fuzzy metric space.

(i) A sequence $\{x_n\}$ is said to be convergent to a point x ∈ X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0.

(ii) A subset A ⊆ X is said to be closed if each convergent sequence $\{x_{n}\}$ with x_n ∈ X (A) and $x_n \to x$, we have x ∈ A.

(iii) A subset A ⊆ X is said to be compact if each sequence in A has a convergent subsequence.

Throughout the paper X will represent the fuzzy metric space (X, M, ∗) and κ(X), the set of compact subsets of X. For A, B ∈ κ(X) and for every t > 0, denote

$$M_{\nabla}(A, B, t) = \min\{\min_{a \in A} M(a, B, t), \min_{b \in B} M(A, b, t)\},$$

$$M(A, y, t) = \max\{M(x, y, t)\};\ x, y \in A$$

Remark: Obviously, $M_{\nabla}(A, B, t) \le M(a, B, t)$ whenever a ∈ A and $M_{\nabla}(A, B, t) = 1 \Leftrightarrow A = B$. Also $M(A, y, t) = 1$ if y ∈ A.

3 Main Results

3.1 Theorems: Let A, B: X → X and S, T: X x X → κ(X) be single and set-valued mappings satisfying the following conditions:

(1) there exist contained coupled weak tangential points (z_1, z_2) to the mappings A and B.

(2) (A, B) is tangential w. r .t (S, T)

(3) $\int_0^{M_{\nabla}(S(x,\ y),\ T(u,\ v),\ t)} \psi(s)ds \ \ge\ \int_0^{m(x,\ y,\ u,\ v,\ t)} \psi(s)ds$

where

m(x, y, u, v, t)

ISSN 2347-1921

$$= \emptyset \left[\begin{array}{c} a \begin{pmatrix} M^{\Delta}(S(x,y), Ax, t) * M^{\Delta}(T(u,v), Bu, t) \\ +M^{\Delta}\left(S(x,y), Bu, \frac{t}{2}\right) * M^{\Delta}\left(T(u,v), Ax, \frac{t}{2}\right) \end{pmatrix} \\ \\ + (1-2a)\, max \begin{Bmatrix} M^{\Delta}(S(x,y), Ax, t), M^{\Delta}(T(u,v), Bu, t), \\ M_{\nabla}(S(x,y), Ax, t), M_{\nabla}(T(u,v), Bu, t) \end{Bmatrix} \end{array} \right]$$

(4) AAa = Aa, BBC = Bc, S(Aa, Ab) = T(Bc, Bd) and

AAb = Ab, BBd = Bd, S(Ab, Aa) = T(Bd, Bc) for (a, b) ϵ C(A, S) and (c, d) ϵ C(B,T),

(5) the pair (A, S) is weakly compatible, for all x, y, u, v ϵ X, $0 \leq a < 1$ and $\emptyset$: 0, 1] → [0,1] be a non-decreasing map such that $\emptyset(t) > t$, $t \geq 0$. Then A, B, S and T have a common coupled fixed point in X.

Proof: Since $z_1, z_2 \,\epsilon\, A(x) \cap B(X)$ so there exist points $w_1, w_2, w_1^{/}, w_2^{/} \,\epsilon\, X$ such that $z_1 = Aw_1 = Bw_1^{/}$, $z_2 = Aw_2 = Bw_2^{/}$. Again (A, B) is tangential w.r.t (S, T) so there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = z_1 = \lim_{n\to\infty} By_n \,\epsilon\, C \,\epsilon\, X \lim_{n\to\infty} S(x_n, y_n) \cap \lim_{n\to\infty} T(y_n, x_n)$$

$$\lim_{n\to\infty} Ay_n = z_2 = \lim_{n\to\infty} Bx_n \,\epsilon\, D \,\epsilon\, X \lim_{n\to\infty} S(y_n, x_n) \cap \lim_{n\to\infty} T(x_n, y_n)$$

Now, we shall prove that

$Aw_1 \,\epsilon\, S(w_1, w_2)$, $Aw_2 \,\epsilon\, S(w_2, w_1)$, $Bw_1^{/} \,\epsilon\, T(w_1^{/}, w_2^{/})$

and

$Bw_2^{/} \,\epsilon\, T(w_2^{/}, w_1^{/})$ If not, putting $x = x_n$, $y = y_n$, $u = w_1^{/}$ and $v = w_2^{/}$ in (3), we get

$m(x_n, y_n, w_1^{/}, w_2^{/}, t)$

$$= \emptyset \left[\begin{array}{c} a \begin{pmatrix} M^{\Delta}(S(x_n, y_n), Ax_n, t) * M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \\ +M^{\Delta}\left(S(x_n, y_n), Bw_1^{/}, \frac{t}{2}\right) * M^{\Delta}\left(T(w_1^{/}, w_2^{/}), Ax_n, \frac{t}{2}\right) \end{pmatrix} \\ \\ + (1-2a)\, max \begin{Bmatrix} M^{\Delta}(S(x_n, y_n), Ax_n, t), M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t), \\ M_{\nabla}(S(x_n, y_n), Ax_n, t), M_{\nabla}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \end{Bmatrix} \end{array} \right]$$

$$\int_0^{M_{\nabla}(S(x_n, y_n),\ T(w_1^{/}, w_2^{/}),\ t)} \psi(s)ds \geq \int_0^{m(x_n, y_n, w_1^{/}, w_2^{/}\ t)} \psi(s)ds$$

Letting $n \to \infty$, we have

$$\int_0^{\lim_{n\to\infty} M_{\nabla}(S(x_n, y_n),\ T(w_1^{/}, w_2^{/}),\ t)} \psi(s)ds \geq \int_0^{\lim_{n\to\infty} m(x_n, y_n, w_1^{/}, w_2^{/}\ t)} \psi(s)ds,$$

but

$\lim_{n\to\infty} m(x_n, y_n, w_1^{/}, w_2^{/}, t)$

$$= \emptyset \left[\begin{array}{c} a \begin{pmatrix} 1 * M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \\ +M^{\Delta}\left(z_1, Bw_1^{/}, \frac{t}{2}\right) * M^{\Delta}\left(T(w_1^{/}, w_2^{/}), z_1, \frac{t}{2}\right) \end{pmatrix} \\ \\ + (1-2a)\, max \begin{Bmatrix} 1, M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t), \\ 1, M_{\nabla}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \end{Bmatrix} \end{array} \right]$$

$$= \emptyset \left[\begin{array}{c} a \begin{pmatrix} M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \\ +M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t) \end{pmatrix} \\ \\ + (1-2a)\, max\{ M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t)\} \end{array} \right]$$

$$= \emptyset\, [2a + (1-2a)]\, M^{\Delta}(T(w_1^{/}, w_2^{/}), Bw_1^{/}, t)$$

$$\int_0^{M_{\nabla}(C, T(w_1^{/},\ w_2^{/}), t)} \psi(s)ds \geq \int_0^{\emptyset(M^{\Delta}(T(w_1^{/},\ w_2^{/}),\ Bw_1^{/},\ t))} \psi(s)ds$$

Since, $z_1 = Bw_1^{/} \epsilon$ C, we have

$$\int_0^{M^{\Delta}(Bw_1^{/},\ T(w_1^{/}, w_2^{/}),\ t))} \psi(s)ds \geq \int_0^{M_{\nabla}(C,\ T(w_1^{/},\ w_2^{/}),\ kt)} \psi(s)ds$$

$$\geq \int_0^{\emptyset(M^{\Delta}(T(w_1^{/},\ w_2^{/}),\ Bw_1^{/},\ t))} \psi(s)ds$$

$$\Rightarrow \int_0^{M^{\Delta}(Bw_1^{/},\ T(w_1^{/}, w_2^{/}),\ t))} \psi(s)ds \geq \int_0^{\emptyset(M^{\Delta}(Bw_1^{/}, T(w_1^{/},\ w_2^{/}),\ t))} \psi(s)ds$$

$$\geq \int_0^{(M^{\Delta}(Bw_1^{/},\ T(w_1^{/},\ w_2^{/}),\ t))} \psi(s)ds$$

which is a contradiction. Hence $Bw_1^{/} \epsilon\ T(w_1^{/}, w_2^{/})$.

Similarly, by putting $x = y_n$, $y = x_n$, $u = w_2^{/}$ and $v = w_1^{/}$ in (3), we get $Bw_2^{/} \epsilon\ T(w_2^{/}, w_1^{/})$.

Again by taking $x = w_1$, $y = w_2$, and $u = y_n$ and $v = x_n$ in (3), we get

$m(w_1, w_2, y_n, x_n, t)$

$$= \emptyset\left[\begin{array}{c} a\left(\begin{array}{c} M^{\Delta}(S(w_1, w_2), Aw_1, t) * M^{\Delta}(T(y_n, x_n), By_n, t) \\ +M^{\Delta}\left(S(w_1, w_2), By_n, \frac{t}{2}\right) * M^{\Delta}\left(T(y_n, x_n), Aw_1, \frac{t}{2}\right) \end{array}\right) \\ + (1-2a)\ max\left\{\begin{array}{c} M^{\Delta}(S(w_1, w_2), Aw_1, t), M^{\Delta}(T(y_n, x_n), By_n, t), \\ M_{\nabla}(S(w_1, w_2), Aw_1, t), M_{\nabla}(T(y_n, x_n), By_n, t) \end{array}\right\} \end{array}\right]$$

$$\int_0^{M_{\nabla}(S(w_1, w_2),\ T(y_n, x_n), t)} \psi(s)ds \geq \int_0^{m(w_1, w_2, y_n, x_n, t)} \psi(s)ds, \text{ hence}$$

$$\int_0^{\lim_{n\to\infty} (M_{\nabla}(S(w_1, w_2), C, t)} \psi(s)ds \geq \int_0^{\lim_{n\to\infty} m(w_1, w_2, y_n, x_n, t)} \psi(s)ds$$

$$= \int_0^{\emptyset(M^{\Delta}S(w_1, w_2), Aw_1, t))} \psi(s)ds$$

As $z_1 = Aw_1$ C, we have

$$\int_0^{(M^{\Delta}S(w_1, w_2), Aw_1, t))} \psi(s)ds \geq \int_0^{M_{\nabla}(S(w_1, w_2), C, t} \psi(s)ds$$

$$\geq \int_0^{\emptyset(M^{\Delta}S(w_1, w_2), Aw_1, t))} \psi(s)ds$$

$$\geq \int_0^{M^{\Delta}S(w_1, w_2), Aw_1, t} \psi(s)ds$$

Which is a contradiction. Hence Aw_1 $S(w_1, w_2)$.

Similarly, by putting $x = w_2$, $y = w_1$, $u = x_n$ and $v = y_n$ in (3),

we get $z_2 = Aw_2$ $S(w_2, w_1)$.

Hence $(w_1, w2)$ $C(A, S)$ and $(w_1^{/}, w_2^{/})$ $C(B, T)$. Now (4), gives

$AAw_1 = Aw_1$, $BBw_1^{/} = Bw_1^{/}$ and $S(Aw_1, Aw_2) = T(Bw_1^{/}, Bw_2^{/})$

$AAw_2 = Aw_2$, $BBw_2^{/} = Bw_2^{/}$ and $S(Aw_2^{/}, Aw_1^{/}) = T(Bw_2^{/}, Bw_1^{/})$.

But we have $z_1 = Aw_1 = Bw_1^{/}$, $z_2 = Aw_2 = B\ w_2^{/}$. This gives,

$Az_1 = z_1 = Bz_1$ and $S(z_1, z_2) = T(z_1, z_2)$

$Az_2 = z_2 = Bz_2$ and $S(z_2, z_1) = T(z_2, z_1)$

Also, weak compatibility of (A, S) gives $AS(w_1, w_2)$ $S(Aw_1, Aw_2)$

$z_1 = Bz_1 = Az_1$ $AS(w_1, w_2)$ $S(Aw_1, Aw_2) = S(z_1, z_2) = T(z_1, z_2)$.

Similarly, we can have $z_2 = Bz_2 = Az_2$ $S(z_2, z_1) = T(z_2, z_1)$. Hence (z_1, z_2) is a common coupled fixed point of the mappings A, B, S and T.

Example: Let X = R and a * b = ab and M (x, y, t) = $\frac{t}{t+d\ (x,y)}$ then (X, M,) is a fuzzy metric space. Define A, B: X → X and S, T: X × X → κ(X) by setting

$$Ax = \begin{cases} 2x-1, x \le 1 \\ 3, x > 1 \end{cases}, \quad Bx = \begin{cases} 2-x, 1 \le x \le 2 \\ 3, otherwise \end{cases} \quad \text{and}$$

$$S(x, y) = \begin{cases} [x+y-4, x+y+2], if\ x, y \in [0, 3] \\ [x-1, y-1], \text{ otherwise} \end{cases}.$$

$$T(x, y) = \begin{cases} [2x-y+1, 3x+y], x < y \\ [x-2y-1, x+3], x \ge y \end{cases}$$

Consider the sequences, $\{x_n\} = 1 - \frac{1}{n}$ and $\{y_n\} = 1 + \frac{1}{n}$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n \to 1 \in \lim_{n\to\infty} S(x_n, y_n) \cap \lim_{n\to\infty} T(y_n, x_n)$$

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n \to 3 \in \lim_{n\to\infty} S(y_n, x_n) \cap \lim_{n\to\infty} T(x_n, y_n)$$

This shows that (A, B) is tangential w.r.t (S, T)

$$A_1 = B_1 = 1 \in [0, 6] = S(1, 3) \cap T(3, 1)$$

Also,

$$A_3 = B_3 = 3 \in [0, 6] = S(3, 1) \cap T(1, 3)$$

Hence all the conditions of above theorems are satisfied and (1, 3) is a coupled fixed point of the maps A, B, S and T.

3.2 Theorem: Let A, B : X → X and S, T : X × X → CB(X) be single and set-valued mappings satisfying the following conditions:

(1) there exist contained coupled weak tangential points (z_1, z_2) to the mappings A and B.

(2) (A, B) is tangential with respect to (S, T).

(3) $\int_0^{M_\nabla(S(x,\ y),\ T(u,\ v),\ t)} \psi(s)ds \ge \int_0^{m(x,\ y,\ u,\ v,\ t)} \psi(s)ds$ where

m(x, y, u, v, t)

$$= \emptyset\left[\begin{array}{c} a\left(M^{\Delta}(S(x,y), Ax, t), \quad M^{\Delta}(T(u,v), Bu, t)\right) \\ +a\left(M^{\Delta}\left(S(x,y), Bu, \frac{t}{2}\right), \quad M^{\Delta}\left(T(u,v), Ax, \frac{t}{2}\right)\right) \\ +(1-2a)\max\left\{\begin{array}{c} M^{\Delta}(S(x,y), Ax, t) * M^{\Delta}(T(u,v), Bu, t), \\ M_\nabla(S(x,y), Ax, t), M_\nabla(T(u,v), Bu, t) \end{array}\right\} \end{array}\right]$$

a. AAa = Aa = BBc, S(Aa, Ab) = T(Bc, Bd) and AAb = Ab = BBd, S(Ab, Aa) = T(Bd, Bc) for (a, b) ϵ C(A, S) and (c, d) ϵ C(B, T)

b. the pair (A,S) is weakly compatible,

for all x, y, u, v ϵ X, 0 ≤ a < 1 and ∅: [0,1] → [0,1] be a non-decreasing map such that ∅(t) > t, t ∈ (0,1). Then A, B, S and T have a common coupled fixed point in X.

Proof The result follows directly from theorem (3.1).

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