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Graceful Labeling for Step Grid Graph

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Abstracts

We investigate a new graph which is called Step Grid Graph. We prove that the step grid graph is graceful. We have investigate some step grid graph related families of connected graceful graphs. We prove that path union of step grid graph, cycle of step grid graph and star of step grid graph are graceful graphs.

Key words: Graceful labeling; path union of graphs; cycle of graphs; star of graphs; step grid graph.

Subject Classification number: 05C78.

Introduction:

The graceful labeling was introduced by A. Rosa [1] during 1967. Golomb [2] named such labeling as graceful labeling, which was called earlier as $\beta-$valuation. In this work we introduce a new graph which is called step grid graph and it is denoted by St_n.

We begin with a simple, undirected finite graph $G=(V,E)$ with $|V|=p$ vertices and $|E|=q$ edges. For all terminology and notations we follows Harary [3]. Here are some of the definitions which are useful in this paper.

Definition -1.1 **:** A function f is called *graceful labeling* of a graph $G=(V,E)$ if $f:V\to\{0,1,\ldots,q\}$ is injective and the induced function $f^*:E\to\{1,2,\ldots,q\}$ defined as $f^*(e) = |f(u)-f(v)|$ is bijective for every edge $e=(u,v)\in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition -1.2 **:** Let G be a graph and $G_1,G_2,\ldots,G_n$, $n\geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(1\leq i\leq n-1)$ is called *path union* of G.

Definition -1.3 **:** For a cycle C_n, each vertex of C_n is replaced by connected graphs $G_1,G_2,\ldots,G_n$ and is known as *cycle of graphs*. We shall denote it by $C(G_1,G_2,\ldots,G_n)$. If we replace each vertex by a graph G, i.e. $G_1=G$, $G_2=G$, $\ldots$, $G_n=G$, such cycle of a graph G is denoted by $C(n\cdot G)$.

Above definition was introduced by Kaneria et al [4].

Definition -1.4 **:** Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G is called a star of G and is denoted by G^*.

Above definition was introduced by Vaidya et al [5].

Definition -1.5 **:** Take $P_n,P_n,P_{n-1},\ldots,P_2$ paths on $n,n,n-1,n-2,\ldots,3,2$ vertices and arrange them vertically.

A graph obtained by joining horizontal vertices of given successive paths is known as a step grid graph of size n, where $n\geq 3$. It is denoted by St_n.

Obviously $|V(St_n)| = \frac{1}{2}(n^2+3n-2)$ and $|E(St_n)| = n^2+n-2$.

In this paper we introduced gracefulness of step grid graph, path union of step grid graph, cycle of step grid graph and star of step grid graph. For detail survey of graph labeling one can refer Gallian [6].

1. Main Results:

Theorem -2.1 **:** A step grid graph St_n is a graceful graph, where $n\geq 3$.

Proof : Let $G=St_n$ be any step grid of size n. Where mention each vertices of n^{th} collum like $u_{1,j}$ $(1\leq j\leq n)$ and $(n-1)^{th}$ collum like $u_{2,j}$ $(1\leq j\leq n)$ and $(n-2)^{th}$ collum like $u_{3,j}$ $(1\leq j\leq n-1)$ and $(n-3)^{th}$ collum like $u_{4,j}$ $(1\leq j\leq n-2)$ similarly the first collum like $u_{n,j}$ $(1\leq j\leq 2)$.

We see that number of vertices in G is $|V(G)|=p=\frac{1}{2}(n^2+3n-2)$ and the number of edges in G is $|E(G)|=q=n^2+n-2$.

We define labeling function $f:V(G)\to\{0,1,\ldots,q\}$ as follows

$$f(u_{1,j})=\frac{q}{2}-\frac{1}{8}+(-1)^{j+1}[\frac{j^2}{4}-\frac{1}{8}],\quad \forall\quad j=1,2\ldots,n;$$

$$f(u_{i,j}) = f(u_{i-1,j-1}) + (-1)^j, \quad \forall \; i = 2,3\ldots,\lceil \frac{n}{2} \rceil, \quad \forall \; j = 1,2\ldots,n-i+1;$$

$$f(u_{i,1}) = (n-i+1)^2 - 1, \quad \forall \; i = n, n-1\ldots,\lceil \frac{n}{2} \rceil;$$

$$f(u_{i,2}) = q - (n-i+1)(n-i), \quad \forall \; i = n, n-1\ldots,\lceil \frac{n}{2} \rceil;$$

$$f(u_{i,j}) = f(u_{i+1,j-2}) + (-1)^{j-1}, \quad \forall \; i = n-1, n-2,\ldots,2, \quad \forall \; j = 3,4,\ldots,n+2-i.$$

Above labeling patten give rise a graceful labeling to the graph G. So G is a graceful graph.

Illustration -2.2 **:** St_8 and its graceful labeling shown in *figure* -1.

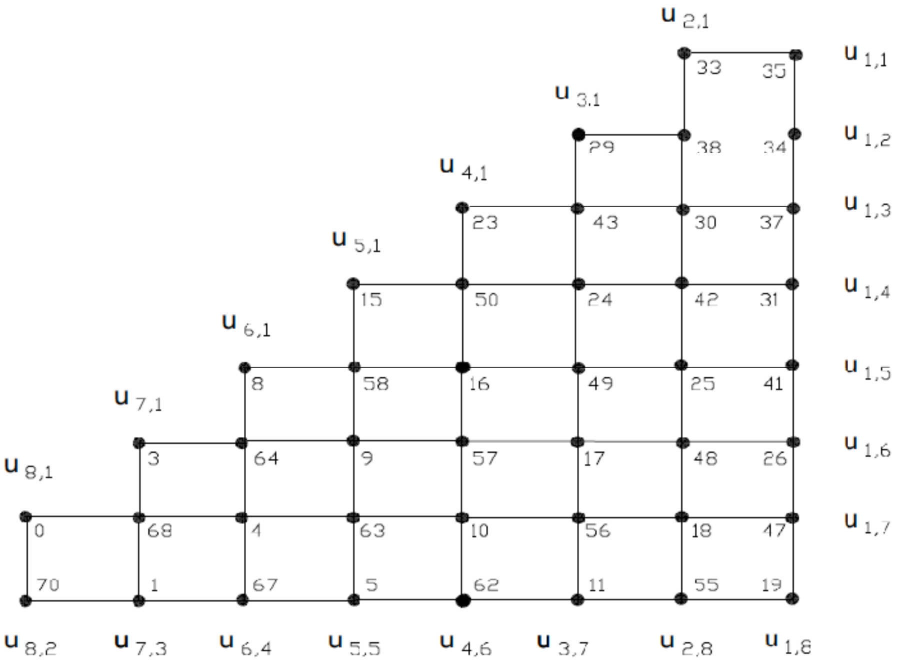

Figure -1 St_8, step grid graph with $n = 8$ and its graceful labeling.

Theorem -2.3 **:** Path union of finite copies of the step grid graph St_n is a graceful graph, where $n \geq 3$.

Proof : Let G be a Path union of r copies of the step grid graph St_n $(r \in N)$. Let f be the function for graceful labeling of St_n as we mentioned in $Theorem-2.1$.

In graph G, we see that the vertices $|V(G)| = \frac{r}{2}(n^2 + 3n - 2) = P$ and the edges $|E(G)| = r(n^2 + n - 1) - 1 = Q$. Let $u_{k,i,j}$ $(1 \leq i, j \leq n)$ be vertices of k^{th} copy of St_n, $\forall$ $k = 1,2,\ldots,r$. Where the vertices of k^{th} copy of St_n is $p = \frac{1}{2}(n^2 + 3n - 2)$ and edges of k^{th} copy of St_n is $q = n^2 + n - 2$.

Join the vertices $u_{k,1,1}$ to $u_{k+1,n,1}$ for $k = 1,2,\ldots,r-1$ by an edge to from the path union of r copies of step grid graph.

We define labeling function $g : V(G) \to \{0,1,\ldots,Q\}$ as follows

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$$g(u_{1,i,j}) = f(u_{i,j}) \qquad \text{if } f(u_{i,j}) < \frac{q}{2};$$

$$= f(u_{i,j}) + (Q+q) \qquad \text{if } f(u_{i,j}) > \frac{q}{2};$$

$$\forall \quad i, j = 1,2,\ldots,n;$$

$$g(u_{k,i,j}) = g(u_{k-1,i,j}) + (\frac{q}{2}) \qquad \text{if } g(u_{k-1,i,j}) < \frac{Q}{2};$$

$$= g(u_{k-1,i,j}) - (\frac{q}{2}+1) \qquad \text{if } g(u_{k-1,i,j}) > \frac{Q}{2}$$

$$\forall \quad i, j = 1,2,\ldots,n, \ \forall \quad k = 2,3,\ldots,r.$$

Above labeling patten give rise a graceful labeling to given graph G. So path union of finite copies of the step grid graph is graceful graph.

Illustration -2.4 **:** Path union of 3 copies of St_4 and its graceful labeling shown in *figure* -2

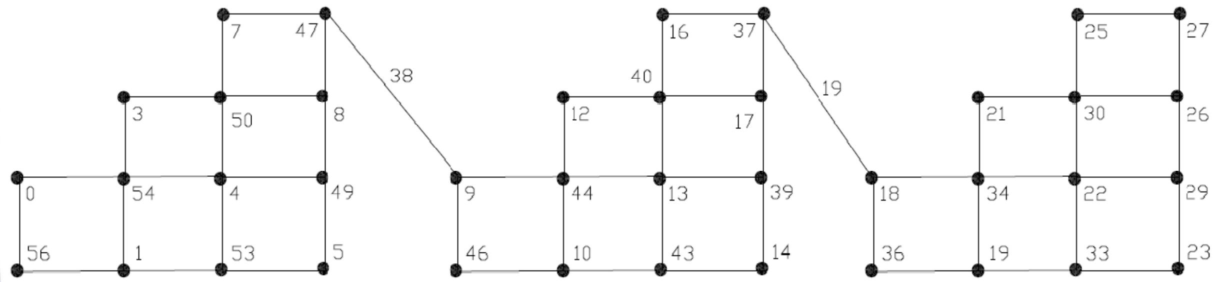

Figure -2 A Path union of 3 copies of St_4 and its graceful labeling.

Theorem -2.5 **:** Cycle of step grid graph $C(r \cdot St_n)$ is graceful, where $n \geq 3$ and $r \equiv 0,3$ (mod 4) is graceful.

Proof : Let $G = C(r \cdot St_n)$ be a cycle of step grid graph St_n with r copies. Let f be the function for graceful labeling of St_n as we mentioned in $Theorem-2.1$.

In graph G, we see that the vertices $|V(G)| = \frac{r}{2}(n^2+3n-2) = p$ and the edges $|E(G)| = r(n^2+n-1) = Q$. Let $u_{i,j} (1 \leq i, j \leq n)$ be the vertices of k^{th} copy of St_n, $\forall \ k = 1,2,\ldots,r$. Where the vertices of k^{th} copy of St_n is $p = \frac{1}{2}(n^2+3n-2)$ and edges of k^{th} copy of St_n is $q = n^2+n-2$. Join the vertices $u_{k,1,1}$ with $u_{k+1,1,1}$ for $k = 1,2,\ldots,r-1$ and $u_{r,1,1}$ with $u_{1,1,1}$ by an edge.

We define labeling function $g : V(G) \to \{0,1,\ldots,Q\}$ as follows

$$g(u_{1,i,j}) = f(u_{i,j}) \qquad \text{if } f(u_{i,j}) < \frac{q}{2};$$

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$$= f(u_{i,j}) + (Q-q) \quad \text{if } f(u_{i,j}) > \frac{q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$g(u_{2,i,j}) = g(u_{1,i,j}) + (Q-q) \quad \text{if } g(u_{1,i,j}) < \frac{Q}{2};$$

$$= g(u_{1,i,j}) - (Q-q) \quad \text{if } g(u_{1,i,j}) > \frac{Q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$g(u_{k,i,j}) = g(u_{k-2,i,j}) - (q+1) \quad \text{if } g(u_{k-2,i,j}) > \frac{Q}{2};$$

$$= g(u_{k-2,i,j}) + (q+1) \quad \text{if } g(u_{k-2,i,j}) < \frac{Q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$\forall \ k = 3,4,\ldots,\lceil \frac{r}{2} \rceil,$$

$$g(u_{\lceil \frac{r}{2} \rceil+1,i,j}) = g(u_{\lceil \frac{r}{2} \rceil-1,i,j}) + (q+2) \quad \text{if } g(u_{\lceil \frac{r}{2} \rceil-1,i,j}) < \frac{Q}{2};$$

$$= g(u_{\lceil \frac{r}{2} \rceil-1,i,j}) - (q+1) \quad \text{if } g(u_{\lceil \frac{r}{2} \rceil-1,i,j}) > \frac{Q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$g(u_{\lceil \frac{r}{2} \rceil+2,i,j}) = g(u_{\lceil \frac{r}{2} \rceil,i,j}) + (q+2) \quad \text{if } g(u_{\lceil \frac{r}{2} \rceil,i,j}) < \frac{Q}{2};$$

$$= g(u_{\lceil \frac{r}{2} \rceil,i,j}) - (q+1) \quad \text{if } g(u_{\lceil \frac{r}{2} \rceil,i,j}) > \frac{Q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$g(u_{k,i,j}) = g(u_{k-2,i,j}) - (q+1) \quad \text{if } g(u_{k-2,i,j}) > \frac{Q}{2};$$

$$= g(u_{k-2,i,j}) + (q+1) \quad \text{if } g(u_{k-2,i,j}) < \frac{Q}{2};$$

$$\forall \ k = \lceil \frac{r}{2} \rceil + 3, \lceil \frac{r}{2} \rceil + 4, \ldots, r.$$

Above labeling patten give rise a graceful labeling of given Cycle of step grid graph G.

Illustration -2.6 **:** $C(4 \cdot St_4)$ and its graceful labeling shown in *figure* -3.

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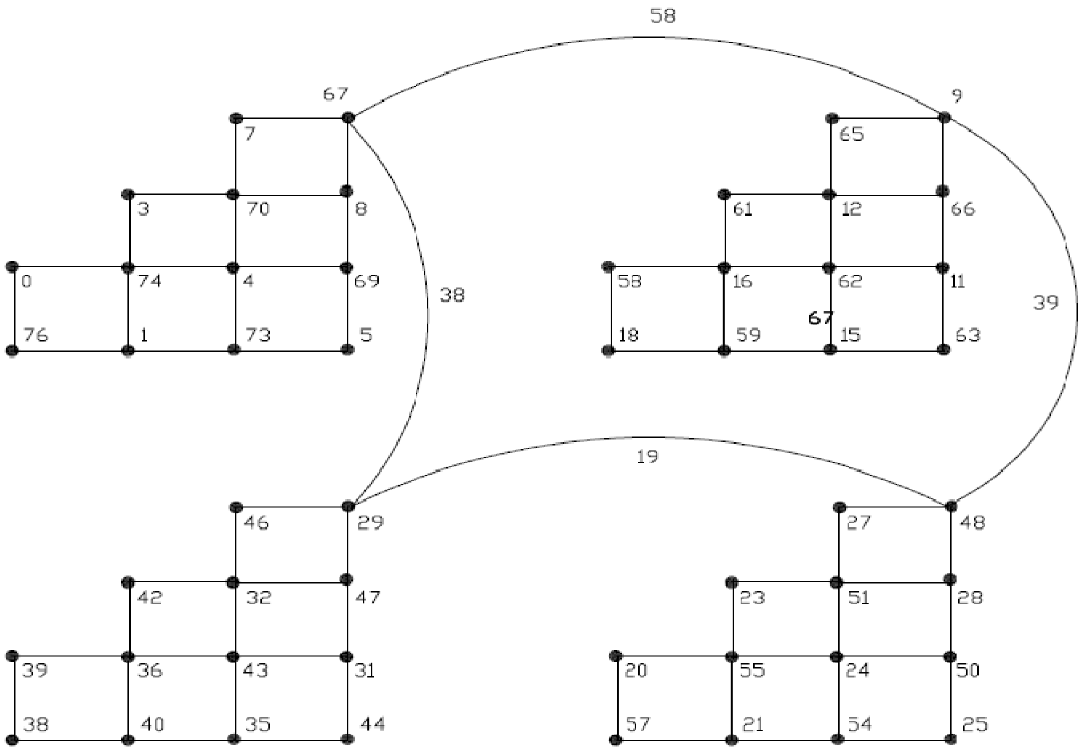

Figure -3 A cycle of St_4 with 4 copies and its graceful labeling.

Theorem -2.7 **:** Star of the step grid graph $(St_n)^*$ is graceful,where $n \geq 3$.

Proof : Let $G = (St_n)^*$ be a star of step grid graph St_n, $n \geq 3$. let f be the graceful labeling function of St_n as we mention in $Theorem-2.1$.

In graph G, we see that the vertices $|V(G)| = p(p+1) = P$ and the edges $|E(G)| = (p+1)(q+1)-1 = Q$, where $p = \frac{1}{2}(n^2+3n-2)$ and $q = n^2+n-2$. Let $u_{i,j}(1 \leq i, j \leq n)$ be the vertices of k^{th} copy of St_n, $\forall$ $k = 1,2,\ldots,p$. Where the vertices of k^{th} copy of St_n is $p = \frac{1}{2}(n^2+3n-2)$ and edges of k^{th} copy of St_n is $q = n^2+n-2$.

We mention that central copy of $(St_n)^* = G$ is $(St_n)^{(0)}$ and other copies of $(St_n)^* = G$ is $(St_n)^{(k)}$, $\forall$ $k = 1,2,\ldots,p$.

We define labeling function $g : V(G) \rightarrow \{0,1,\ldots,Q\}$ as follows

$$g(u_{0,i,j}) = f(u_{i,j}) \qquad \text{if } f(u_{i,j}) < \frac{q}{2};$$

$$= f(u_{i,j}) + (Q-q) \qquad \text{if } f(u_{i,j}) > \frac{q}{2}, \forall \ i, j = 1,2,\ldots,n$$

$$g(u_{1,i,j}) = g(u_{0,i,j}) + p(q+1) \qquad \text{if } g(u_{0,i,j}) < \frac{Q}{2};$$

$$= g(u_{0,i,j}) - p(q+1) \qquad \text{if } g(u_{0,i,j}) > \frac{Q}{2}, \forall \; i,j = 1,2,\ldots,n$$

$$g(u_{k,i,j}) = g(u_{k-2,i,j}) + (q+1) \qquad \text{if } g(u_{k-2,i,j}) < \frac{Q}{2};$$

$$= g(u_{k-2,i,j}) - (q+1) \qquad \text{if } g(u_{k-2,i,j}) > \frac{Q}{2}, \forall \; i,j = 1,2,\ldots,n,$$

$\forall \; k = 2,3,\ldots,p$.

We see that difference of vertices for the central copy $(St_n)^{(0)}$ of G and its other copies $(St_n)^{(k)}$ $(1 \le k \le p)$ is precisely following sequence

$$p(q+1)$$
$$(q+1)$$
$$(p-1)(q+1)$$
$$\vdots$$
$$\lfloor \frac{p}{2} \rfloor (q+1).$$

Using this sequence we can produce required edge label by joining corresponding vertices of $(St_n)^{(0)}$ with its other copy $(St_n)^{(k)}$ $(1 \le k \le p)$ in G. Thus G admits graceful labeling.

Illustration- 2.8 : Star graph of St_3 and its graceful labeling shown in *figure*-4.

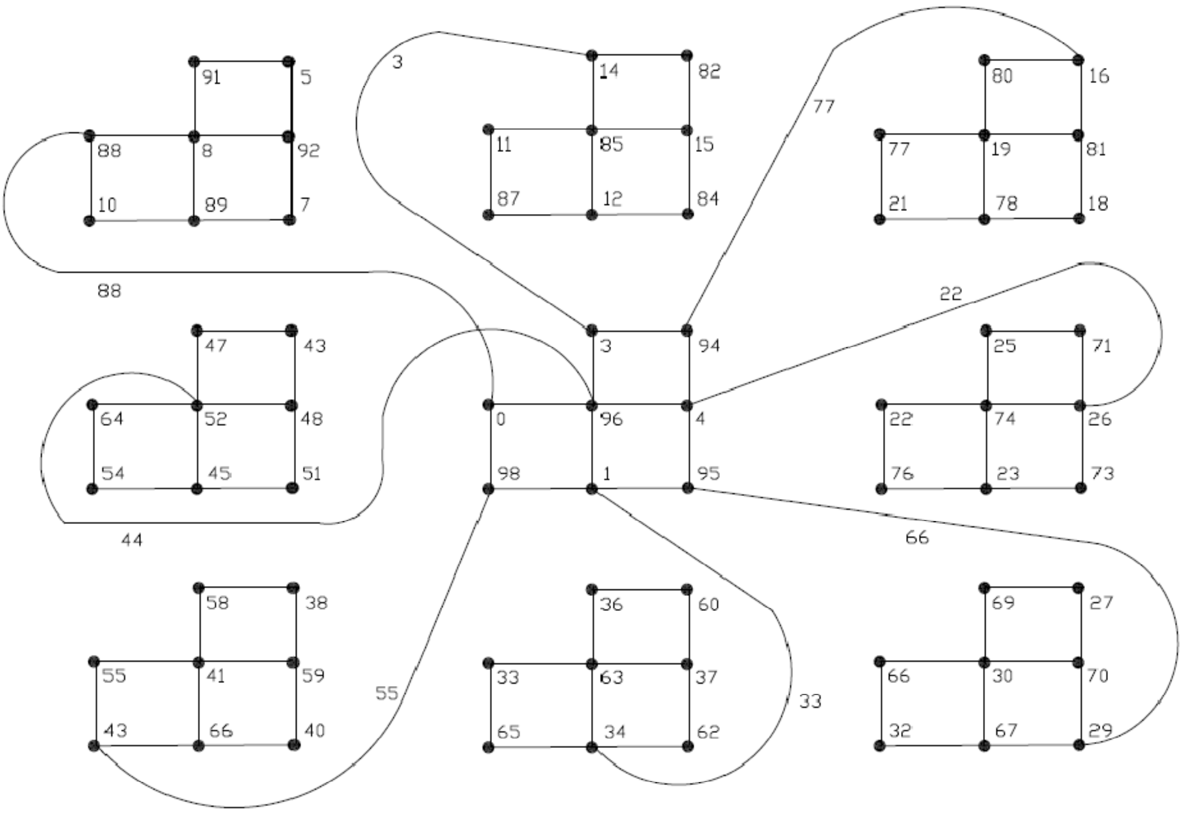

Figure-4 A star graph of St_3 and its graceful labeling.

2. Concluding Remarks:

Here we introduced a new graph is called step grid graph. Present work contributes four new results. We discussed gracefulness of step grid graphs, path union of step grid graphs, cycle of step grid graphs and star of step grid graphs. The labeling patten is demonstrated by means of illustrations which provide better understanding of derived results.

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