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BAYESIAN TWO-SAMPLE PREDICTION OF THE GENERALIZED PARETO DISTRIBUTION WITH FIXED AND RANDOM SAMPLE SIZES BASED ON GENERALIZED ORDER STATISTICS

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ABSTRACT

Bayesian predictive intervals for future observations from a future sample from the generalized Pareto distribution (GPD) based on generalized order statistics (GOS) are obtained when the shape parameter θ is unknown. We consider two cases: (i) fixed sample size (FSS), and (ii) random sample size (RSS).Some closed forms for the Bayesian predictive functions are obtained. Finally examples are calculated for the lower and the upper bounds of the future observations in cases when the future sample is ordinary order statistics (OOS), record values and progressive type II censoring with different values for the scale parameter σ.

Indexing terms/Keywords

Two-samples Bayesian prediction, generalized order statistics, generalized Pareto distribution, ordinary order statistics, progressive type II censoring, random sample size.

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1. INTRODUCTION

Generalized Pareto distribution (GPD) was firstly introduced by Pikands [1]. After that it has been applied to many areas including physical and biological processes, reliability studies and the analysis of environmental extremes, estimation of the stable index *α* to measure tail thickness, modeling the peaks over a threshold stream flows and rainfall series and analyzing flood frequencies and wave heights and others. Considering a random variable *X* has GPD, the cumulative distribution function of *X* can be expressed as

$$F(x;\sigma,\theta)=1-\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}} \qquad ;\theta>0, 0<x<\sigma, \tag{1}$$

and the probability density function (pdf) of GPD is given by

$$f(x;\sigma,\theta)=\frac{1}{\sigma\theta}\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}-1} \qquad ;\theta>0, 0<x<\sigma, \tag{2}$$

where θ is the shape parameter and σ is the scale parameter. See [2].

In many practical problems such as medicine, engineering and business, observers wish to use previous data to predict a future observation from the same population. Due to this, prediction bounds for future observations have been discussed by many authors, [3-9] among others.

Suppose that $X_{1,n,\tilde{m},k_n}, X_{2,n,\tilde{m},k_n}, \ldots, X_{r,n,\tilde{m},k_n}$, $k_n>0,$ are the first *r* GOS failure times from a GOS sample of size *n* drawn from GPD, whose pdf is given by (1) where $\tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in\Re^{n-1}$, and let $Z_{1,N,\tilde{M},K}, Z_{2,N,\tilde{M},K},\ldots, Z_{N,N,\tilde{M},K}, K>0$, be a second independent GOS sample of size *N* of future observations from the same distribution where $\tilde{M}=(M_1,M_2,\ldots,M_{N-1})\in\Re^{N-1}$.Let Z_s denotes the s[th] GOS in the future sample where $1\le s\le N$, our aim is to predict bounds of Z_s based on the observed sample in cases of the sample size N is fixed and a random variable distributed as (i) Poisson distribution (ii) binomial distribution. The pdf of the GOS Z_s was introduced by Kamps [10] in case of $M_1=M_2=\ldots=M_{N-1}=M$ for *M* = -1 and *M* ≠ -1 then for $M_i\neq M_j, i\neq j$ by Kamps and Gather [11]

$$\varphi(z_s\,|\,\theta,\sigma)=\begin{cases}\dfrac{K^s}{(s-1)!}\left[1-F(z_s)\right]^{K-1}f(z_s)\left[-\ln(1-F(z_s))\right]^{s-1}, & when\ M=-1.\\[2ex] \dfrac{C^*_{s-1}}{(s-1)!}\left[1-F(z_s)\right]^{\gamma^*_s-1}f(z_s)\left[\dfrac{-(1-F(z_s))^{M+1}}{M+1}+\dfrac{1}{M+1}\right]^{s-1}, & when\ M\neq-1.\\[2ex] C^*_{s-1}f(z_s)\sum_{i=1}^{s}a_{i,s}(1-F(z_s))^{\gamma^*_i-1}, & M_i\neq M_j,\end{cases} \tag{3}$$

where $C^*_{s-1}=\prod_{i=1}^{s}\gamma^*_i$, $\gamma^*_i=K+N-i+\sum_{j=i}^{N-1}M_i$ and $a_{i,s}=\prod_{\substack{j=1\\ j\neq i}}^{s}\frac{1}{\gamma^*_j-\gamma^*_i}, 1\le i\le N-1.$ (4)

The likelihood function of the given *r* GOS is given by

$$L(\sigma,\theta;x)=C_{r-1}\left[\prod_{i=1}^{r-1}(1-F(x_i))^{m_i}f(x_i)\right]\left[(1-F(x_r))^{\gamma_r-1}f(x_r)\right]. \tag{5}$$

See [10], where $f(x_i)$ and $F(x_i)$ are the pdf and the cdf of X_i , and

$$C_{j-1}=\prod_{i=1}^{j}\gamma_i,\quad \gamma_i=k_n+n-i+\sum_{j=i}^{n-1}m_i \ \text{ for } r=1,2,\ldots,n-1 . \tag{6}$$

Substituting (1) and (2) into (5), we get

$$L(\sigma,\theta;x)=C_{r-1}\eta(\sigma;x)(\sigma\theta)^{-r}\,e^{\frac{T}{\theta}}, \tag{7}$$

where

$$\eta(\sigma;x)=\prod_{i=1}^{i=r}(1-\frac{x_i}{\sigma})^{-1}\ and\ T\equiv T(\sigma;x)=\sum_{i=1}^{i=r-1}(m_i+1)\ln(1-\frac{x_i}{\sigma})+\gamma_r\ln(1-\frac{x_r}{\sigma})\ . \tag{8}$$

Assuming that the shape parameter *θ* is the only unknown parameter, and *θ* has the inverted gamma distribution defined by

$$g(\theta)=\begin{cases}\dfrac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\frac{\beta}{\theta}}, & if\quad \theta>0,\ (\alpha,\beta)>0\\ 0, & o.w,\end{cases} \tag{9}$$

where *α* and *β* are assumed to be known. Using (7), the posterior distribution of θ can be obtained as

$$f(\theta\,|\,\underset{-}{x})=\frac{(\beta-T)^{\alpha+r}}{\Gamma(\alpha+r)}\theta^{-(r+\alpha+1)}e^{-\frac{\beta-T}{\theta}}. \tag{10}$$

2. PREDICTION BOUNDS WHEN THE SAMPLE SIZE IS FIXED

In this section Bayes predictive density function $\hbar(z_s|\underline{x})$ for the future observation z_s from the future sample with fixed sample size *N* is obtained in three cases of $\tilde{M}$.

THEORM 2.1 *Suppose* $\underline{X}=(X_{1,n,\tilde{m},k_n},X_{2,n,\tilde{m},k_n},...,X_{r,n,\tilde{m},k_n}),\ k_n>0,$ *is a known GOS sample and* $\underline{Z}=(Z_{1,N,\tilde{M},K},Z_{2,N,\tilde{M},K},...,Z_{N,N,\tilde{M},K}),\ K>0$ *is a future independent GOS sample both are from the same population having the GPD, Then the 100τ% Bayesian prediction bounds for the* s^{th} *GOS* Z_s, *1≤ s≤ N are obtained by solving the following equations with respect to t:*

$$\frac{1\pm\tau}{2}=\begin{cases}R_{K,s,n}\left[\dfrac{-Y^{s-1}\left[\beta-T-KY\right]^{1-(s+\alpha+r)}}{K(s+\alpha+r-1)}+\dfrac{(s-1)Y^{s-2}\left[\beta-T-KY\right]^{2-(s+\alpha+r)}}{K^2(s+\alpha+r-1)(s+\alpha+r-2)}+\right.\\ \qquad\left.\ldots+\dfrac{(-1)^s(s-1)!\left[\beta-T-KY\right]^{-(\alpha+r)}}{K^s(s+\alpha+r-1)(s+\alpha+r-2)\ldots(\alpha+r)}\right], & M=-1\\ \dfrac{C^*_{s-1}(\beta-T)^{\alpha+r}}{(s-1)!\left[M+1\right]^{s-1}}\displaystyle\sum_{j=0}^{s-1}b_j(s)\dfrac{\left(\beta-T-(\gamma^*_s+(M+1)j)Y\right)^{-(r+\alpha)}}{(\gamma^*_s+(M+1)j)}, & M\neq-1\\ C^*_{s-1}(\beta-T)^{\alpha+r}\displaystyle\sum_{i=1}^{s}\dfrac{a_{i,s}}{\gamma^*_i}\left(\beta-T-\gamma^*_iY\right)^{-(r+\alpha)}, & M_i\neq M_j,i\neq j,\end{cases} \tag{11}$$

where $$R_{K,s,n}=\frac{(-K)^s(\beta-T)^{\alpha+r}\Gamma(s+r+\alpha)}{\Gamma(\alpha+r)(s-1)!}\ \ and\ \ Y=\ln\left(1-\frac{t}{\sigma}\right). \tag{12}$$

Proof Substituting (1) and (2) into (3), the pdf of the GOS Z_s reduces to

$$\varphi(z_s|\theta,\sigma)=\begin{cases}\dfrac{K^s}{(s-1)!\sigma\theta}\left[1-\dfrac{z_s}{\sigma}\right]^{\frac{K-\theta}{\theta}}\left[-\dfrac{A}{\theta}\right]^{s-1}, & M=-1\\ \dfrac{C^*_{s-1}}{\sigma(s-1)![M+1]^{s-1}}\left[1-\dfrac{z_s}{\sigma}\right]^{\frac{\gamma^*_s}{\theta}-1}\dfrac{1}{\theta}\left[1-(1-\dfrac{z_s}{\sigma})^{\frac{M+1}{\theta}}\right]^{s-1}, & M\neq -1\\ \dfrac{C^*_{s-1}}{\sigma\theta}\left[1-\dfrac{z_s}{\sigma}\right]^{\frac{1}{\theta}-1}\sum_{i=1}^{s}a_{i,s}\left[1-\dfrac{z_s}{\sigma}\right]^{\frac{\gamma^*_i-1}{\theta}}, & M_i\neq M_j,\end{cases}$$

(13)

where $A=\ln(1-\frac{z_s}{\sigma})$.

Substituting (10) and (13), noting that, for the case when $M \neq -1$ using the binomial expansion, the Bayesian predictive density function of the future GOS Z_s takes the form

$$\psi(z_s\,|\,\underline{x})=\int_0^\infty \varphi(z_s\,|\,\sigma,\theta)\,f(\theta\,|\,\underline{x})d\theta$$

$$=\begin{cases}\dfrac{K^s(\beta-T)^{\alpha+r}\Gamma(s+r+\alpha)}{(s-1)!\Gamma(\alpha+r)\sigma}\dfrac{[-A]^{s-1}e^{-A}}{(\beta-T-KA)^{s+\alpha+r}}, & M=-1\\ \dfrac{C^*_{s-1}(\alpha+r)}{\sigma(s-1)![M+1]^{s-1}}\left[1-\dfrac{z_s}{\sigma}\right]^{-1}\sum_{j=0}^{s-1}\dfrac{b_j(s)(\beta-T)^{\alpha+r}}{\left(\beta-T-(\gamma^*_s+(M+1)j)A\right)^{(r+\alpha+1)}}, & M\neq -1\\ C^*_{s-1}\dfrac{(\alpha+r)(\beta-T)^{\alpha+r}}{\sigma}\sum_{i=1}^{s}\dfrac{a_{i,s}e^{A}}{(\beta-T-A\gamma^*_i)^{(r+\alpha+1)}}, & M_i\neq M_j,\end{cases}$$

(14)

Let *L*(*x*) and *U*(*x*) are the lower and the upper limits of the 100τ% Bayesian prediction interval for the GOS z_s such that *pr* [$L\leq z_s\leq U$] $=\tau$ where,

$$\Pr(z_s\geq L\,|\,\underline{x})=\frac{1+\tau}{2}\quad and\ \Pr(z_s\geq U\,|\,\underline{x})=\frac{1-\tau}{2},\qquad (15)$$

The Bayesian prediction bounds for the GOS Z_s are obtained by evaluating $\Pr(z_s\geq t\,|\,\underline{x})=\int_t^\sigma\psi(z_s\,|\,\underline{x})dz_s$ for some value of *t* and so the required result follows. ■

2.1 Bayesian prediction bounds when the future sample is a record sample

We can get Bayesian prediction bounds for the upper record z_s from the future upper record sample when the given sample assumed to be a GOS sample by putting *K=1* in equation (11).

$$\frac{(\beta-T)^{\alpha+r}(-1)^S\Gamma(s+r+\alpha)}{\Gamma(\alpha+r)(s-1)!}\left[\frac{-Y^{s-1}(\beta-T-Y)^{1-(s+\alpha+r)}}{(s+\alpha+r-1)}\right.$$
$$\left.+\frac{(s-1)Y^{s-2}(\beta-T-Y)^{2-(s+\alpha+r)}}{(s+\alpha+r-1)(s+\alpha+r-2)}+\ldots+\frac{(-1)^S(s-1)!(\beta-T-Y)^{-(\alpha+r)}}{(s+\alpha+r-1)(s+\alpha+r-2)\ldots(\alpha+r)}\right]=\frac{1\pm\tau}{2}.\qquad (16)$$

2.2 Bayesian prediction bounds when the future sample is OOS.

Given a GOS sample and assuming that the future sample is OOS sample, Then by putting *K=1* and *M=0*, (11) reduce to

$$s\binom{N}{s}(\beta-T)^{\alpha+r}\sum_{j=0}^{s-1}b_j(s)\frac{\left(\beta-T-(N-s+j+1)Y\right)^{-(r+\alpha)}}{(N-s+j+1)}=\frac{1\pm\tau}{2}. \qquad (17)$$

2.3 Bayesian prediction bounds when future progressively type-II censored sample

Bayesian prediction bounds of the progressively Type-II censored order statistic z_s can be obtained by setting $M_i = R_i$ for $1 \le i \le l-1$, $l \in \{1,2,...,N-1\}$ and $\gamma_l = R_l + 1$, $s \in \{1,2,...,l\}$ in (11),see [11, 12].

$$D^{\bullet}_{s-1}(\beta-T)^{\alpha+r}\sum_{i=1}^{s}\frac{a^{\bullet}_{i,s}}{\delta_i}\left(\beta-T-\delta_i Y\right)^{-(r+\alpha)}=\frac{1\pm\tau}{2}, \qquad (18)$$

where $\delta_i = \sum_{j=i}^{l}(R_j+1) = N-\sum_{j=1}^{i-1}(R_j+1),\ D^{\bullet}_{s-1}=\prod_{i=1}^{s}\delta_i \ and \ a^{\bullet}_{i,s}=\prod_{\substack{j=1\\ j\neq i}}^{s}\frac{1}{\delta_j-\delta_i}$. (19)

3. PREDICTION BOUNDS WHEN THE SAMPLE SIZE IS A RANDOM VARIABLE

Assuming the sample size *N* is a random variable, the predictive density function of z_s is given by

$$q(z_s|\underline{x},N)=\frac{1}{\Pr(N\ge s)}\sum_{N=s}^{\infty}v(N)\psi(z_s|\underline{x}), \qquad (20)$$

where $v(N)$ is the probability mass function (pmf) of N and $\psi(z_s|\underline{x})$ is Bayes predictive density function of the future GOS z_s, $s=1,2,\ldots,N$, see [13].

Remark 3.1 When *K* is independent of the sample size *N*, the 100τ% Bayesian prediction bounds for *the* s^{th} GOS Z_s when *M =-1* are independent of the sample size too. Hence Bayesian prediction bounds for the record value Z_s which obtained in (16) will have the same formula when the sample size is a random variable.

3.1 When N has a Poisson distribution

Here, setting the sample size *N* as a random variable having a Poisson distribution with pmf given by

$$p(N;\lambda)=\frac{e^{-\lambda}\lambda^N}{N!}, \qquad N=0,1,2,\ldots and\ \lambda>0. \qquad (21)$$

Replacing *v*(*N*) in (20) by *p(N; λ)* defined in (21)

$$q(z_s|\underline{x},N)=\frac{1}{e^{\lambda}-\sum_{N=0}^{s-1}\frac{\lambda^N}{N!}}\sum_{N=s}^{\infty}\frac{\lambda^N}{N!}\psi(z_s|\underline{x}). \qquad (22)$$

THEORM 3.1.1 *Suppose* $\underline{X}=(X_{1,n,\tilde{m},k_n},X_{2,n,\tilde{m},k_n},\ldots,X_{r,n,\tilde{m},k_n})$, $k_n>0$, *is a known GOS sample and an unknown independent GOS sample with unknown sample size N distributed as a Poisson distribution* $\underline{Z}=(Z_{1,N,\tilde{M},K},Z_{2,N,\tilde{M},K},\ldots,Z_{N,N,\tilde{M},K})$, $K>0$ *both are from the same population GPD, Then the 100τ% Bayesian prediction bounds for the* s^{th} *GOS* Z_s, *1≤ s≤ N are obtained by solving the following equations with respect to t:*

$$\frac{1\pm\tau}{2}=\begin{cases}\frac{\Gamma(s1)P_1}{\Gamma(\alpha+r)(s-1)!}\sum_{N=s}^{\infty}\frac{\lambda^N(-K)^s\beta_1^{\alpha+r}}{N!}\left[\frac{-Y^{s-1}[\beta_1-KY]^{1-(s+\alpha+r)}}{K(s1-1)}\right. \\ \left.+\frac{(s-1)Y^{s-2}[\beta_1-KY]^{2-(s+\alpha+r)}}{K^2(s1-1)(s1-2)}+\ldots+\frac{(-1)^S(s-1)![\beta_1-KY]^{-(\alpha+r)}}{K^s(s1-1)(s1-2)\ldots(\alpha+r)}\right], & M=-1 \\ P_1\sum_{N=s}^{\infty}\left[\frac{\lambda^N}{N!}\frac{C^*_{s-1}\beta_1^{\alpha+r}}{(s-1)![M+1]^{s-1}}\sum_{j=0}^{s-1}b_j(s)\frac{\left(\beta_1-(\gamma_s^*+(M+1)j)Y\right)^{-(r+\alpha)}}{\left(\gamma_s^*+(M+1)j\right)}\right], & M\neq-1 \\ P_1\sum_{N=s}^{\infty}\left(\frac{\lambda^N C^*_{s-1}\beta_1^{\alpha+r}}{N!}\sum_{i=1}^{s}\frac{a_{i,s}}{\gamma_i^*\left(\beta_1-\gamma_i^*Y\right)^{(r+\alpha)}}\right), & M_i\neq M_j,\end{cases}$$

(23)

where

$$s1=s+r+\alpha,\quad P_1=\frac{1}{\left(e^{\lambda}-\sum_{N=0}^{s-1}\frac{\lambda^N}{N!}\right)}\quad\text{and}\quad \beta_1=\beta-T. \tag{24}$$

Proof Substituting $\psi(z_s|x)$ obtained in (14) into (22) and then evaluating $\Pr(z_s>t|x)$ for some t and hence the theorem follows. ∎

3.1.2 Bayesian prediction when the future sample is OOS

The Bayes predictive bounds of the future OOS z_s when the sample size is distributed as a Poisson distribution is obtained by putting $K=1$ and $M=0$ in (23)

$$\Pr(z_s\geq t\,|\,\underset{-}{x})=\frac{1}{e^{\lambda}-\sum_{N=0}^{s-1}\frac{\lambda^N}{N!}}\sum_{N=s}^{\infty}\left[\frac{\lambda^N}{(N-s)!}\frac{(\beta-T)^{\alpha+r}}{(s-1)!}\sum_{j=0}^{s-1}b_j(s)\frac{\left(\beta-T-(N_s+j)Y\right)^{-(r+\alpha)}}{(N_s+j)}\right], \tag{25}$$

where $N_s = N - s+1$. (26)

3.1.3 Bayesian prediction bounds of progressively type-II censored order statistics

For progressively Type-II censored order statistic sample when random sample size N is having a Poisson distribution, we have

$$\Pr(z_s\geq t\,|\,\underset{-}{x})=\frac{1}{e^{\lambda}-\sum_{N=0}^{s-1}\frac{\lambda^N}{N!}}\sum_{N=s}^{\infty}\left(\frac{\lambda^N D^{\bullet}_{s-1}(\beta-T)^{\alpha+r}}{N!}\sum_{i=1}^{s}\frac{a_{i,s}}{\delta_i\left(\beta-T-\delta_i Y\right)^{(r+\alpha)}}\right). \tag{27}$$

3.2 When N has a binomial distribution

Here we assume that, the sample size N has a binomial distribution $b(N;N^*;p)$ with pmf given by

$$b(N;N^*;\mathrm{p})=\binom{N^*}{\mathrm{N}}p^N(1-p)^{N^*-N},\quad N=0,1,\ldots N^*. \tag{28}$$

Replacing $v(N)$ in (20) by $b(N;N^*;p)$ defined in (28), we get

$$q(z_s|\underline{x},N)=P_2\sum_{N=s}^{N^*}\binom{N^*}{N}p^N(1-p)^{N^*-N}\psi(z_s|\underline{x}). \quad (29)$$

where $$P_2=\frac{1}{1-\sum_{N=0}^{s-1}\binom{N^*}{N}p^N(1-p)^{N^*-N}}$$

THEORM 3.2.1 *Suppose* $\underline{X}=(X_{1,n,\tilde{m},k_n},X_{2,n,\tilde{m},k_n},...,X_{r,n,\tilde{m},k_n})$, $k_n>0$, *is a known GOS sample and an unknown independent GOS sample with unknown sample size N distributed as a binomial distribution* $\underline{Z}=(Z_{1,N,\tilde{M},K},Z_{2,N,\tilde{M},K},...,Z_{N,N,\tilde{M},K})$, $K>0$ *the two samples are from the same population GPD, Then the 100τ% Bayesian prediction bounds for the* s^{th} *GOS* Z_s, *1≤ s≤ N are obtained by solving the following equations with respect to t:*

$$\frac{1\pm\tau}{2}=\begin{cases}\frac{\Gamma(s1)P_2}{\Gamma(\alpha+r)(s-1)!}\sum_{N=s}^{\infty}\left(b(N;N^*;\mathrm{p})(-K)^s\beta_1^{\alpha+r}\left[\frac{-Y^{s-1}[\beta_1-KY]^{1-(s+\alpha+r)}}{K(s1-1)}\right.\right.\\ \left.\left.+\frac{(s-1)Y^{s-2}[\beta_1-KY]^{2-(s+\alpha+r)}}{K^2(s1-1)(s1-2)}+...+\frac{(-1)^S(s-1)![\beta_1-KY]^{-(\alpha+r)}}{K^s(s1-1)(s1-2)...(\alpha+r)}\right]\right), & M=-1\\ P_2\sum_{N=s}^{\infty}\left[\frac{b(N;N^*;\mathrm{p})C_{s-1}^*\beta_1^{\alpha+r}}{(s-1)![M+1]^{s-1}}\sum_{j=0}^{s-1}b_j(s)\frac{\left(\beta_1-(\gamma_s^*+(M+1)j)Y\right)^{-(r+\alpha)}}{\left(\gamma_s^*+(M+1)j\right)}\right], & M\neq-1\\ P_2\sum_{N=s}^{\infty}\left(b(N;N^*;\mathrm{p})C_{s-1}^*\beta_1^{\alpha+r}\sum_{i=1}^{s}\frac{a_{i,s}}{\gamma_i^*\left(\beta_1-\gamma_i^*Y\right)^{(r+\alpha)}}\right), & M_i\neq M_j,\end{cases}$$

(30)

Proof Substituting ψ(z_s|x) obtained in (14) into (29) and then evaluating Pr(z_s> t|x) for some t and hence the theorem follows. ■

3.2.2 Bayesian prediction when the future sample is OOS

The Bayes predictive bounds of the future OOS z_s when the sample size is distributed as a binomial distribution is obtained by putting *K=1* and *M=0* in (30)

$$\Pr(z_s\geq t\mid\underline{x})=\frac{1}{1-\sum_{N=0}^{s-1}\binom{N^*}{N}p^N(1-p)^{N^*-N}}\sum_{N=s}^{N^*}\left[\frac{N!}{(N-s)!}\frac{(\beta-T)^{\alpha+r}}{(s-1)!}\binom{N^*}{N}\right.$$
$$\left.\times p^N(1-p)^{N^*-N}\sum_{j=0}^{s-1}b_j(s)\frac{(\beta-T-(N_s+j)Y)^{-(r+\alpha)}}{(N_s+j)}\right]. \quad (31)$$

3.2.3 Bayesian Prediction Bounds of progressively Type-II censored order statistics

For progressively Type-II censored order statistic z_s when the random sample size *N* having a binomial distribution, the Bayes predictive bounds for some *t*, will be

$$\Pr(z_s\geq t\mid\underline{x})=\frac{1}{\sum_{N=s}^{N^*}\binom{N^*}{N}p^N(1-p)^{N^*-N}}\sum_{N=s}^{N^*}\left(D_{s-1}^{\bullet}\binom{N^*}{N}\sum_{i=1}^{s}\frac{a_{i,s}(\beta-T)^{\alpha+r}p^N(1-p)^{N^*-N}}{\delta_i\left(\beta-T-\delta_iY\right)^{(r+\alpha)}}\right). \quad (32)$$

4. NUMERICAL ANALYSIS

In this section, Bayesian two-sample prediction bounds for the s^{th} future observation in samples of ordered statistic, progressively type-II censored and upper record samples with fixed and random sample sizes are obtained when the given sample is an ordinary ordered statistics.

(1) For arbitrary given values of the prior parameters (α, β) = (1, 0.1), we generate θ using (7).

(2) Using generated θ = 0.209633 obtained in step (1), we generate two samples of order statistics from GPD given in (1) with σ = 0.1 and 0.5 and sample size n = 20. The two generated samples are {0.00508552, 0.0126332, 0.0156405, 0.0172508, 0.026927, 0.0300092, 0.048717, 0.0490063, 0.0495334, 0.0642638, 0.0669125, 0.0743327, 0.0773543, 0.0848635, 0.0955009, 0.108275, 0.112579, 0.118333, 0.161378, 0.205822} and {0.00049019, 0.0048263, 0.00487353, 0.00513226, 0.00530827, 0.00631538, 0.00753004, 0.00824338, 0.00830022, 0.00887286, 0.0092671, 0.0103991, 0.0134561, 0.014651, 0.0199454, 0.0201259, 0.0236925, 0.0290986, 0.031884, 0.0345258}.

(3) Taking the number of observed order statistics is r = 15 and 17,using Mathematica-8 via the routine NSolve the 90% and 95% BPI for the future OOS observation $z_{s:N}$, s=1, 2 and 3 by evaluating (17) with respevt to t, are obtained and are shown in Table 1, the future progressive type II order statistic $z_{s:N}$, s=1, 2 by evaluating (18), these are given in Table 5 and the future upper record observation $z_{s:N}$, s=1, 2 by evaluating (16) ,The results are shown in Table 3.

(4) For sample size N having Poisson distribution with λ = *0.65*, we construct 90% and 95% predictive intervals for the future OOS observation $z_{s:N}$, s=1, 2 by (25), Table 2 represents these results and the future observation based on progressive type-II censoring statistics $z_{s:N}$, s=1, 2 by evaluating (27), these are given in Table 6.

5. With sample size N having binomial distribution $b(N;7;0.6)$, the 90% and 95% predictive intervals for the future observation OOS observation $z_{s:N}$, s=1, 2, were obtained and shown in Table 4 and the future observation based on progressive type II censoring statistics $z_{s:N}$, s=1, 2 by evaluating (32), Table 7 shows these results.

Remark 4.1 From Tables 1- 7, it is noticed that the width of the Bayesian predictive interval increases by increasing σ and s and decreases by increasing the number of known observations r.

Table 1 : BPI of the OOS Z_s in case of fixed sample size

r	s	σ =0.5		σ =0.1	
		90% L,U Width	**95% L,U width**	**90% L,U width**	**95% L,U width**
15	1	0.00043297, 0.027019 0.026586	0.00021358, 0.033784 0.03357	0.000082, 0.005145 0.005063	0.000041, 0.006436 0.006395
	2	0.00308827, 0.04534 0.042251	0.00209962, 0.054177 0.052077	0.00058741, 0.008642 0.008055	0.00039934, 0.010332 0.009932
	3	0.00737882, 0.063683 0.056304	0.00555868, 0.074416 0.068857	0.00140379, 0.012151 0.010747	0.00105742, 0.014207 0.013149
17	1	0.00041399, 0.025602 0.025188	0.00020424, 0.031944 0.031739	0.00007872, 0.004874 0.004795	0.00003884, 0.006083 0.006045
	2	0.002960, 0.042905 0.039945	0.002013, 0.051149 0.049135	0.00056288, 0.008176 0.007613	0.00038279, 0.009751 0.009368
	3	0.00708479, 0.060237 0.053153	0.00534018, 0.070222 0.064882	0.00134755, 0.011489 0.010141	0.00101563, 0.013401 0.012385

Table 2: BPI of OOS Z_s in case of N having Poisson distribution

r	s	σ =0.5 90% L,U Width	σ =0.5 95% L,U width	σ =0.1 90% L,U width	σ =0.1 95% L,U width
15	1	0.00319315, 0.194666 0.191473	0.00157279, 0.234197 0.232624	0.00060736, 0.037436 0.036828	0.00029913, 0.045164 0.044865
	2	0.0170233, 0.238723 0.2217	0.0115253, 0.274916 0.263391	0.00324017, 0.046052 0.042812	0.00219309, 0.053183 0.05099
	3	0.0318251, 0.262235 0.23041	0.0238477, 0.29633 0.272482	0.00606207, 0.050679 0.044617	0.00454068, 0.057429 0.052888
17	1	0.00305342, 0.186538 0.183485	0.00150402, 0.224709 0.223205	0.00058066, 0.035847 0.035267	0.00028599, 0.043297 0.043011
	2	0.0163221, 0.229184 0.212862	0.0110527, 0.264294 0.253241	0.00310596, 0.044174 0.041068	0.00210268, 0.051078 0.048975
	3	0.0305759, 0.252046 0.22147	0.0229197, 0.285225 0.262305	0.00582258, 0.048664 0.042841	0.00436289, 0.055217 0.050854

Table 3: BPI of the record value Z_s in case of fixed sample size

r	s	σ =0.5 90% L,U Width	σ =0.5 95% L,U width	σ =0.1 90% L,U width	σ =0.1 95% L,U width
15	1	0.00431284, 0.213117 0.208804	0.00213171, 0.251603 0.249471	0.00082038, 0.041036 0.040216	0.00040545, 0.048584 0.048178
	2	0.0285368, 0.296699 0.268162	0.0195651, 0.331118 0.311553	0.00543479, 0.057502 0.052067	0.0037245, 0.064374 0.06065
17	1	0.00412447, 0.204401 0.200277	0.00203859, 0.241624 0.239585	0.00078437, 0.039328 0.038543	0.00038765, 0.046615 0.046227
	2	0.0273805, 0.286153 0.258773	0.0187721, 0.319941 0.301169	0.00521322, 0.055401 0.050188	0.00357262, 0.062128 0.058555

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Table 4 : BPI of OOS Z_s in case of N having binomial distribution

r	s	σ =0.5		σ =0.1	
		90% L,U Width	95% L,U width	90% L,U width	95% L,U width
15	1	0.0010312, 0.073167 0.072136	0.00050819, 0.094806 0.094298	0.00019612, 0.013968 0.013771	0.000096647, 0.01812 0.018023
	2	0.00757299, 0.136892 0.129319	0.00512455, 0.169819 0.164694	0.00144074, 0.026228 0.024788	0.00097482, 0.032604 0.031629
	3	0.0185405, 0.197641 0.179101	0.0138517, 0.234175 0.220323	0.00352923, 0.038016 0.034486	0.00263609, 0.045159 0.042523
17	1	0.000986, 0.069544 0.068558	0.00048595, 0.090077 0.089591	0.000096647, 0.01812 0.018023	0.0000924, 0.017208 0.017116
	2	0.0072585, 0.130501 0.123243	0.00491331, 0.162055 0.157142	0.00097482, 0.032604 0.031629	0.00093443, 0.031093 0.030158
	3	0.0178026, 0.189147 0.171344	0.0133073, 0.22444 0.211133	0.00263609, 0.045159 0.042523	0.00253189, 0.043245 0.040713

Table 5: BPI of Z_s progressive type II censoring in case of fixed sample size

r	s	σ =0.5		σ =0.1	
		90% L,U Width	95% L,U width	90% L,U width	95% L,U width
15	1	0.00043297, 0.027019 0.026586	0.00021358, 0.033784 0.03357	0.00008234, 0.005145 0.005063	0.00007872, 0.004874 0.004795
	2	0.00327701, 0.048284 0.045007	0.00222764, 0.057714 0.055487	0.00062331, 0.009205 0.008582	0.00059728, 0.00871 0.008112
17	1	0.00041399, 0.025602 0.025188	0.00020423, 0.031944 0.031739	0.00004062, 0.006436 0.006395	0.00003883, 0.006083 0.006045
	2	0.00314083, 0.045701 0.04256	0.00213578, 0.054504 0.052368	0.00042369, 0.011008 0.010585	0.00040613, 0.010392 0.009986

Table 6: BPI of Z_s progressive type II censoring in case of N is having Poisson distribution

r	s	σ =0.5		σ =0.1	
		90% L,U Width	95% L,U width	90% L,U width	95% L,U width
15	1	0.0031932, 0.194666 0.191473	0.00157279, 0.234197 0.232624	0.00060736, 0.037436 0.036828	0.00029913, 0.045164 0.044865
	2	0.0150116, 0.227288 0.212276	0.0101492, 0.264189 0.25404	0.00285698, 0.043809 0.040952	0.0019311, 0.051064 0.049133
17	1	0.00305342, 0.186538 0.183485	0.00150402, 0.224709 0.223205	0.00058066, 0.035847 0.035267	0.00028599, 0.043297 0.043011
	2	0.0143917, 0.218062 0.20367	0.00973236, 0.253815 0.244083	0.00273836, 0.041997 0.039258	0.00185137, 0.049012 0.047161

Table 7: BPI of Z_s progressive type II censoring in case of N is having binomial distribution

r	s	σ =0.5		σ =0.1	
		90% L,U Width	95% L,U width	90% L,U width	95% L,U width
15	1	0.0010312, 0.073167 0.072136	0.00050818, 0.094806 0.094298	0.00019612, 0.013968 0.013771	0.000096647, 0.01812 0.018023
	2	0.00858877, 0.161496 0.152907	0.00580253, 0.198448 0.192645	0.00163408, 0.03099 0.029356	0.00416324, 0.038173 0.03401
17	1	0.000986, 0.069544 0.068558	0.00048595, 0.090077 0.089591	0.00018748, 0.013271 0.013083	0.0000924, 0.017208 0.017116
	2	0.00823206, 0.154285 0.146053	0.00556331, 0.18985 0.184287	0.00156586, 0.029587 0.028021	0.00105808, 0.036492 0.035434

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