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FASt Iterative Solver for the 2-D Convection-Diffusion Equations

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ABSTRACT

In this paper, we introduce the preconditioned Explicit Decoupled Group (EDG) for solving the two dimensional Convection-Diffusion equation with initial and Dirichlet boundary conditions. The purpose of this paper is to accelerate the convergence rate of the Explicit Decoupled Group (EDG) method by using suitable preconditioned iterative scheme for solving the Convection-Diffusion. The robustness of these new formulations over the existing EDG scheme demonstrated through numerical experiments.

Keywords

Convection-Diffusion Equation, Preconditioning method, Explicit Decoupled Group (EDG) method.

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1. INTRODUCTION

Consider the two dimensional Convection-Diffusion equation:

$$\frac{\partial U}{\partial t}=\alpha_x\frac{\partial^2 U}{\partial x^2}+\alpha_y\frac{\partial^2 U}{\partial y^2}-\beta_x\frac{\partial U}{\partial x}-\beta_y\frac{\partial U}{\partial y} \tag{1.1}$$

with initial and boundary conditions:

$$\begin{cases} u(x,y,0)=f(x,y) \\ u(0,y,t)=g_1(y,t),\ u(X,y,t)=g_2(y,t) \\ u(x,0,t)=h_1(x,t),\ \ u(x,Y,t)=h_2(x,t). \end{cases} \tag{1.2}$$

where $\alpha_x,\alpha_y,\beta_x$ and β_y are positive constants, which can be solved numerically on a rectangular grid with constant spacing Δx and Δy in the x and y direction respectively, with $x_i=x_0+i\Delta x, y_i=y_0+j\Delta y$ and $t_n=n\Delta t$ (for all $i=0,1,2,...,nx$, $j=0,1,2,...,ny$, $n=0,1,2,...$), $X=x_0+nx\Delta x, Y=y_0+ny\Delta y$. By using Crank-Nicolson formulation, about any point (x_i,y_j,t_n) on the above Convection-Diffusion equation, the following formula is obtained:

$$\begin{aligned}\frac{u_{i,j,n+1}-u_{i,j,n}}{\Delta t}&=\frac{\alpha_x}{2}\left[\frac{u_{i-1,j,n+1}-2u_{i,j,n+1}+u_{i+1,j,n+1}}{\Delta x^2}+\frac{u_{i-1,j,n}-2u_{i,j,n}+u_{i+1,j,n}}{\Delta x^2}\right]\\ &+\frac{\alpha_y}{2}\left[\frac{u_{i,j-1,n+1}-2u_{i,j,n+1}+u_{i,j+1,n+1}}{\Delta y^2}+\frac{u_{i,j-1,n}-2u_{i,j,n}+u_{i+1,j+1,n}}{\Delta y^2}\right]\\ &-\frac{\beta_x}{2}\left[\frac{u_{i+1,j,n+1}-u_{i-1,j,n+1}}{2\Delta x}+\frac{u_{i+1,j,n}-u_{i-1,j,n}}{2\Delta x}\right]\\ &-\frac{\beta_y}{2}\left[\frac{u_{i,j+1,n+1}-u_{i,j-1,n+1}}{2\Delta y}+\frac{u_{i,j+1,n}-u_{i,j-1,n}}{2\Delta y}\right]\end{aligned} \tag{1.3}$$

Let the Courant and diffusion numbers be defined as

$$\begin{cases} c_x=\beta_x\dfrac{\Delta t}{\Delta x},\ \ c_y=\beta_y\dfrac{\Delta t}{\Delta y}, \\ s_x=\alpha_x\dfrac{\Delta t}{\Delta x^2},\ s_y=\alpha_y\dfrac{\Delta t}{\Delta y^2}. \end{cases} \tag{1.4}$$

Thus (1.3) can be written as

$$\begin{aligned}&(1+S_x+S_y)u_{i,j,n+1}-(\frac{S_x}{2}+\frac{C_x}{4})u_{i-1,j,n+1}-(\frac{S_x}{2}-\frac{C_x}{4})u_{i+1,j,n+1}\\ &-(\frac{S_y}{2}-\frac{C_y}{4})u_{i,j-1,n+1}-(\frac{S_y}{2}-\frac{C_y}{4})u_{i,j+1,n+1}\\ &=(1-S_x-S_y)u_{i,j,n}+(\frac{S_x}{2}+\frac{C_x}{4})u_{i-1,j,n}+(\frac{S_x}{2}-\frac{C_x}{4})u_{i+1,j,n}\\ &+(\frac{S_y}{2}+\frac{C_y}{4})u_{i,j-1,n}+(\frac{S_y}{2}-\frac{C_y}{4})u_{i,j+1,n}\end{aligned} \tag{1.5}$$

By rotating the x-y axis clockwise by 45^0 and using Taylor series expansion, the rotated Crank-Nicolson formula for (1.1) can be shown as [1]:

$$
\begin{aligned}
&(1+\frac{S_x}{2}+\frac{S_y}{2})u_{i,j,n+1}-(\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8})u_{i-1,j+1,n+1}-(\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8})u_{i+1,j+1,n+1}\\
&-(\frac{S_y}{4}+\frac{C_x}{8}-\frac{C_y}{8})u_{i+1,j-1,n+1}-(\frac{S_y}{4}+\frac{C_x}{8}+\frac{C_y}{8})u_{i-1,j-1,n+1}\\
&=(1-\frac{S_x}{2}+\frac{S_y}{2})u_{i,j,n}+(\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8})u_{i-1,j+1,n}+(\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8})u_{i+1,j+1,n}\\
&+(\frac{S_y}{4}-\frac{C_x}{8}+\frac{C_y}{8})u_{i+1,j-1,n}+(\frac{S_y}{4}+\frac{C_x}{8}+\frac{C_y}{8})u_{i-1,j-1,n} \qquad (1.6)
\end{aligned}
$$

It is well known that the application of either (1.3) or (1.6) at each time step will result in a large and sparse linear system,

$$Au_{n+1}=Bu_n \qquad (1.7)$$

Where A and B are square nonsingular matrices, while u_{n+1} and u_n are specific column matrices. The iterative methods are the suitable methods to obtain the solution of (1.7) compared to the other direct methods ([2], [3], [4], [5], [6], [7]). Among these iterative methods, the Explicit Group (EG) and Explicit Decoupled Group (EDG) can be formulated based on Equations (1.5) and (1.6) respectively. Abdullah [8] constructed the EDG method which was shown to be more efficient computationally than the EG method for solving two dimensional elliptic equation. The aim of this paper is to propose new preconditioned iterative scheme and apply it to the (EDG) iterative method for solving the two dimensional Convection-Diffusion equation. The paper is organized in five sections: The formulation of proposed preconditioned EDG iterative method will be given in section 2. Sections 3 discuss the truncation Error and Consistency. Stability Analysis for the proposed method will be justified in section 4. The numerical results are presented in Section 5 in order to show the efficiency of the new preconditioned method. Finally, the conclusion is given in Section 6.

2. THE PROPOSED PRECONDITIONED EDG METHOD

EDG scheme can be constructed by applying (1.6) to any group of four points on the solution domain at each time step. As a result of that, at any particular time level (n+1), a (4×4) system will be obtained with the form:

$$
\begin{bmatrix}
1+\frac{S_x}{2}+\frac{S_y}{2} & -(\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8}) & 0 & 0\\
-(\frac{S_y}{4}+\frac{C_x}{8}+\frac{C_y}{8}) & 1+\frac{S_x}{2}+\frac{S_y}{2} & 0 & 0\\
0 & 0 & 1+\frac{S_x}{2}+\frac{S_y}{2} & -(\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8})\\
0 & 0 & -(\frac{S_y}{4}-\frac{C_x}{8}+\frac{C_y}{8}) & 1+\frac{S_x}{2}+\frac{S_y}{2}
\end{bmatrix}
\begin{bmatrix} u_{i,j,n+1}\\ u_{i+1,j+1,n+1}\\ u_{i+1,j,n+1}\\ u_{i,j+1,n+1}\end{bmatrix}
=
\begin{bmatrix} rh_1\\ rh_2\\ rh_3\\ rh_4\end{bmatrix} \qquad (2.1)
$$

where

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$$\begin{bmatrix} rh_1 \\ rh_2 \\ rh_3 \\ rh_4 \end{bmatrix} = \begin{bmatrix} bu_{i-1,j+1,n+1} + du_{i+1,j-1,n+1} + eu_{i-1,j-1,n+1} + T_{i,j} \\ bu_{i,j+2,n+1} + cu_{i+2,j+2,n+1} + du_{i+2,j,n+1} + T_{i+1,j+1} \\ cu_{i+2,j+1,n+1} + du_{i+2,j-1,n+1} + eu_{i,j-1,n+1} + T_{i+1,j} \\ bu_{i-1,j+2,n+1} + cu_{i+1,j+2,n+1} + eu_{i-1,j,n+1} + T_{i,j+1} \end{bmatrix} \qquad (2.2)$$

$$\begin{bmatrix} T_{i,j} \\ T_{i+1,j+1} \\ T_{i+1,j} \\ T_{i,j+1} \end{bmatrix} = \begin{bmatrix} au_{i,j,n} + bu_{i-1,j+1,n} + cu_{i+1,j+1,n} + du_{i+1,j-1,n} + eu_{i-1,j-1,n} \\ au_{i+1,j+1,n} + bu_{i,j+2,n} + cu_{i+2,j+2,n} + + du_{i+2,j,n} + eu_{i,j,n} \\ au_{i+1,j,n} + bu_{i,j+1,n} + cu_{i+2,j+1,n} + + du_{i+2,j-1,n} + eu_{i,j-1,n} \\ au_{i,j+1,n} + bu_{i-1,j+2,n} + cu_{i+1,j+2,n} + du_{i+1,j,n} + eu_{i-1,j,n} \end{bmatrix} \qquad (2.3)$$

This system leads to a decoupled system of (2×2) equations which can be made explicit by imposing iteratively as follows:

$$\begin{bmatrix} 1+\frac{S_x}{2}+\frac{S_y}{2} & -(\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8}) \\ -(\frac{S_y}{4}+\frac{C_x}{8}+\frac{C_y}{8}) & 1+\frac{S_x}{2}+\frac{S_y}{2} \end{bmatrix} \begin{bmatrix} u_{i,j,n+1} \\ u_{i+1,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rhs_1 \\ rhs_2 \end{bmatrix} \qquad (2.4)$$

$$\begin{bmatrix} 1+\frac{S_x}{2}+\frac{S_y}{2} & -(\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8}) \\ -(\frac{S_y}{4}-\frac{C_x}{8}+\frac{C_y}{8}) & 1+\frac{S_x}{2}+\frac{S_y}{2} \end{bmatrix} \begin{bmatrix} u_{i+1,j,n+1} \\ u_{i,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rhs_3 \\ rhs_4 \end{bmatrix} \qquad (2.5)$$

Fig. (1-1) showed that the iterative evaluation of (2.4) at any time level involves points of type ○ only, while the evaluation of Equation (2.5) involves points of type □ only(see Fig. (1-2)). Thus, the iterations may be chosen to involve only one type of points. If we choose to iterate on points of type ○, the EDG scheme corresponds to generation of iterations on these points using the group formula (2.4) until a convergence test is satisfied. After convergence is achieved, the solutions at the points of type ○ are evaluated directly once using the Crank-Nicolson formula (1.5) before proceeding to the next time level.

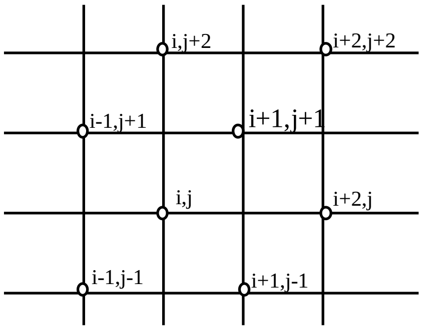

Fig.(1-1): computational molecule of Equation (2.4)

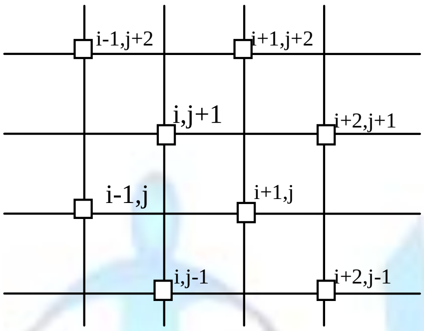

Fig (1-2): computational molecule of Equation (2.5)

This method was found to be much more superior to the existing Crank Nicolson scheme because of its lower computational complexity and yet it has the same level of accuracy [1].

Now, from the linear system of equations (2.1) which formed when heat equation is solved by EDG method, matrix A can be write as $A = D - L - U$ where *D* is diagonal matrix *A*, $-L$ is strictly lower triangular parts of *A* and $-U$ is strictly upper triangular parts of *A*. A preconditioner $(I + PU)$, where: $1 \leq P < 2$ is used to modify the original system (2.1) to

$$(I + PU)A\bar{u} = (I + PU)\bar{b} \tag{2.6}$$

The resulted system of (2.6) called preconditioned EDG Iterative method (PEDG).

3. TRUNCATION ERROR AND CONSISTENCY

The local truncation for the Crank-Nicolson scheme may be obtained by using the Taylor series expansion about the point $(x_i, y_j, t_{n+\frac{1}{2}})$:

$$T_{CN} = -\frac{\Delta t^2}{24}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2}{8}\Big(\alpha_x \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\Big|_{i,j,n+0.5} + \alpha_y \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial y^2}\Big|_{i,j,n+0.5}$$

$$-\beta_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\Big|_{i,j,n+0.5} - \beta_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\Big|_{i,j,n+0.5} + \ldots\Big) + \Delta x^2\Big(\frac{\alpha_x}{12}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} - \frac{\beta_x}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5}\Big)$$

$$+\Delta y^2\Big(\frac{\alpha_y}{12}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \frac{\beta_y}{6}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5}\Big) + \frac{\Delta x^2 \Delta t^2}{48}\Big(\frac{\alpha_x}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5}\Big)$$

$$+\frac{\Delta y^2 \Delta t^2}{48}\Big(\frac{\alpha_y}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \beta_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5}\Big) + \ldots$$

Hence,

$$T_{CN} = O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2) \tag{3.1}$$

Let $h = \Delta x = \Delta y,\ k = \Delta t$, the local truncation error for this scheme is then

$$T_{CN} = -\frac{k^2}{24}\frac{\partial^3 u}{\partial t^3} + \frac{k^2}{8}(\alpha_x \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\Big|_{i,j,n+0.5} + \alpha_y \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\Big|_{i,j,n+0.5}$$
$$-\beta_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\Big|_{i,j,n+0.5} + ...) + h^2(\frac{\alpha_x}{12}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} + \frac{\alpha_y}{12}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \frac{\beta_x}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5})$$
$$-\frac{\beta_y}{6}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5}) + \frac{h^2k^2}{48}(\frac{\alpha_x}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5}) + \frac{h^2k^2}{48}(\frac{\alpha_x}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5}$$
$$+\frac{\alpha_y}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5} - \beta_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5}) + ...$$

Hence,

$$T_{CN} = O(k^2) + O(h^2) \qquad (3.2)$$

As, $\Delta x, \Delta y, \Delta t \to 0$ the truncation error T_{CN} tends to zero. Hence, as the grid spacing $\Delta x, \Delta y, \Delta t \to 0$ in the limit sense, the Crank-Nicolson formula (1.5)is equivalent to the convection-diffusion equation and thus is consistent. Explicit Group (EG) method is also consistent and its truncation error is similar with Crank-Nicolson scheme since it is derived from the same formula [?].

Assuming that $\alpha = \alpha_x = \alpha_y$, the truncation error for the rotated Crank-Nicolson Scheme becomes:

$$T_{R-CN} = -\frac{k^2}{24}\frac{\partial^3 u}{\partial t^3} + \frac{k^2}{8}(\alpha \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\Big|_{i,j,n+0.5} + \alpha \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\Big|_{i,j,n+0.5}$$
$$-\beta_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\Big|_{i,j,n+0.5} + ...) + \frac{h^2}{2}(\frac{\alpha}{12}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} + \frac{\alpha}{2}\frac{\partial^4 u}{\partial x^2 \partial y^2}\Big|_{i,j,n+0.5}$$
$$+\frac{\alpha}{12}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \frac{\beta_x}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5} - \frac{\beta_x}{2}\frac{\partial^3 u}{\partial x \partial y^2}\Big|_{i,j,n+0.5} - \frac{\beta_y}{6}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5}$$
$$-\frac{\beta_y}{2}\frac{\partial^3 u}{\partial x^2 \partial y}\Big|_{i,j,n+0.5}) + \frac{h^2k^2}{48}(\frac{\alpha}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\Big|_{i,j,n+0.5} + 4\alpha \frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^2 \partial y^2}\Big|_{i,j,n+0.5}$$
$$+\frac{\alpha}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\Big|_{i,j,n+0.5} - \beta_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\Big|_{i,j,n+0.5} - 4\beta_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x \partial y^2}\Big|_{i,j,n+0.5}$$
$$-\beta_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\Big|_{i,j,n+0.5} - 4\beta_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^2 \partial y}\Big|_{i,j,n+0.5}) + ...$$

Thus,

$$T_{R-CN} = O(k^2) + O(h^2) \qquad (3.3)$$

Therefore, the rotated Crank-Nicolson formula (1.6) is consistent. EDG is also consistent and its truncation error is similar with the rotated Crank-Nicolson scheme since it is derived from the same formula.

4. STABILITY ANALYSIS

Equation (2.1) can be written explicitly in difference form as:

$$u_{n+1} = Tu_n \qquad where: \quad T = A^{-1}B .$$

The matrix A is of the form:

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$$A=\begin{bmatrix} R_1 & R_2 & & & \\ R_3 & R_1 & R_2 & & \\ & \ddots & \ddots & \ddots & \\ & & R_3 & R_1 & R_2 \\ & & & R_3 & R_1 \end{bmatrix}$$

where:

$$R_1=\begin{bmatrix} G_1 & G_2 & & & \\ G_3^T & G_1 & G_2 & & \\ & \ddots & \ddots & \ddots & \\ & & G_3^T & G_1 & G_2 \\ & & & G_3^T & G_1 \end{bmatrix}, R_2=\begin{bmatrix} G_3 & G_4 & & & \\ & G_3 & G_4 & & \\ & & \ddots & \ddots & \\ & & & G_3 & G_4 \\ & & & & G_3 \end{bmatrix}$$

$$R_3=\begin{bmatrix} G_2^T & & & & \\ G_5 & G_2^T & & & \\ & \ddots & \ddots & & \\ & & G_5 & G_2^T & \\ & & & G_5 & G_2^T \end{bmatrix}, G_1=\begin{bmatrix} a & -e \\ -d & a \end{bmatrix}$$

$$G_2=\begin{bmatrix} 0 & 0 \\ -c & 0 \end{bmatrix}, G_3=\begin{bmatrix} 0 & 0 \\ -b & 0 \end{bmatrix}, G_4=\begin{bmatrix} 0 & 0 \\ -e & 0 \end{bmatrix}, G_5=\begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix}$$

with $a=1+\frac{S_x}{2}+\frac{S_y}{2}, b=\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8}, c=\frac{S_x}{4}-\frac{C_x}{8}+\frac{C_y}{8},$

$$d=\frac{S_y}{4}+\frac{C_x}{8}+\frac{C_y}{8}, e=\frac{S_y}{4}-\frac{C_x}{8}-\frac{C_y}{8}, f=1-\frac{S_x}{2}-\frac{S_y}{2}.$$

$$\|A\|_\infty=(\left|1+\frac{S_x}{2}+\frac{S_y}{2}\right|+\left|\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8}\right|+\left|\frac{S_x}{4}-\frac{C_x}{8}+\frac{C_y}{8}\right|$$

$$+\left|\frac{S_x}{4}+\frac{C_x}{8}+\frac{C_y}{8}\right|+\left|\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8}\right|$$

$$B=\begin{bmatrix} S_1 & S_2 & & & \\ S_3 & S_1 & S_2 & & \\ & \ddots & \ddots & \ddots & \\ & & S_3 & S_1 & S_2 \\ & & & S_3 & S_1 \end{bmatrix},$$

where

$$S_1=\begin{bmatrix} H_1 & H_2 & & & \\ H_3^T & H_1 & H_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & H_1 & H_2 \\ & & & H_3^T & H_1 \end{bmatrix}, S_2=\begin{bmatrix} H_3 & H_4 & & & \\ & H_3 & H_4 & & \\ & & \ddots & \ddots & \\ & & & H_3 & H_4 \\ & & & & H_3 \end{bmatrix}$$

$$S_3=\begin{bmatrix} H_2^T & & & \\ H_5 & H_2^T & & \\ & \ddots & \ddots & \\ & & H_2^T & \\ & & H_5 & H_2^T \end{bmatrix},$$

$$H_1=\begin{bmatrix} f & e \\ d & f \end{bmatrix}, H_2=\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, H_3=\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix},$$

$$H_4=\begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, H_5=\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}.$$

$$\|B\|_\infty=(\left|1-\frac{S_x}{2}-\frac{S_y}{2}\right|+\left|\frac{S_x}{4}+\frac{C_x}{8}-\frac{C_y}{8}\right|+\left|\frac{S_x}{4}-\frac{C_x}{8}+\frac{C_y}{8}\right|$$

$$+\left|\frac{S_x}{4}+\frac{C_x}{8}+\frac{C_y}{8}\right|+\left|\frac{S_x}{4}-\frac{C_x}{8}-\frac{C_y}{8}\right|$$

Since the amplification matrix $T = A^{-1} B$,

$$\|T\|_\infty=\|A^{-1}B\|_\infty\le\|A^{-1}\|_\infty\|B\|_\infty\le\frac{1}{\|A^{-1}\|_\infty}\|B\|_\infty<1$$

for all $C_x, C_y, S_x, S_y \ge 0$. Therefore, the EDG iterative scheme is unconditionally stable. Furthermore, if we apply suitable preconditioner to these iterative scheme resulted from solving the Equation (1.1) with EDG method; we will get another iterative system which is unconditionally stable by using same manner above.

5. NUMERICAL RESULTS

As a model of 2 dimensional Convection-Diffusion equations, we consider the following example ($\alpha_x=\alpha_y=\beta_x=\beta_y=1$):

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}-\frac{\partial U}{\partial x}-\frac{\partial U}{\partial y}\quad 0\le x\le 1, 0\le y\le 1,\ \ 0\le t\le T$$

with initial and boundary conditions satisfying its exact solution

$$U(x,y,t)=\frac{1}{4t+1}\left\{\frac{-(x-t-0.5)^2}{4t+1}-\frac{-(y-t-0.5)^2}{4t+1}\right\}, t>0.$$

Several numerical experiments are made to justify our results and confirm the superiority of the proposed preconditioned method. Throughout the experiments, a tolerance of $\varepsilon=10^{-10}$ was used as the termination criteria. The computer processing unit is Intel(R) Core(TM) i5 with memory of 4Gb and the software used to implement and generate the results was Developer C++ Version 4.9.9.2. Comparisons for Number of iterations at all the required time levels and Elapsed time of Explicit Decoupled Group (EDG) and Preconditioned Proposed Explicit Decoupled Group (PEDG) methods are made for the particular mesh size with number of time step equals to 100 and $\Delta t=0.01$ shown in table 1.

Table 1. Comparison of Number of iterations at all the required time levels and Elapsed time of EDG and PEDG method for solving 2-D Convection-Diffusion Equation

N	Explicit Decoupled Group (EDG) method			Preconditioned Explicit Decoupled Group (PEDG) method		
	Number of iterations	Elapsed time (sec.)	Average absolute Error	Number of iterations	Elapsed time (sec.)	Average absolute Error
82	2734	0.231	0.00758326	2513	0.191	0.00567284
114	2927	0.283	0.00638204	2769	0.214	0.00443202
142	3098	0.311	0.00415255	2915	0.295	0.00382168
186	3221	0.394	0.00312163	3103	0.341	0.00248501

6. CONCLUSION

From the numerical results obtained in [1,2], it is apparent that the EDG method has the least computation time compared to Classical and rotated Crank-Nicolson. As the extension work of these results, new PEDG method to accelerate the convergence rate of the original EDG method. The results reveal that the proposed preconditioned method faster than the original EDG method due to the lowest Number of iterations at all the required time levels and Elapsed time. Therefore, it can be concluded that the PEDG scheme may be a good alternative to solve 2 dimensional Convection-Diffusion equations.

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