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THE BESSEL-TYPE WAVELET CONVOLUTION PRODUCT

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Abstract:

In this paper the convolution product associated with the Bessel-type Wavelet transformation is investigated.Further,certain norm inequalities for the convolution product are established.

Keywords: Bessel-type Wavelet transform; Convolution product; Hankel-type transformation.

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1 INTRODUCTION

The Hankel Convolution was studied by many authors from time to time, Cholewinski[1], Haimo[2], Hirshman Jr.[3] studied the Hankel-type convolution for the following form of the Hankel-type transformation of a function $f \in L^1_\sigma(I)$, where $I=(0,\infty)$ and

$L^1_\sigma(I)=\{f : | f(x) | d\sigma(x) < \infty\}$. Namely

$$(h_{\alpha,\beta} f)(x)=\hat{f}(x)=\int_0^\infty j_{\alpha-\beta}(xt) f(t) d\sigma(t) \tag{1}$$

where

$$j_{\alpha-\beta}(x)=2^{\alpha-\beta}\Gamma(\alpha-\beta)x^{-(\alpha-\beta)}J_{\alpha-\beta}(x)$$

Here $J_{\alpha-\beta}(x)$ is the Bessel-type function of order $\alpha-\beta$, and

$$d\sigma(t)=\frac{t^{2(\alpha-\beta)}}{2^{\alpha-\beta}\Gamma(\alpha-\beta+1)}dt.$$

We say that $f \in L^p_\sigma(I), 1 \le p < \infty$, if

$$\| f \|_{p,\sigma}=\left(\int_0^\infty | f(x) |^p \, d\sigma(x)\right)^{1/p} < \infty.$$

If $f \in L^1_\sigma(I)$ and $h_{\alpha,\beta} f \in L^1_\sigma(I)$ then the inverse Hankel-type transform is given by

$$f(x)=(h^{-1}_{\alpha,\beta}[\hat{f}])(x)=\int_0^\infty j_{\alpha-\beta}(xt)\left(h_{\alpha,\beta} f\right)(t) d\sigma(t) \tag{2}$$

If $f \in L^1_\sigma(I)$ and $g \in L^1_\sigma(I)$ then the Hankel-type convolution is defined by

$$\left(f \# g\right)(x)=\int_0^\infty \left(\tau_x f\right)(y) g(y) d\sigma(y), \tag{3}$$

where the Hankel-type translation τ_x is given by

$$\left(\tau_x f\right)(y)=\hat{f}(x,y)=\int_0^\infty D(x,y,z) f(x) d\sigma(z), \tag{4}$$

$$D(x,y,z)=\int_0^\infty j_{\alpha-\beta}(xt) j_{\alpha-\beta}(yt) j_{\alpha-\beta}(zt) d\sigma(t)$$

$$=2^{5\alpha-\beta}\pi^{\alpha-5\beta}\left[\Gamma(7\alpha+5\beta)\right]^2\left[\Gamma(\alpha-\beta+1/2)\right]^{-1}(xyz)^{-2(\alpha-\beta)}\times\left[\Delta(x,y,z)\right]^{-2(\alpha-\beta)}$$

For $\alpha-\beta \ge -1/2$,where $\Delta(x,y,z)$ is the area of a triangle with sides x,y,z, if such a triangle exists and zero otherwise.Here we note that $D(x,y,z)$ is symmetric in x,y,z.

Applying (2) to (4) we get the formula

$$\int_0^\infty j_{\alpha-\beta}(zt) D(x,y,z) d\sigma(z)=j_{\alpha-\beta}(xt) j_{\alpha-\beta}(yt).$$

Setting $t=0$, we get

$$\int_0^\infty D(x,y,z) d\sigma(z)=1$$

Therefore in view of (4),

$$\| \hat{f}(x,y) \|_{1,\sigma} \leq \| f \|_{1,\sigma} \qquad (5)$$

Now, using (4) we can write (3) in the following form,

$$(f\#g)(x) = \int_0^\infty \int_0^\infty D(x,y,z) f(z) g(y) d\sigma(z) d\sigma(y).$$

Some important properties of the Hankel-type convolution are:

1. If $f, g \in L^1_\sigma(I)$ then from [2]

$$\| f\#g \|_{1,\sigma} \leq \| f \|_{1,\sigma} \| g \|_{1,\sigma} \qquad (6)$$

2. With the same assumptions,

$$h_{\alpha,\beta}(f\#g)(x) = (h_{\alpha,\beta} f)(x)(h_{\alpha,\beta} g)(x) \qquad (7)$$

3. Let $f \in L^1_\sigma(I)$ and $g \in L^p_\sigma(I), p \geq 1$,then (f# g) exists,is continuous and from [7] we get the inequality

$$\| f\#g \|_{p,\sigma} \leq \| f \|_{1,\sigma} \| g \|_{p,\sigma} \qquad (8)$$

4. Let $f \in L^p_\sigma(I)$ and $g \in L^q_\sigma(I), 1/p + 1/q = 1$, then f# g exists, is conntinuos and form [7] we have

$$\| f\#g \|_{\infty,\sigma} \leq \| f \|_{p,\sigma} \| g \|_{q,\sigma} \qquad (9)$$

5. Let $f \in L^p_\sigma(I)$ and $g \in L^q_\sigma(I), 1/p + 1/q - 1 = 1/r$ then f# g exists, is conntinuos and form [7] we have

$$\| f\#g \|_{r,\sigma} \leq \| f \|_p \| g \|_q \qquad (10)$$

6. Let $f \in L^p_\sigma(I)$ and $g \in L^q_\sigma(I)$ and $h \in L^r_\sigma(I)$, then the weighted norm inequality

$$\left| \int_0^\infty f(x)(g\#h)(x) d\sigma(x) \right| \leq \| f \|_{p,\sigma} \| g \|_{q,\sigma} \| h \|_{r,\sigma} \qquad (11)$$

holds for $1/p + 1/q + 1/r = 2$.

As indicated above, the proof of properties 1-5 are well known. Hence we next give the proof of 6. Using Holder's inequality,we get

$$\left| \int_0^\infty f(x)(g\#h)(x) d\sigma(x) \right| \leq \| f \|_{p,\sigma} \| g\#h \|_{s,\sigma}, 1/p + 1/s = 1.$$

thus by using (9) we get,

$$\left| \int_0^\infty f(x)(g\#h)(x) d\sigma(x) \right| \leq \| f \|_{p,\sigma} \| g \|_{q,\sigma} \| h \|_{s,\sigma}, 1/s = 1/q + 1/r - 1$$

From [4], $h_{\alpha,\beta}$ is isometric on $L^2_\sigma(I)$, $h^{-1}_{\alpha,\beta} h_{\alpha,\beta} f = f$ then Parseval's formula of the Hankel-type transformation for $f, g \in L^2_\sigma(I)$ is given by

$$\int_0^\infty f(x) g(x) d\sigma(x) = \int_0^\infty (h_{\alpha,\beta} f)(h_{\alpha,\beta} g)(y) d\sigma(y). \qquad (12)$$

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Furthermore, this relation also holds for $f, g \in L^1_\sigma(I)$,see [8].

For $\psi \in L^1_\sigma(I)$,using translation τ_x given in [7] and dialation $D_a f(x,y) = f(ax, ay)$, the Bessel-type wavelet [6] is defined by

$$\psi\left(\frac{t}{a},\frac{b}{a}\right) = D_{1/a}\tau_b\psi(t) = \int_0^\infty \psi(z) D\left(\frac{t}{a},\frac{b}{a},z\right) d\sigma(z) \tag{13}$$

The continuous Bessel-type transform [6] of a function $f \in L^1_\sigma(I)$ with respect to wavelet $\psi \in L^1_\sigma(I)$ is defined by

$$(B_\psi f)(b,a) = a^{-2(\alpha-\beta+1)} \int_0^\infty \psi\left(\frac{t}{a},\frac{b}{a}\right) f(t) d\sigma(t), a > 0$$

by simple modification we get

$$(B_\psi f)(b,a) = (f \# \psi)\left(\frac{b}{a}\right), a > 0$$

From [6] and [7] the continuous Bessel-type wavelet transform of a function $f \in L^1_\sigma(I)$ can be written in the form:

$$(B_\psi f)(b,a) = \int_0^\infty j_{\alpha-\beta}(bw)(h_{\alpha,\beta} f)(w)(h_{\alpha,\beta}\psi)(aw) d\sigma(w) \tag{14}$$

Now we state the Parseval formula of the Bessel-type wavelettransform from [6,pp. 245],

$$\int_0^\infty \int_0^\infty (B_\psi f)(b,a)(B_\psi g)(b,a) \frac{d\sigma(b) d\sigma(a)}{a^{2(\alpha-\beta+1)}} = C_\psi < f, g > \tag{15}$$

for $f \in L^2_\sigma(I)$ and $g \in L^2_\sigma(I)$. Now we also state from [3,Theorem 2c,pp.312] and [3,corollay 2c,pp.313] which is useful for our approximation results:

Theorem 1.1: Suppose that

1. $K_n(x) \geq 0, 0 < x < \infty$
2. $\int_0^\infty K_n(x) d\sigma(x) = 1, n = 0,1,2,3...$
3. $\lim_{n\to\infty} \int_\delta^\infty K_n(x) d\sigma(x) = 0$,for each $\delta > 0$,
4. $\phi(x) \in L^\infty_\sigma(I)$
5. ϕ is continuous at $x_0, x_0 \in [x-\delta, x+\delta]$ and $\delta > 0$

Then

$$\lim_{n\to\infty} (\phi \# K_n)(x_0) = \phi(x_0).$$

Corollary 1.1: With the same assumptions on k_n(x),if f(x) $\in L^1_\sigma(I)$ then _n||f#k_n-f||_1=0.

Motivated from [5,pp.129-136], we define convolution product for Bessel-type wavelet transform and study some of its properties.

2 THE BESSEL-TYPE WAVELET CONVOLUTION PRODUCT

In this section,using properties(5),(12) and (13) we formally define the convolution product for Bessel-type wavelet transformation by the relation

$$B_\psi(f \otimes g)(b,a) = (B_\psi f)(b,a)(B_\psi g)(b,a) \tag{16}$$

and investigate its boundedness and approximation properties.This in turn implies the product of two Bessel-type wavelet transforms could be wavelet transform under certain conditions.

Theorem 2.1: Let $f, g, \psi \in L^1_\sigma(I)$ and $h_{\alpha,\beta}(\psi)(w) \neq 0$. Then the bessel-type wavelet convolution can be written in the form

$$(f \otimes g)(z) = \int_0^\infty (\tau_{z,a} f)(y) g(y) d\sigma(y),$$

where

$$(\tau_{z,a} f)(y) = \int_0^\infty f(x) D_a(x, y, z) d\sigma(x),$$

$$D_a(x, y, z) = \int_0^\infty \int_0^\infty (h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi) j_{\alpha-\beta}(xt) j_{\alpha-\beta}(y\xi) L_a(t, \xi, z) d\sigma(t) d\sigma(\xi) \quad (17)$$

and

$$L_a(t, \xi, z) = \int_0^\infty j_{\alpha-\beta}(yt) j_{\alpha-\beta}(y\xi) Q_a(y, z) d\sigma(y), \quad (18)$$

$$Q_a(y, z) = \int_0^\infty \frac{j_{\alpha-\beta}(wz) j_{\alpha-\beta}(wy)}{(h_{\alpha,\beta}\psi)(aw)} d\sigma(w) \quad (19)$$

Proof: From (14) we have

$$h_{\alpha,\beta}\left[(B_\psi f)(b,a)\right](w) = (h_{\alpha,\beta}\psi)(aw)(h_{\alpha,\beta} f)(w)$$

therefore

$$h_{\alpha,\beta}\left[(B_\psi f \otimes g)(b,a)\right](w) = h_{\alpha,\beta}\left[(B_\psi f)(b,a)(B_\psi g)(b,a)\right](w)$$

$$= h_{\alpha,\beta} h^{-1}_{\alpha,\beta}((h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} f)(\cdot) h^{-1}_{\alpha,\beta}\left((h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} g)(\cdot)\right)(w)$$

By property (7) of the Hankel-type convolution we have,

$$h_{\alpha,\beta}\left[(B_\psi f \otimes g)(b,a)\right](w) = \left[(h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} f)(\cdot) \times \# (h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} g)(\cdot)\right](w)$$

Therefore we get

$$(h_{\alpha,\beta}\psi)(aw) h_{\alpha,\beta}\left[(f \otimes g)\right](w) = \left[(h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} f)(\cdot) \times \# (h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} g)(\cdot)\right](w) \quad (20)$$

This gives a relation between Bessel-type wavelet transform convolution and the Hankel-type transform convolution.

Let us set

$$F_a = (h_{\alpha,\beta}\psi)(a\cdot)(h_{\alpha,\beta} f)(\cdot)$$

$$G_a = (h_{\alpha,\beta}\psi)(\cdot)(h_{\alpha,\beta} g)(\cdot)$$

Then by (3) and (4) we get

$$h_{\alpha,\beta}\left[(B_\psi f \otimes g)(b,a)\right](w) = \int_0^\infty (\tau_w G_a)(\eta) F_a(\eta) d\sigma(\eta)$$

$$= \int_0^\infty F_a(\eta) \left(\int_0^\infty D(w, \eta, \xi) G_a(\xi) d\sigma(\xi) \right) d\sigma(\eta)$$

$$= \int_0^\infty \int_0^\infty F_a(\eta) G_a(\xi) D(w, \eta, \xi) d\sigma(\xi) d\sigma(\eta)$$

$$= \int_0^\infty \int_0^\infty F_a(\eta) G_a(\xi) \int_0^\infty j_{\alpha-\beta}(wy) j_{\alpha-\beta}(\eta y) j_{\alpha-\beta}(\xi y) d\sigma(y) d\sigma(\xi) d\sigma(\eta)$$

$$= \int_0^\infty \left(\int_0^\infty F_a(\eta) j_{\alpha-\beta}(\eta y) d\sigma(\eta) \right) \left(\int_0^\infty G_a(\xi) j_{\alpha-\beta}(\xi y) d\sigma(\xi) \right) j_{\alpha-\beta}(wy) d\sigma(y)$$

$$= \int_0^\infty (h_{\alpha,\beta} F_a)(y)(h_{\alpha,\beta} G_a)(y) j_{\alpha-\beta}(wy) d\sigma(y)$$

Therefore by the inversion formula of the Hankel-type Transformation we have,

$$(f \otimes g)(z) = \int_0^\infty \frac{j_{\alpha-\beta}(wz)}{(h_{\alpha,\beta}\psi)(aw)} \left(\int_0^\infty (h_{\alpha,\beta} F_a)(y)(h_{\alpha,\beta} G_a)(y) j_{\alpha-\beta}(wy) d\sigma(y) \right) d\sigma(w)$$

$$= \int_0^\infty (h_{\alpha,\beta} F_a)(y)(h_{\alpha,\beta} G_a)(y) Q_a(y,z) d\sigma(y),$$

where $Q_a(y,z)$ is given by (19).

Then by the definition of Hankel-type transformation (1)

$$(f \otimes g)(z) = \int_0^\infty \int_0^\infty j_{\alpha-\beta}(yt)(h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta} f)(t) d\sigma(t) \left(\int_0^\infty j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(a\xi)(h_{\alpha,\beta} g)(\xi) d\sigma(\xi) \right) Q_a(y,z)$$

$$= \int_0^\infty \int_0^\infty (h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi)(h_{\alpha,\beta} f)(t)(h_{\alpha,\beta} g)(\xi) \left(\int_0^\infty j_{\alpha-\beta}(y\xi) j_{\alpha-\beta}(yt) Q_a(y,z) d\sigma(y) \right) d\sigma(t) d\sigma(\xi)$$

$$= \int_0^\infty \int_0^\infty (h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi)(h_{\alpha,\beta} f)(t)(h_{\alpha,\beta} g)(\xi) L_a(t,\xi,z) d\sigma(t) d\sigma(\xi)$$

Therefore

$$(f \otimes g)(z) = \int_0^\infty \int_0^\infty (h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi) \left(\int_0^\infty j_{\alpha-\beta}(xt) f(x) d\sigma(x) \right) \left(\int_0^\infty j_{\alpha-\beta}(y\xi) g(y) d\sigma(y) \right) L_a(t,\xi,z) d\sigma(t) d\sigma(\xi)$$

$$= \int_0^\infty \int_0^\infty f(x) g(y) \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt) j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi) L_a(t,\xi,z) d\sigma(t) d\sigma(\xi) d\sigma(x) d\sigma(y)$$

$$= \int_0^\infty \int_0^\infty f(x) g(y) D_a(x,y,z) d\sigma(x) d\sigma(y),$$

where

$$D_a(x,y,z) = \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt) j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi) L_a(t,\xi,z) d\sigma(t) d\sigma(\xi)$$

If we define the generalized translation by

$$F_a(z,y) = ({}_{z,a} f)(y) = \int_0^\infty D_a(x,y,z) f(x) d\sigma(x),$$

then

$$(f \otimes g)(z) = \int_0^\infty ({}_{z,a} f)(y) g(y) d\sigma(y)$$

Thus proof is completed.

Theorem 2.2: Assume that $\inf_w |(h_{\alpha,\beta}\psi)(aw)| = B_\psi(a) > 0$. Then

$$\| D_a(x,y,z)\| \le \frac{1}{B_\psi(a)} a^{-2(\alpha-\beta+1)} \|\psi\|_{1,\sigma}^2$$

Proof: From (17) we have

$$D_a(x,y,z) = \int_0^\infty\int_0^\infty (h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi)\, j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi)$$

$$= \int_0^\infty\int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi)\left(\int_0^\infty j_{\alpha-\beta}(\eta t)\, j_{\alpha-\beta}(\eta\xi) Q_a(\eta,z)\,d\sigma(\eta)\right)d\sigma(t)\,d\sigma(\xi)$$

$$= \int_0^\infty\int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(at)(h_{\alpha,\beta}\psi)(a\xi)\int_0^\infty j_{\alpha-\beta}(\eta t)\, j_{\alpha-\beta}(\eta\xi)\left(\int_0^\infty \frac{j_{\alpha-\beta}(wz)\, j_{\alpha-\beta}(\eta w)}{(h_{\alpha,\beta}\psi)(aw)}\right.$$

$$\left.\times d\sigma(w)\right)d\sigma(\eta)\,d\sigma(t)\,d\sigma(\xi)$$

$$= \int_0^\infty\left(\int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(\eta t)(h_{\alpha,\beta}\psi)(at)\,d\sigma(t)\right)\left(\int_0^\infty j_{\alpha-\beta}(y\xi)\, j_{\alpha-\beta}(\eta\xi)(h_{\alpha,\beta}\psi)(a\xi)\,d\sigma(\xi)\right)Q_a(z,\eta)\,d\sigma(\eta)$$

$$= \int_0^\infty h_{\alpha,\beta}\left[j_{\alpha-\beta}(xt)(h_{\alpha,\beta}\psi)(at)\right](\eta)\, h_{\alpha,\beta}\left[j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(a\xi)\right](\eta)$$

$$\times Q_a(z,\eta)\,d\sigma(\eta)$$

$$= \int_0^\infty\int_0^\infty h_{\alpha,\beta}\left[j_{\alpha-\beta}(xt)(h_{\alpha,\beta}\psi)(at)\# j_{\alpha-\beta}(y\xi)(h_{\alpha,\beta}\psi)(a\xi)\right](\eta)\, j_{\alpha-\beta}(wz)\, j_{\alpha-\beta}(\eta w)\left[(h_{\alpha,\beta}\psi)(aw)\right]^{-1} d\sigma(w)\,d\sigma(\eta)$$

$$= \int_0^\infty \left[j_{\alpha-\beta}(x\cdot)(h_{\alpha,\beta}\psi)(a\cdot)\# j_{\alpha-\beta}(y\cdot)(h_{\alpha,\beta}\psi)(a\cdot)\right](w)\, j_{\alpha-\beta}(wz)\left[(h_{\alpha,\beta}\psi)(aw)\right]^{-1} d\sigma(w).$$

Now set $F_a(t) = j_{\alpha-\beta}(xt)(h_{\alpha,\beta}\psi)(at)$ and assume that $\inf_w |(h_{\alpha,\beta}\psi)(aw)| = B_\psi(a) > 0$

Since $|j_{\alpha-\beta}(z)| \le 1, [2, pp. 336]$, we have

$$|D_a(x,y,z)| \le \frac{1}{B_\psi(a)}\int_0^\infty |(F_a \# F_a)(w)|\,d\sigma(w)$$

Using (6) we have

$$|D_a(x,y,z)| \le \frac{1}{B_\psi(a)}\|F\|_{1,\sigma}\|F\|_{1,\sigma}$$

$$\le \frac{1}{B_\psi(a)}\left[\int_0^\infty |j_{\alpha-\beta}(xv)(h_{\alpha,\beta}\psi)(av)|\,d\sigma(v)\right]^2$$

$$\le \frac{1}{B_\psi(a)}\left[\int_0^\infty |\psi(av)|\,d\sigma(v)\right]^2$$

$$\le \frac{1}{B_\psi(a)}\left[\|\psi_a\|_{1,\sigma}\right]^2$$

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$$\le \frac{a^{-2(\alpha-\beta+1)}}{B_\psi(a)}\left[\| \psi_a \|_{1,\sigma}\right]^2$$

This completes the proof.

In order to obtain Plancheral formula for the Bessel-type wavelet transform,we define the space

$$W^2(I\times I)=\left\{ g(b,a):\| g\|_{W^2}=\left(\int_0^\infty\int_0^\infty | g(b,a)|^2 \frac{d\sigma(b)d\sigma(a)}{a^{2(\alpha-\beta+1)}}\right)^{1/2}<\infty\right\}$$

Theorem2.3: Let $f\in L_\sigma^2(I),\psi\in L_\sigma^2(I)$ then

$$\| (B_\psi f)(b,a)\|_{W^2}=\sqrt{C_\psi}\,\| f\|_{2,\sigma}$$

proof: Proof can be completed by just putting $f=g$ in (15).

Theorem2.4: Let $f,g\in L_\sigma^2(I),\psi\in L_\sigma^2(I)$ be a Bessel-type wavelet which satisfies

$$C_\psi=\int_0^\infty |(h_{\alpha,\beta}\psi)(aw)|^2 \frac{d\sigma(a)}{a^{2(\alpha-\beta+1)}}>0.$$

Then $\| f\otimes g\|_{2,\sigma}\le\| f\|_{2,\sigma}\| g\|_{2,\sigma}\|\psi\|_{2,\sigma}$

Proof: Using formula (16) and (18),we have

$$\sqrt{C_\psi}\,\| f\otimes g\|_{2,\sigma}=\| B_\psi(f\otimes g)\|_{W^2}$$

$$=\| B_\psi f(b,a)B_\psi g(b,a)\|_{W^2}$$

$$=\left(\int_0^\infty\int_0^\infty | B_\psi f(b,a)B_\psi g(b,a)|^2\frac{d\sigma(b)d\sigma(a)}{a^{2(\alpha-\beta+1)}}\right)$$

From (15) and (9) we have

$$| B_\psi g(b,a)|\le\left|\left(g(a\cdot)\#\psi(\cdot)\right)(b/a)\right|$$

$$\le\| g\|_{2,\sigma}\|\psi\|_{2,\sigma}$$

Applying above results we obtain

$$\sqrt{C_\psi}\,\| f\otimes g\|_{2,\sigma}\le\| g\|_{2,\sigma}\|\psi\|_{2,\sigma}\left(\int_0^\infty\int_0^\infty | B_\psi f(b,a)B_\psi g(b,a)|^2\frac{d\sigma(b)d\sigma(a)}{a^{2(\alpha-\beta+1)}}\right)$$

From theorem (2.3) we get

$$\sqrt{C_\psi}\,\| f\otimes g\|_{2,\sigma}\le\| g\|_{2,\sigma}\|\psi\|_{2,\sigma}\sqrt{C_\psi}\,\| f\|_{2,\sigma}$$

Thus

$$\| f\otimes g\|_{2,\sigma}\le\| g\|_{2,\sigma}\|\psi\|_{2,\sigma}\| f\|_{2,\sigma}$$

Thus proof is completed.

3 WEIGHTED SOBOLEV-TYPE SPACE

In this section we study certain properties of the bessel-type wavelet convolution on a weighted Sobolev-type space defined as below:

Definition 3.1: The Zemanian space $H_{\alpha,\beta}(I),I=(0,\infty)$ is the set of all infinitely differentiable functions ϕ on $(0,\infty)$ such that

$$\rho_{m,k}^{\alpha,\beta}(\phi) = \sup_{x \in I} \left| x^m \left(x^{-1} \frac{d}{dx} \right)^k x^{-(\alpha-\beta+1)} \phi(x) \right| < \infty, \quad (21)$$

for all $m,k \in N_0$. Then $f \in H'_{\alpha,\beta}(I)$ is defined by the following way:

$$< f,\phi >= \int_0^\infty f(x)\phi(x)dx, \phi \in H_{\alpha,\beta}(I)$$

Definition 3.2: Let $k(w)$ be an arbitrary weight function. Then a function $\Phi \in [H_{\alpha,\beta}(I)]'$ is said to belong to weighted Sobolev space $G^p_{\alpha,\beta,k}(I)$ for $\alpha - \beta \in R, 1 \le p < \infty$, if it satisfies

$$\| \Phi \|_{p,\alpha,\beta,\sigma,k} = \left(\int_0^\infty | k(w) \left(H_{\alpha,\beta}\Phi\right)(w) |^p \, d\sigma(w) \right)^{1/p}, \Phi \in L^p_\sigma(I).$$

In what follows we shall assume that $k(w) = \left| \left(H_{\alpha,\beta}\psi\right)(aw) \right|$

Theorem 3.1: Let $f \in G^1_{\alpha,\beta,k}(I)$ and $g \in G^p_{\alpha,\beta,k}(I), p \ge 1$. Then

$$\| f \otimes g \|_{p,\alpha,\beta,\sigma,k} \le \| f \|_{1,\alpha,\beta,\sigma,k} \| g \|_{p,\alpha,\beta,\sigma,k}$$

Proof: In view of (21), we have

$$\| f \otimes g \|_{p,\alpha,\beta,\sigma,k} \le \| F_a(w) \|_{1,\alpha,\beta,\sigma,k} \| G_a(w) \|_{p,\alpha,\beta,\sigma,k} \quad (22)$$

$$\le \| \left(h_{\alpha,\beta}\psi\right)(aw)\left(h_{\alpha,\beta} f\right)(w) \|_{1,\alpha,\beta,\sigma,k} \| \left(h_{\alpha,\beta}\psi\right)(aw)\left(h_{\alpha,\beta} g\right)(w) \|_{p,\alpha,\beta,\sigma,k} \quad (23)$$

From definition 3.2, we get

$$\| f \otimes g \|_{p,\alpha,\beta,\sigma,k} \le \| f \|_{1,\alpha,\beta,\sigma,k} \| g \|_{p,\alpha,\beta,\sigma,k}$$

Thus proof is completed.

Theorem 3.2: $f \in G^p_{\alpha,\beta,k}(I), p \ge 1$ and $g \in G^q_{\alpha,\beta,k}(I), 1 \le p,q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then

$$\| f \otimes g \|_{r,\alpha,\beta,\sigma,k} \le \| f \|_{p,\alpha,\beta,\sigma,k} \| g \|_{q,\alpha,\beta,\sigma,k}$$

Proof: Using (10) and (21) we get (23). Thus proof is completed.

Approximation properties of the Bessel-type wavelet convolution are given next.

Theorem 3.3: Let $\Psi_{n,a}(w) = \Psi_n(aw), n = 0,1,2\ldots$ be the sequence of the basic wavelet functions such that

$$\Psi_{n,a}(w) \ge 0, 0 < w < \infty \int_0^\infty \Psi_{n,a}(w) d\sigma(w) = 1 \lim_{n\to\infty} \int_\varepsilon^\infty \Psi_{n,a}(w) d\sigma(w) = 0 \; for each \varepsilon > 0 \left(h_{\alpha,\beta}\Psi_{n,a}\right)(w) \in L^1_\sigma(I) h^{-1}_{\alpha,\beta}\left[(h_{\alpha,\beta}\Psi_{n,\right.$$

Then

$$\lim_{n\to\infty} \| f(b) - (B_{\Psi_n} f)(b,a) \|_{1,\sigma} = 0$$

Proof: Proof follows from [3,pp.315-316].

Theorem 3.4: Let $K_n(w) = (h_{\alpha,\beta}\psi)(aw)(h_{\alpha,\beta} g_n)(w)$ for fixed $a > 0, n \in N$ and $\phi(w) = (h_{\alpha,\beta}\psi)(aw)(h_{\alpha,\beta} f)(w)$ satisfy:

$$K_n(w) \geq 0, 0 < w < \infty \int_0^\infty K_n(w) d\sigma(w) = 1, w = 0,1,2,3 \ldots \lim_{n\to\infty} \int_\delta^\infty K_n(w) d\sigma(w) = 0\, foreach \delta > 0 \phi(w) \in L_\sigma^\infty(I) \phi iscontinuousatw_0, and (h_{\alpha,\beta}\psi)(aw$$

Proof: In view of relation (20) we have

$$(h_{\alpha,\beta}\psi)(aw)h_{\alpha,\beta}(f \otimes g_n)(w) = (\phi \# K_n)(w).$$

Now using theorem 1.1 we have

$$\lim_{n\to\infty}(h_{\alpha,\beta}\psi)(aw_0)h_{\alpha,\beta}(f \otimes g_n)(w_0) = \lim_{n\to\infty}(\phi \# K_n)(w_0)$$

$$= \phi(w_0)$$

$$= (h_{\alpha,\beta}\psi)(aw_0)(h_{\alpha,\beta} f)(w_0).$$

This implies that

$$\lim_{n\to\infty} h_{\alpha,\beta}(f \otimes g_n)(w_0) = (h_{\alpha,\beta} f)(w_0).$$

Thus proof is completed.

Theorem 3.5: Let $f, \psi \in L_\sigma^1(I)$ and $K_n(w)$ be the same as theorem 3.4, which satisfies all the four properties of theorem 3.3.Then

$$\lim_{n\to\infty} \| (h_{\alpha,\beta}\psi)(aw_0)(h_{\alpha,\beta} f)(w_0) - (h_{\alpha,\beta}\psi)(aw_0)h_{\alpha,\beta}(f \otimes g_n)(w_0) \|_{1,\sigma} = 0$$

Proof: Using (20) we have

$$\lim_{n\to\infty} \| (h_{\alpha,\beta}\psi)(aw_0)(h_{\alpha,\beta} f)(w_0) - (h_{\alpha,\beta}\psi)(aw_0)h_{\alpha,\beta}(f \otimes g_n)(w_0) \|_{1,\sigma} = \lim_{n\to\infty} \| \phi(w_0) - (\phi \# K_n)(w_0) \|_{1,\sigma}$$

Since $f, \psi_a \in L_\sigma^1(I), \phi(w) = (h_{\alpha,\beta}\psi_a)(h_{\alpha,\beta} f) = h_{\alpha,\beta}(f \# \psi_a)$.

Therefore using the tools of [3,corollary 2c,pp.313-314] we have

$$\lim_{n\to\infty} \| (h_{\alpha,\beta}\psi)(aw_0)(h_{\alpha,\beta} f)(w_0) - (h_{\alpha,\beta}\psi)(aw_0)h_{\alpha,\beta}(f \otimes g_n)(w_0) \|_{1,\sigma} = 0$$

Thus proof is completed.

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