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SEMI-ROUGH CONNECTED TOPOLOGIZED APPROXIMATION SPACES

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Abstract

The topological view for connectedness is more general and is applied for topologies on discrete sets. Rough thinking is one of the topological connections to uncertainty. The purpose of this paper is to introduce connectedness in approximation spaces using semi-open sets and rough set notions. The definition of semi-rough connected topologized approximation space is introduced.

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1. Preliminaries

This section presents a review of some fundamental notions of topological spaces and rough set theory.

A topological space [4] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.

(T2) τ is closed under arbitrary union.

(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all $\tau-$closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology for X. A subset A of X in a topological space (X,τ) is said to be clopen if it is both open and closed in (X,τ).

A family $\mathrm{B} \subseteq \tau$ is called a base for (X,τ) iff every nonempty open subset of X can be represented as a union of subfamily of B. Clearly, a topological space can have many bases. A family $\mathrm{S} \subseteq \tau$ is called a subbase iff the family of all finite intersections of S is a base for (X,τ).

The τ-closure of a subset A of X is denoted by A^- and it is given by $A^- = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$. The τ-interior of a subset A of X is denoted by A° and it is given by $A^\circ = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A° is the largest open subset of X which contained in A. Note that A is open iff $A = A^\circ$.

A subset A of X in a topological space (X,τ) is called semi-open [6] (briefly s-open) if $A \subseteq A^{\circ -}$. The complement of a s-open set is called s-closed. The family of all s-open (resp. s-closed) sets is denoted by $SO(X)$ (resp. $SC(X)$). The s-closure of a subset A of X is denoted by A^{s-} and it is defined by $A^{s-} = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in SC(X)\}$. Evidently, A^{s-} is the smallest s-closed subset of X which contains A. Note that A is s-closed iff $A = A^{s-}$. The s-interior of a subset A of X is denoted by $A^{s\circ}$ and it is defined by $A^{s\circ} = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in SO(X)\}$. Evidently, $A^{s\circ}$ is the largest s-open subset of X which contained in A. Note that A is s-open iff $A = A^{s\circ}$ .0

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X,R)$, where X is a set called the universe and R is an equivalence relation [7, 8]. The equivalence classes of R are also known as the granules, elementary sets or blocks. We shall use R_x to denote the equivalence class containing $x \in X$, and X/R to denote the set of all elementary sets of R. In the approximation space $K = (X,R)$, the upper (resp. lower) approximation of a subset A of X is given by

$$\overline{R}A = \{x \in X : R_x \cap A \neq \phi\} \ (\text{resp. } \underline{R}A = \{x \in X : R_x \subseteq A\}).$$

Pawlak noted [8] that the approximation space $K = (X,R)$ with equivalence relation R defines a uniquely topological space (X,τ) where τ is the family of all clopen sets in (X,τ) and X/R is a base of τ. Moreover, the upper (resp. lower) approximation of any subset A of X is exactly the closure (resp. interior) of A.

If R is a general binary relation, then the approximation space $K = (X,R)$ defines a uniquely topological space (X,τ_K) where τ_K is the topology associated to K (i.e. τ_K is the family of all open sets in (X,τ_K) and $\mathrm{S} = \{xR : x \in X\}$ is a subbase of τ_K, where $xR = \{y \in X : x R y\}$) [2, 5].

Definition 1.1 [2]. Let $K=(X,R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the triple $\kappa=(X,R,\tau_K)$ is called a topologized approximation space.

Definition 1.2 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The upper (resp. lower) approximation of A is denoted by $\overline{R}A$ (resp. $\underline{R}A$) and it is defined by

$$\overline{R}A=A^{-} \text{ (resp. } \underline{R}A=A^{\circ}\text{)}.$$

Proposition 1.1 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then

i) $\underline{R}A\subseteq A\subseteq\overline{R}A$.

ii) $\underline{R}\phi=\overline{R}\phi=\phi$ and $\underline{R}X=\overline{R}X=X$.

iii) If $A\subseteq B$, then $\underline{R}A\subseteq\underline{R}B$.

iv) If $A\subseteq B$, then $\overline{R}A\subseteq\overline{R}B$.

v) $\underline{R}(X-A)=X-\overline{R}A$.

vi) $\overline{R}(X-A)=X-\underline{R}A$.

Definition 1.3 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The s-upper (resp. s-lower) approximation of A is denoted by $\overline{R}_s A$ (resp. $\underline{R}_s A$) and it is defined by

$$\overline{R}_s A=A^{s-} \text{ (resp. } \underline{R}_s A=A^{s\circ}\text{)}.$$

Proposition 1.2 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then

i) $\underline{R}_s A\subseteq A\subseteq\overline{R}_s A$

ii) $\underline{R}_s\phi=\overline{R}_s\phi=\phi$ and $\underline{R}_s X=\overline{R}_s X=X$.

iii) If $A\subseteq B$, then $\underline{R}_s A\subseteq\underline{R}_s B$.

iv) If $A\subseteq B$, then $\overline{R}_s A\subseteq\overline{R}_s B$.

v) $\underline{R}_s(X-A)=X-\overline{R}_s A$.

vi) $\overline{R}_s(X-A)=X-\underline{R}_s A$.

2. Semi-rough connected topologized approximation spaces

The present section is devoted to introduce the concept of semi-rough connectedness in approximation spaces with general binary relations. The following two definitions introduce concepts of definability for a subset A of X in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 2.1 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. Then

i) A is called totally $R-$definable (exact) set if $\underline{R}A=A=\overline{R}A$,

ii) A is called internally $R-$definable set if $A=\underline{R}A$,

iii) A is called externally $R-$definable set if $A=\overline{R}A$,

iv) A is called $R-$indefinable (rough) set if $A\neq\underline{R}A$ and $A\neq\overline{R}A$.

Definition 2.2 [1]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. Then

i) A is called totally s-definable (s-exact) set if $\underline{R}_s A=A=\overline{R}_s A$,

ii) A is called internally s-definable set if $A=\underline{R}_s A$,

iii) A is called externally s-definable set if $A=\overline{R}_s A$,

iv) A is called s-indefinable (s-rough) set if $A\neq\underline{R}_s A$ and $A\neq\overline{R}_s A$.

Remark 2.1. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$.

- If A is exact set, then it is both internally $R-$definable and externally $R-$definable set.
- If A is s-exact set, then it is both internally s-definable and externally s-definable set.
- $\underline{R}A$ is the largest internally $R-$definable set contained in A.
- $\underline{R}_s A$ is the largest internally s-definable set contained in A.
- $\overline{R}A$ is the smallest externally $R-$definable set contains A.
- $\overline{R}_s A$ is the smallest externally s-definable set contains A.

Lemma 2.1. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. Then

i) A is exact set if and only if $X-A$ is exact.

ii) A is s-exact set if and only if $X-A$ is s-exact.

iii) A is internally $R-$definable (resp. externally $R-$definable) set if and only if $X-A$ is externally $R-$definable (resp. internally $R-$definable) set.

iv) A is internally s-definable (resp. externally s-definable) set if and only if $X-A$ is externally s-definable (resp. internally s-definable) set.

Proof. By using Proposition 1.1 and Proposition 1.2, the proof is obvious. □

The following definition introduces the concept of semi-rough disconnected topologized approximation space.

Definition 2.3. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. Then κ is said to be semi-rough (briefly s-rough) disconnected if there are two nonempty subsets A and B of X such that

$$A\cup B=X \text{ and } A\cap\overline{R}_s B=\overline{R}_s A\cap B=\phi.$$

The space $\kappa=(X,R,\tau_K)$ is said to be s-rough connected if it is not s-rough disconnected.

Proposition 2.1. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If X has a nonempty s-exact proper subset A, then $\kappa=(X,R,\tau_K)$ is s-rough disconnected.

Proof. Suppose that A is a nonempty s-exact proper subset of X. Then by Lemma 2.1, we get $B=X-A$ is also a nonempty s-exact proper subset of X. Hence $A\cup B=X$ and $A\cap\overline{R}_s B=A\cap B=\overline{R}_s A\cap B=\phi$.

Thus $\kappa=(X,R,\tau_K)$ is s-rough disconnected. □

Example 2.1. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space such that $X=\{a,b,c,d\}$ and $R=\{(a,a),(b,b),(d,d),(a,b),(b,a)\}$. Then $aR=\{a,b\}=bR$, $cR=\phi$ and $dR=\{d\}$. Hence

$$\mathrm{S}=\{\phi,\{d\},\{a,b\}\},\ \mathrm{B}=\{X,\phi,\{d\},\{a,b\}\},\ \tau_K=\{X,\phi,\{d\},\{a,b\},\{a,b,d\}\},$$

$$SO(X)=\{X,\phi,\{d\},\{a,b\},\{c,d\},\{a,b,c\},\{a,b,d\}\}$$

and

$$SC(X)=\{\phi,X,\{a,b,c\},\{c,d\},\{a,b\},\{d\},\{c\}\}.$$

Since $A=\{c,d\}$ is a nonempty S-exact proper subset of X, then the space $\kappa=(X,R,\tau_K)$ is S-rough disconnected.

Proposition 2.2. Let $\kappa=(X,R,\tau_K)$ be a S-rough disconnected topologized approximation space, then there is a nonempty S-exact proper subset of X.

Proof. Let $\kappa=(X,R,\tau_K)$ be a S-rough disconnected topologized approximation space. Then there exist two nonempty subsets A and B of X such that

$A\cup B=X$ and $A\cap\overline{R}_s B=\overline{R}_s A\cap B=\phi$. But $A\subseteq\overline{R}_s A$, hence $A\cap B=\phi$. Thus $A=X-B$. Also $A=X-\overline{R}_s B$, since $A\cap\overline{R}_s B=\phi$ and $A\cup\overline{R}_s B\supseteq A\cup B=X$. Hence $A=\underline{R}_s A$ and $B=\overline{R}_s B$. Similarly $B=\underline{R}_s B$ and $A=\overline{R}_s A$. Therefore there exists a nonempty S-exact proper subset A of X. □

Theorem 2.1. A topologized approximation space $\kappa=(X,R,\tau_K)$ is S-rough disconnected if and only if there exists a nonempty S-exact proper subset of X.

Proof. By using Proposition 2.1 and Proposition 2.2, the proof is obvious. □

Definition 2.4 [3]. Let $\kappa=(X,R_1,\tau_K)$, $\mathcal{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces. Then a mapping $f:\kappa\to\mathcal{Q}$ is called S-rough continuous if $f^{-1}(\underline{R_2}V)\subseteq\underline{R_1}_s f^{-1}(V)$ for every subset V of Y in $\mathcal{Q}$.

In Definition 2.4, f^{-1} does not mean the inverse function, but it means the inverse image.

Theorem 2.2. Let $f:\kappa\to\mathcal{Q}$ be a mapping from a topologized approximation space $\kappa=(X,R_1,\tau_K)$ to a topologized approximation space $\mathcal{Q}=(Y,R_2,\tau_Q)$. Then the following statements are equivalent.

i) f is S-rough continuous.

ii) The inverse image of each internally R_2 $-$definable set in $\mathcal{Q}$ is internally S-definable set in κ.

iii) The inverse image of each externally R_2 $-$definable set in $\mathcal{Q}$ is externally S-definable set in κ.

iv) $f(\overline{R_1}_s A)\subseteq\overline{R_2}\,f(A)$ for every subset A of X in κ.

v) $\overline{R_1}_s f^{-1}(B)\subseteq f^{-1}(\overline{R_2}B)$ for every subset B of Y in $\mathcal{Q}$.

Proof.

(i)$\Rightarrow$(ii) Let f be S-rough continuous and let V be an internally R_2 $-$definable set in $\mathcal{Q}$. Then $\underline{R_2}V=V$ and $f^{-1}(V)$ is a subset of X in κ. By (i), we get

$f^{-1}(V)=f^{-1}(\underline{R_2}V)\subseteq\underline{R_1}_s f^{-1}(V)$. Then

$f^{-1}(V)\subseteq\underline{R_1}_s f^{-1}(V)$. But $\underline{R_1}_s f^{-1}(V)\subseteq f^{-1}(V)$. Hence

$f^{-1}(V)=\underline{R}_{1_s} f^{-1}(V)$. Therefore $f^{-1}(V)$ is internally s-definable set in κ.

(ii)$\Rightarrow$(i) Let A be a subset of Y in $\mathcal{Q}$. Since $\underline{R_2}A\subseteq A$, then $f^{-1}\left(\underline{R_2}A\right)\subseteq f^{-1}(A)$. But $\underline{R_2}A$ is internally R_2-definable set in $\mathcal{Q}$, then by (ii), we get $f^{-1}\left(\underline{R_2}A\right)$ is internally s-definable set in κ contained in $f^{-1}(A)$. Hence $f^{-1}\left(\underline{R_2}A\right)\subseteq \underline{R}_{1_s} f^{-1}(A)\subseteq f^{-1}(A)$, since $\underline{R}_{1_s} f^{-1}(A)$ is the largest internally s-definable set contained in $f^{-1}(A)$. Thus $f^{-1}\left(\underline{R_2}A\right)\subseteq \underline{R}_{1_s} f^{-1}(A)$ for every subset A of Y in $\mathcal{Q}$. Therefore f is s-rough continuous.

(ii)$\Rightarrow$(iii) Let F be an externally R_2-definable set in $\mathcal{Q}$, then by Lemma 2.1, we get $Y-F$ is internally R_2-definable. Thus by (ii), we have $f^{-1}(Y-F)$ is internally s-definable set in κ.

Since $f^{-1}(Y-F)=X-f^{-1}(F)$, then $X-f^{-1}(F)$ is internally s-definable set in κ. Hence $f^{-1}(F)$ is externally s-definable set in κ.

Similarly we can prove (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iv) Let A be a subset of X in κ, then $\overline{R_2}f(A)$ is an externally R_2-definable set in $\mathcal{Q}$. Hence $Y-\overline{R_2}f(A)$ is internally R_2-definable set in $\mathcal{Q}$. Thus by (ii), we get $f^{-1}\left(Y-\overline{R_2}f(A)\right)=X-f^{-1}\left(\overline{R_2}f(A)\right)$ is internally s-definable set in κ, and so $f^{-1}\left(\overline{R_2}f(A)\right)$ is externally s-definable set containing A in κ. Thus $A\subseteq\overline{R}_{1_s}A\subseteq f^{-1}\left(\overline{R_2}f(A)\right)$, since $\overline{R}_{1_s}A$ is the smallest externally s-definable set containing A in κ. Hence

$$f\left(\overline{R}_{1_s}A\right)\subseteq f\left[f^{-1}\left(\overline{R_2}f(A)\right)\right]\subseteq\overline{R_2}f(A).$$

Therefore $f\left(\overline{R}_{1_s}A\right)\subseteq\overline{R_2}f(A)$ for every subset A in κ.

(iv)$\Rightarrow$(v) Let B be a subset of Y in $\mathcal{Q}$. Let $A=f^{-1}(B)$, then A is a subset of X in κ. By (iv), we get

$$f\left(\overline{R}_{1_s}A\right)\subseteq\overline{R_2}f(A)=\overline{R_2}f\left(f^{-1}(B)\right)\subseteq\overline{R_2}B.$$

Hence $\overline{R}_{1_s}A\subseteq f^{-1}\left(\overline{R_2}B\right)$. Thus $\overline{R}_{1_s}A=\overline{R}_{1_s}f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}B\right)$.

Therefore $\overline{R}_{1_s}f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}B\right)$ for every subset B of Y in $\mathcal{Q}$.

(v)$\Rightarrow$(ii) Let G be an internally R_2-definable set in $\mathcal{Q}$, then $B=Y-G$ is externally R_2-definable set in $\mathcal{Q}$. Thus by (v), we get

$$\overline{R}_{1_s}f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}B\right).$$

Since B is externally R_2-definable set, then $f^{-1}\left(\overline{R_2}B\right)=f^{-1}(B)$. Thus $\overline{R}_{1_s}f^{-1}(B)\subseteq f^{-1}(B)$. But $f^{-1}(B)\subseteq\overline{R}_{1_s}f^{-1}(B)$, then $\overline{R}_{1_s}f^{-1}(B)=f^{-1}(B)$. Hence $f^{-1}(B)$ is externally s-definable set in κ.

Since $f^{-1}(B)=f^{-1}(Y-G)=X-f^{-1}(G)$, then $X-f^{-1}(G)$ is externally s-definable set in κ. Therefore $f^{-1}(G)$ is internally s-definable set in κ. □

Example 2.2. Let $\kappa=(X,R_1,\tau_K)$, $\mathcal{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces such that $X=\{a,b,c,d\}$, $Y=\{y_1,y_2,y_3,y_4\}$,

$R_1=\{(a,a),(b,b),(d,d),(a,b),(b,a)\}$ and $R_2=\{(y_1,y_1),(y_4,y_4),(y_1,y_2)\}$. Then

$\tau_K = \{X, \phi, \{d\}, \{a,b\}, \{a,b,d\}\}$ and $\tau_Q = \{Y, \phi, \{y_4\}, \{y_1, y_2\}, \{y_1, y_2, y_4\}\}$. Hence

$$SO(X) = \{X, \phi, \{d\}, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}\}.$$

Define a mapping $f : \kappa \to \mathcal{Q}$ such that

$$f(a) = y_1,\ f(b) = y_2,\ f(c) = y_4 \text{ and } f(d) = y_3.$$

Then f is not a S-rough continuous mapping, since $V = \{y_4\}$ is an internally R_2-definable set in $\mathcal{Q}$, but $f^{-1}(V) = \{c\}$ is not an internally S-definable set in κ.

Proposition 2.3. Let $\kappa = (X, R_1, \tau_K)$ and $\mathcal{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. If $f : \kappa \to \mathcal{Q}$ is a S-rough continuous mapping, then the inverse image of each exact set in $\mathcal{Q}$ is S-exact set in κ.

Proof. Let A be an exact set in $\mathcal{Q}$, then A is both internally and externally R_2-definable set in $\mathcal{Q}$. Hence by Theorem 2.2, we get $f^{-1}(A)$ is both internally and externally S-definable set in κ. Therefore $f^{-1}(A)$ is a S-exact set in κ. □

3. Conclusions

In this paper, we used S-open sets to introduce the definition of S-rough connected topologized approximation space.

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