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ONE CONTRACTIVE INEQUALITY ON QUASI-NORMED SPACE

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ABSTRACT

We analyze the existence of fixed points for mappings defined on quasi normed Banach spaces $(X,\|.\|)$ satisfying a general contractive inequality of integral type. We are affected from the similar results achieved by A. Branciari and B. E. Rhoades. This condition is extension to Banach-Caccioppoli's one. We study mappings $f: X \to X$ for which there exists a real number $c \in]0,1[$ such that for each $x, y \in X$ we have $\int_0^{\|f(x)-f(y)\|} \|f(t)\| dt \le (2-c)c \int_0^{\|x-y\|} \|f(t)\| dt$ where f is a strong measurable mapping which is summable [7], [8, p.1] as well as $\|\phi\|$ on each compact subset of $[0,+\infty[$ and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \|f(t)\| dt > 0$. We give a general condition which enables one to easily establish fixed point theorems for a pair of maps satisfying a contractive inequality of integral type.

Keywords:
fixed points; contractive-type mapping; integral-type inequality; quasi normed space

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Introduction

The first important result on fixed points for contractive-integral-type mappings was the well-known Banach-Caccioppoli [4] A. Branciari [3] and B. E. Rhoades [7] obtained a fixed point result for a single mapping satisfying an analogue of Banach's [1] contraction principle for an integral-type inequality. Their main condition has the form $\int_0^{d(fx,fy)} f(t)dt \leq c\int_0^{d(x,y)} f(t)dt$ with the same above conditions We substitute this one with

$$\int_0^{\|f(x)-f(y)\|} \|f(t)\|dt \leq (2-c)c\int_0^{\|x-y\|} \|f(t)\|dt$$

where ||.|| is a quasi-norm; that is provided with property of usual norm expect the triangle inequality which is substitute with

$$\|x+y\| \leq K(\|x\|+\|y\|), K>1$$

From the inequality

$$0<c<(2-c)c<1$$

we claim to have a better result than Branciari one.

The example 3.2 shows that.

In the general setting of complete quasi-normed spaces this theorem runs as follows .

Theorem 1.1. Let (X,d) be a complete normed space, $c\in\left]0,1\right[$ and let $f:X\rightarrow X$ be a mapping such that for each

$$x,y\in X\ \|fx-fy\|\leq c\times\|x-y\| \quad (1.1)$$

then f has a unique fixed point $a\in X$ such that for each $x\in X$, $\lim_{n\rightarrow\infty} f^n x=a$

After this classical result Kannan in [5] analyzed a substantially new type of contractive condition through integral. The aim of this paper is to analyze the existence of fixed points for mappings f defined on a complete quasi normed space (X,d) satisfying a contractive condition of integral type (see (2.1) below).

Section 2 contains the main result. At the end of the paper some remarks and two examples concerning this kind of contractions are given.

In the sequel, $\mathbb{N}$ will represent the set of natural numbers (starting from 1), $\Re$ the set of real numbers, and $\Re^+$ the set of nonnegative real numbers

1. Main Results

The following theorem is the main result of this paper; the proof, which proceeds by steps, is based on an argument similar to the one used by Boyd and Wong [2, Theorem 1].

Theorem 2.1. Let (X,d) be a quasi normed complete space, $c\in\left]0,1\right[$ and let $f:X\rightarrow X$ be a mapping such that for each $x,y\in X$

$$\int_0^{\|f(x)-f(y)\|} \|f(t)\|dt \leq (2-c)c.\int_0^{\|x-y\|} \|f(t)\|dt \quad (2.1)$$

where ϕ is Bochner-integrable mapping which is summable (i.e., with finite integral) on each compact subset of X and such that for each $\varepsilon>0$

$$\int_0^{\varepsilon} \|f(t)\| dt > 0$$

then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$

Proof

Step 1. We have

$$\int_0^{\|f^n x - f^{n+1}x\|} \|f(t)\| \, dt \le (2-c)^n c^n \int_0^{\|x - fx\|} \|f(t)\| dt \tag{2.2}$$

This follows immediately by iterating (2.1) n times:

$$\int_0^{\|f^n x - f^{n+1}x\|} \|f(t)\| \, dt \le (2-c)c \int_0^{\|f^{n-1} x - f^{n-2}x\|} \|f(t)\| dt \le \cdots$$
$$\le (2-c)^n c^n \int_0^{\|x - fx\|} \|f(t)\| dt \le (2-c)^n c^n \cdot \| x - fx \| \cdot M \tag{2.3}$$

where $\varphi(t)$ is bounded as an integrable function.

As a consequence, since $c \in \left]0,1\right[$ we further have

$$\int_0^{\|f^n x - f^{n+1}x\|} \|f(t)\| dt \to 0^+ \text{ as } n \to +\infty \tag{2.4}$$

Step 2. We have $\| f^n x - f^{n+1} x \| \to 0^+$ as $n \to +\infty$. Suppose that

$$\lim_{n\to+\infty} \sup \| f^n x - f^{n+1} x \| = \varepsilon > 0 \tag{2.5}$$

then there exists an $\upsilon_\varepsilon \in \mathrm{N}$ and a sequence $\left(f^{n_\upsilon} x\right)_{\upsilon \ge \upsilon_\varepsilon}$ such that $\| f^{n_\upsilon} x - f^{n_\upsilon+1} x \| \to \varepsilon > 0$ as $\upsilon \to +\infty$ $\| f^{n_\upsilon} x - f^{n_\upsilon+1} x \| \ge \frac{\varepsilon}{2}$ for each $\upsilon \ge \upsilon_\varepsilon$,

thus (by Step 1 and the sign of $\|\varphi(t))\|$) we have the following contradiction:

$$0 = \lim_{\upsilon\to+\infty} \int_0^{\|f^{n_\upsilon} x - f^{n_\upsilon+1}x\|} \|f(t)\| \, dt \ge \int_0^{\frac{\varepsilon}{2}} \|f(t)\| dt > 0 \tag{2.6}$$

Step 3. For $x \in X$ $\left\{f^n x\right\}_{n \in \mathrm{N}}$ is a Cauchy sequence, that is

$$\forall \varepsilon > 0 \quad \exists \upsilon_\varepsilon \in \mathbb{N} \mid \forall m, n \in \mathbb{N}, \quad m > n > \upsilon_\varepsilon \quad \| f^m x - f^n x \| < \varepsilon \tag{2.7}$$

Suppose that there exists an $\varepsilon > 0$ such that for each ν∈N there are $m_\upsilon, n_\upsilon \in \mathbb{N}$ with $m_\upsilon > n_\upsilon > \upsilon$, such that $\| f^{m_\upsilon} x - f^{n_\upsilon} x \| \ge \varepsilon$, then we choose the sequences $\left(m_\upsilon\right)_{\upsilon \in \mathrm{N}}$ and $\left(n_\upsilon\right)_{\upsilon \in \mathrm{N}}$ such that for each $\upsilon \in \mathbb{N}$ m_υ is "minimal" in the sense that $\| f^{m_\upsilon} x - f^{n_\upsilon} x \| \ge \varepsilon$ but $\| f^h x - f^{n_\upsilon} x \| < \frac{\varepsilon}{K}$ for each $h \in \left\{n_\upsilon + 1, \cdots, m_\upsilon - 1\right\}$

Now we analyze the properties of $\| f^{m_u}x - f^{n_u}x \|$ and $\| f^{m_u+1}x - f^{n_u+1}x \|$. First of all, we have $\| f^{m_u}x - f^{n_u}x \| \to e$ as $\upsilon \to +\infty$, in fact by the triangular inequality and Step 2

$$e \le \| f^{m_u}x - f^{n_u}x \|$$
$$\le K[\| f^{m_u}x - f^{m_u-1}x \| + \| f^{m_u-1}x - f^{n_u}x \|] \qquad (2.8)$$
$$< K \| f^{m_u}x - f^{m_u-1}x \| + e \xrightarrow{u \to +\infty} e^{+}$$

further there exists $\mu \in N$ such that for each natural number $\upsilon > \mu$, one has $\| f^{m_u+1}x - f^{n_u+1}x \| < \frac{e}{K^2}$; in fact, if there exists a subsequence $(\upsilon_k)_{k \in N} \subseteq \mathbb{N}$ such that $\| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| \ge e$, then

$$e \le \| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| \le K \| f^{m_{u_k}+1}x - f^{m_{u_k}}x \|$$
$$+ K^2 \| f^{m_{u_k}}x - f^{n_{u_k}} \| + K^2 \| f^{n_{u_k}}x - f^{n_{u_k}+1}x \| \xrightarrow{k \to \infty} e \qquad (2.9)$$

And from (2.1)

$$\int_0^{\| f^{m_{uk}+1}x - f^{n_{uk}+1}x \|} \| f(t) \| \, dt \le (2-c)c \int_0^{\| f^{m_{uk}}x - f^{n_{uk}}x \|} \| f(t) \| \, dt \qquad (2.10)$$

Let (e), $e > 0$ be a decreasing sequence convergent to zero. Letting now $k \to +\infty$ in both sides of 2.10 we have $0 < \int_0^e \| f(t) \| dt \le (2-c)c \int_0^e \| f(t) \| dt$ which is a contradiction because $(2-c)c < 1$ being $c \in]0,1[$ and the integral being positive. Therefore for a certain $\mu \in \mathbb{N}$ one has $\| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| < e$ for all $\upsilon > \mu$.

Finally we prove the stronger property. That there exists $s \in]0, e[$ and $\upsilon_\varepsilon \in \mathbb{N}$ such that for each $\upsilon > \upsilon_\varepsilon$ $(\upsilon \in \mathbb{N})$ we have $\| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| < \frac{e-s}{K^2}$ for every $n \in \mathbb{N}$ suppose the existence of a subsequence $(\upsilon_k)_{k \in N} \subseteq N$ such $\| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| \to e$ as $k \to +\infty$, then from

$$\int_0^{\| f^{m_{uk}+1}x - f^{n_{uk}+1}x \|} \| f(t) \| dt \le (2-c)c \int_0^{\| f^{m_{uk}}x - f^{n_{uk}}x \|} \| f(t) \| \, dt \qquad (2.11)$$

letting $k \to +\infty$ we have again the contradiction $0 < \int_0^e \| f(t) \| dt \le (2-c)c \int_0^e \| f(t) \| \, dt$. In conclusion of this step we can prove the Cauchy character of $(f^n x)_{n \in N}$ $(x \in X)$ in fact for each natural number $\upsilon > \upsilon_\varepsilon$ $(\upsilon_\varepsilon$ as above$)$

$$e \le \| f^{m_{u_k}}x - f^{n_{u_k}}x \| \le K \| f^{m_{u_k}}x - f^{m_{u_k}+1}x \|$$
$$+ K^2 \| f^{m_{u_k}+1}x - f^{n_{u_k}+1}x \| + K^2 \| f^{n_{u_k}+1}x - f^{n_{u_k}}x \| \qquad (2.12)$$
$$< K \| f^{m_{u_k}}x - f^{m_{u_k}+1}x \| + (e - s) + K^2 \| f^{n_{u_k}+1}x - f^{n_{u_k}}x \| \xrightarrow{u \to +\infty} e - s.$$

Thus $e < e - s$ which is a contradiction. This proves the step 3.

Step 4. Existence of a fixed point. Since $(X, \|.\|)$ is a complete quasi-normed space, there exists a point $x \in X$ such that $\lim_{n\to\infty} f^n x = x$. Further x is a fixed point, in fact suppose that ||x-fx||>0, thus

$$0 < \|x - fx\| \le K(\|x - f^{n+1}x\| + \|f^{n+1}x - fx\|) \to 0 \qquad (2.13)$$

In fact both $\|x - f^{n+1}x\|$ and $\|f^{n+1}x - fx\|$ converge to 0 when $n \to +\infty$. For the first one it is obvious while for the second one we have

$$\int_0^{\|f^{n+1}x - fx\|} \|f(t)\| dt \le (2-c)c \int_0^{\|f^n x - x\|} \|f(t)\| dt \xrightarrow{n\to+\infty} 0 \qquad (2.14)$$

Step 5. Uniqueness of the fixed point. Suppose that there are two distinct points $x_1, x_2 \in X$ such that $fx_1 = x_1$, $fx_2 = x_2$ and $\lim_{n\to\infty} f^n x_1 = x_1$ $\lim_{n\to\infty} f^n x_2 = x_2$

From the inequality

$$\|x_1 - x_2\| \le K\|x_1 - f^n x_1\| + K^2\|f^n x_1 - f^n x_2\| + K^2\|f^n x_2 - x_2\|$$

we have

$$0 < \int_0^{\|x_1 - x_2\|} \|f(t)\| dt \le \int_0^{K\|x_1 - f^n x_1\|} \|f(t)\| dt + \int_0^{K^2\|f^n x_1 - f^n x_1\|} \|f(t)\| dt + \int_0^{K^2\|f^n x_2, x_2\|} \|f(t)\| dt \qquad (2.15)$$

and inequality

$$\int_0^{\|f^n x_1 - f^n x_2\|} \|f(t)\| dt \le (2-c)^n c^n \int_0^{\|x_1 - x_2\|} \|f(t)\| dt \to 0$$

then by (2.1) bring us to the contradiction.

This final step also proves that for each $x \in X$ such that $\lim_{n\to\infty} f^n x = x_1 = fx_1$ .The proof is thus completed.

2. Examples

In this section, we give some remarks and examples concerning contractive mappings of this integral type, similar with the examples given and proved by from Branciari which clarify the connection between our result and the classical ones.

Remark 1. Theorem 2.1 is a generalization of the Banach-Caccioppoli principle, letting $\varphi(t) = 1$ for each $t \ge 0$ in (2.1), we have

$$\int_0^{\|fx - fy\|} f(t) dt = \|fx - fy\| \le (2-c)c \cdot \|x - y\| = (2-c)c \int_0^{\|x-y\|} f(t) dt \qquad (3.1)$$

thus a Banach-Caccioppoli contraction also satisfies (2.1). The converse is not true.

The condition that for each $e > 0$ we must have $\int_0^{\varepsilon} \varphi(t) dt > 0$ is essential for the existence of fixed point as we will show in example below.

Example 2 . In order to compare our condition with above Branciari condition and to see their difference we can construct one example that their condition is not fulfilled but our conditions stand.

a) Let we take $\phi(t) = t$, $t > 0$, $c = \frac{1}{3}$, $\|x - y\| \le 1$ and $f(x) = \begin{cases} 1 \text{ for x} \ne 1 \\ \frac{3}{2} \text{ for x=1} \end{cases}$.

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We have that $\| fx - fy \| = \frac{1}{2}$ and

$$\int_0^{\|fx-fy\|} t\,dt = \int_0^{\frac{1}{2}} t\,dt = \frac{1}{8} \leq (2-c)\int_0^{\|x-y\|} t\,dt = \frac{5}{3}\frac{1}{3}\int_0^1 t\,dt = \frac{5}{18}$$

On the other side in these conditions the Branciari result is not satisfied.

$$\int_0^{\|fx-fy\|} t\,dt = \int_0^{\frac{1}{2}} t\,dt = \frac{1}{8} \geq c\int_0^{\|x-y\|} dt = \frac{1}{3}\int_0^1 t\,dt = \frac{1}{6}.$$

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