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# SOME FUNCTIONAL RELATIONS OF ERROR! BOOKMARK NOT DEFINED.I-FUNCTIONS USING SAIGO FRACTIONAL INTEGRAL OPERATOR

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**ABSTRACT:** In the present paper we seek to establish functional relations between digamma functions and I-functions by using Saigo fractional integral operator. This paper is extension of the paper of functional relations of I-function by using Riemann-Liouville & Weyl fracatinal integral operators and Erdėlyi-Kober fracational integral operators.

**KEY WORDS**: Gamma function; I-function; Saigo fractional integral operator; Laplace transform; digamma function.

**MATHEMATICS SUBJECT CLASSIFICATION**: Primary33C60, Secondary33E50.

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## INTRODUCTION

In a fractional calculus there are many types of fractional integral operators like as Riemann-Liouville & Weyl, Erdėlyi-Kober, pathway operator but in this paper we use Saigo fractional integral operator.The Saigo fractional integral operator was introduced by Saigo [6, 8] which is defined as follows;

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta-\eta;\alpha;1-\frac{t}{x}\right) f(t)\,dt \quad (1)$$ where $Re(\alpha) \geq 0$, and

$$J_{x,\infty}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma\alpha} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta-\eta;\alpha;1-\frac{x}{t}\right) f(t)\,dt \quad (2)$$

where $Re(\alpha) > 0$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function which is defined as:

$${}_2F_1(a,b;c;x) = {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (3)$$

The Saigo fractional integral operator (1) contains both the Riemann-Liouville and Erdėlyi-Kober fractional integral operators by the following relationships:

$$R_{0,x}^{\alpha} f(x) = I_{0,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} f(t)\,dt \quad (4)$$

and

$$E_{0,x}^{\alpha,\eta} f(x) = I_{0,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\,dt \quad (5)$$

and the operator (2) contains the Weyl and Erdėlyi-Kober fractional integral operators by the following relationships:

$$W_{x,\infty}^{\alpha} f(x) = J_{x,\infty}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma\alpha} \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt \quad (6)$$

and

$$K_{x,\infty}^{\alpha,\eta} f(x) = J_{x,\infty}^{\alpha,0,\eta} f(x) = \frac{x^{\eta}}{\Gamma\alpha} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,dt \quad (7)$$

## The I-Function

The $I$-function is the generalization of $H$-function which was introduced by Saxena [9], while is defines as follows:

$$I(x) = I_{p_i,q_i;r}^{m,n}[x] = I_{p_i,q_i;r}^{m,n}\left[x \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix} \right.\right] \quad (8)$$

$$= \frac{1}{2\pi i} \int_L \mathcal{X}(s)\, x^s\, ds \quad (9)$$

Where $x \neq 0$ and $x = exp\{s\, Log\,|x| + i \arg x\}$ in which $Log\,|x|$ represents the natural logarithmic of $|x|$ and $\arg x$ is not necessarily the principal value. Here

$$\mathcal{X}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1-a_j+A_j s)}{\sum_{i=1}^{r}\{\prod_{j=m+1}^{q_i} \Gamma(1-b_{ji}+B_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji}-A_{ji} s)\}} \quad (10)$$

with all other conditions as already detailed in [18].

The following results [10] are required in this continuation.

$$\frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x t^{\rho-1}(x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{t}{x}\right) I_{p_i,q_i;r}^{m,n}\left[at \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix} \right.\right] dt =$$

$$x^{\rho-\beta-1} I_{p_i+2,q_i+2;r}^{m,n+2}\left[ax \left| \begin{matrix} (1-\rho,1),(1-\rho-\eta+\beta,1),(a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i}, (1-\rho+\beta,1)(1-\rho-\alpha-\eta,1) \end{matrix} \right.\right] \quad (11)$$

$$\frac{1}{\Gamma\alpha} \int_x^\infty t^{\rho-1}(t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{x}{t}\right) I_{p_i,q_i;r}^{m,n}\left[at \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix} \right.\right] dt =$$

$$x^{\rho-\beta-1} I_{p_i+2,q_i+2;r}^{m+2,n}\left[ax \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i}, (1-\rho,1)(1-\rho+\alpha+\beta+\eta,1) \\ (1-\rho+\beta,1)(1-\rho+\eta,1), (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix} \right.\right] \quad (12)$$

where $s = (b_h + v)/B_{h'}$ $Re(\rho + \min b_j/B_j) > 0,\ j = 1,2,\ldots,m;\ \ Re(\alpha) \geq 0;\ |\arg ax| < (1/2)\pi\lambda_i,\ \lambda_i = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_i} B_{ji} > 0$, where $i = 1,2,\ldots,r$; $\mu_i = \sum_{j=1}^{n} A_j + \sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{m} B_j + \sum_{j=m+1}^{q_i} B_{ji} \leq 0$. or $\mu_i = 0$ ; $i = 1,2,\ldots,r$ and $0 < |at| < \beta_i^{-1}$; $\beta_i = \prod_{j=1}^{p_i} (A_{ji})^{A_{ji}} \prod_{j=1}^{q_i} (B_{ji})^{-B_{ji}}$ , $i = 1,2,\ldots,r$.

Equations (11) and (12) find by using the following integral:

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$$\int_0^x t^{\rho+s-1}(x-t)^{\sigma+k-1}\,dt = x^{\rho+s+\alpha+k-1}\beta(\rho+s,\alpha+k) \tag{13}$$

$$\int_x^\infty t^{\rho-\alpha-\beta+s-k-1}(t-x)^{\alpha+k-1}\,dt = x^{\rho-\beta+s-1}\frac{\Gamma(1-\rho+\beta-s)\Gamma(\alpha+k)}{\Gamma(1-\rho+\alpha+\beta+k-s)} \tag{14}$$

where, $Re(\rho) > 0, Re(\alpha) > 0$.

The computable form of I-function can be written as:

$$I_{p_i,q_i;r}^{m,n}[x] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v x^s \mathcal{X}(s)}{v!\,B_h} \tag{15}$$

where $s = (b_h + v)/B_h$ exist for all $x \neq 0$ if $\mu_i < 0$ and $0 < |x| < \beta_i^{-1}$ if $\mu_i = 0$, where $\mu_i = \sum_{j=1}^{n} A_j + \sum_{j=n+1}^{p_i} A_{ji} -$ *j=1mBj+j=m+1qiBji* and *βi=j=1piAjiAjij=1qiBji−Bji, i=1,2,….,r.* Here $\chi(s)$ is given by (10).

## THE FUNCTIONAL RELATIONS

### 1. In this section, we will establish the following relations by using Saigo fractional integral operator:

$$\sum_{k=1}^{\infty}\frac{(\alpha)_k}{k}\times$$
$$\left\{\left(\sum_{l=0}^{\infty}\frac{(\alpha+\beta)_l(-\eta)_l(\alpha+k)_l}{l!(\alpha)_l}\right)\times\left(I_{p_i+2,q_i+2;r}^{m,n+2}\left[ax\left|\begin{matrix}(1-\rho,1),(1-\rho-\alpha-\beta-\eta,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta,1),(1-\rho-\alpha-\beta-l-k,1)\end{matrix}\right.\right]\right)\right\}=$$
$$\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\mathcal{X}(s)}{v!\,B_h}\{\psi(\rho+s)+\psi(\rho+\sigma+\beta+\eta+s)-\psi(\rho+\beta+s)-\psi(\rho+\eta+s)\} \tag{16}$$

where $s = (b_h + v)/B_h$ , $Re(\rho + \min b_j/B_j) > 0,\ j = 1,2,\ldots,m;\ Re(\alpha) \geq 0, |\arg ax| < (1/2)\pi\lambda_i,\ \lambda_i > 0;\ \mu_i \leq 0.\ \lambda_i$ and $\mu_i$ are defined where $\psi$ is the logarithmic derivative of Gamma function.

**Proof:** On differentiating both sides of (11) with respect to $\rho$ according to Leibnitz's rule it is found that

$$\frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x t^{\rho-1}\ln t\,(x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{t}{x}\right)I_{p_i,q_i;r}^{m,n}\left[at\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]dt=$$
$$x^{\rho-\beta-1}\left[\ln x\ I_{p_i+2,q_i+2;r}^{m,n+2}\left[ax\left|\begin{matrix}(1-\rho,1),(1-\rho-\eta+\beta,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho+\beta,1)(1-\rho-\alpha-\eta,1)\end{matrix}\right.\right]+\right.$$

*h=1mv=0∞−1vaxsXsΓρ+sΓρ+η−β+sv! BhΓρ−β+sΓρ+α+η+sψρ+s+ψρ+η−β+s−ψρ−β+s−ψρ+α+η+s*
(17)
where $s = (b_h + v)/B_h$.

The R. H. S. of the equation (17) will be designated by $J(a_{p_i}, A_{p_i}; b_{q_i}, B_{q_i}; \rho, \alpha, \beta, \eta, a)$.

In term of Saigo fractional integral operator, equation (17) can be written as:

$$I_{0,x}^{\alpha,\beta,\eta}f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{t}{x}\right)f(t)\,dt \tag{18}$$

On the account of the property of analyticity and continuity at $\alpha = 0, \beta = 0$ and $\eta = 0$, we interchanging the role of $\eta$ by $\eta - \alpha$, $\beta$ by $-\beta$ and $\alpha$ by $-\alpha$. Hence for the differentiation of

$$x^{\rho-1}\ln x\,I_{p_i,q_i;r}^{m,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

to an arbitrary order, we find that

$$I_{0,x}^{-\alpha,-\beta,\eta-\alpha}(x^{\rho-1}\ln x\ I(ax)) = x^{\rho+\beta-1}\left[\ln x\ I_{p_i+2,q_i+2;r}^{m,n+2}\left[ax\left|\begin{matrix}(1-\rho,1),(1-\rho-\eta-\alpha-\beta,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\beta,1)(1-\rho-\eta,1)\end{matrix}\right.\right]+\right.$$

*h=1mv=0∞−1vaxsXsΓρ+sΓρ+α+β+η+sv! BhΓρ+β+sΓρ+η+sψρ+s+ψρ+α+β+η+s−ψρ+β+s−ψρ+η+s*
(19)

Now we consider the following integral equation of Volterra type:

$$\frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x t^{\rho-1}(x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt = x^{\rho-1}\ln x\, I^{m,n}_{p_i,q_i;r}(ax) \qquad (20)$$

where, $Re\left(\rho+\min\ b_j/B_j\right)>0, Re\,(\sigma)\geq 0, |\arg a|<(1/2)\pi\lambda_i,\ \lambda_i>0;\mu_i\leq 0.$

Since (20) is of convolution type, it can be solved by applying Laplace transform. However, we use the technique of fractional integration operator to solve it, due to its elegance and simplicity.

On writing (20) in the operator form, we have

$$I^{\alpha,\beta,\eta}_{0,x}f(x) = x^{\rho-1}\ln x\, I^{m,n}_{p_i,q_i;r}(ax) \qquad (21)$$

Operating on both side of (21) with $I^{-\alpha,-\beta,\eta-\alpha}_{0,x}$

$$f(x) = I^{-\alpha,-\beta,\eta-\alpha}_{0,x}\left(x^{\rho-1}\ln x\, I^{m,n}_{p_i,q_i;r}(ax)\right) \qquad (22)$$

In view of (19), then we can write the solution of the integral (22) as

$$f(x) = x^{\rho+\beta-1}\times \left[\ln x\ I^{m,n+2}_{p_i+2,q_i+2;r}\left[ax\left|\begin{matrix}(1-\rho,1),(1-\rho-\eta-\alpha-\beta,1),\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i},(1-\rho-\beta,1)(1-\rho-\eta,1)\end{matrix}\right.\right]+\right.$$

*h=1mv=0∞−1vaxsXsΓρ+sΓρ+α+β+η+sv!   BhΓρ+β+sΓρ+η+sψρ+s+ψρ+α+β+η+s−ψρ+β+s−ψρ+η+s*

(23)

to verify the solution, we substitutes (23) in to (20) in terms of argument $t$.

On writing

$$t = x+t-x = x\left(1+\frac{t-x}{x}\right) = x\left(1-\frac{x-t}{x}\right)$$

Where $x$ and $t$ are real and $x>0$, so we obtain a series expansion of $\ln t$ in the form

$$\ln t = \ln x + \ln\left(1+\frac{t-x}{x}\right) \qquad (24)$$

when $|(t-x)/x|<1$, $\ln\big(1+((t-x)/x)\big)$ can be expanded in to a Taylor's series expansion. Thus

$$\ln t = \ln x - \sum_{k=1}^{\infty}\frac{(x-t)^k}{kx^k} \qquad (25)$$

with the interval of the convergence $0<t\leq 2x$ .

If we substitute (23) and (25) in (20) and evaluate the corresponding beta type integrals, the desired result (16) is achieved.

### Special Case:

Putting $l=0,\eta=0,\beta=0$ and $\rho=\rho-\alpha$ in equation (16), then we get the functional relations by using Riemann Liouville operator [3]

$$\sum_{k=1}^{\infty}\frac{(\alpha)_k}{k}I^{m,n+1}_{p_i+1,q_i+1;r}\left[ax\left|\begin{matrix}(1-\rho,1),\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i},(1-\rho-k,1)\end{matrix}\right.\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\mathcal{X}(s)}{v!\,B_h}\{\psi(\rho+s)-\psi(\rho-\alpha+s)\}$$

(26)

If we putting $p_i=p$ , $q_i=q$ for all values of $i$ and $r=1$ in equation (26). Then we get the functional relation of H-function [5].

$$\sum_{k=1}^{\infty}\frac{(\alpha)_k}{k}H^{m,n+1}_{p+1,q+1}\left[ax\left|\begin{matrix}(1-\rho,1),\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i},(1-\rho-k,1)\end{matrix}\right.\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\mathcal{X}(s)}{v!\,B_h}\{\psi(\rho+s)-\psi(\rho-\alpha+s)\}$$

(27)

## 2. In this section, we establish the following functional relations by using Saigo fractional integral operator:

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$\sum_{k=1}^{\infty} \frac{(-1)^k(\alpha)_k}{k} \left( \sum_{l=0}^{\infty} \frac{(\alpha+\beta)_l(-\eta)_l(\alpha+k)_l}{(\alpha)_l\, l!} \right) \times$

$I_{p_i+3,q_i+3;r}^{m+3,n} \left[ ax \left| \begin{matrix} (a_j,A_j)_{1,n}; (a_{ji},A_{ji})_{n+1,p_i}, (1-\rho,1)\,(1-\rho-\beta+\eta,1)(1-\rho+\alpha+l,1) \\ (1-\rho-\beta,1)(1-\rho+\alpha+\eta,1)(1-\rho-k,1), (b_j,B_j)_{1,m}; (b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] = \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v(ax)^s \mathcal{X}(s)}{v!\, B_h} \{\psi(1-\rho-$

$\beta-s+\psi 1-\rho+\alpha+\eta-s-\psi 1-\rho-s-\psi 1-\rho-\beta+\eta-s$ (28)

where $s=(b_h+v)/B_h$ , $Re(\rho+\min b_j/B_j)>0,\ j=1,2,\dots,m;\ \ Re(\alpha)\ge 0, |\arg ax|<(1/2)\pi\lambda_i,\ \ \lambda_i>0;\ \ \mu_i\le 0.$ $\lambda_i$ *and* $\mu_i$ are defined where $\psi$ is the logarithmic derivative of Gamma function.

**Proof:** On differentiating both sides of (12) with respect to $\rho$ according to Leibnitz's rule it is found that

$\frac{1}{\Gamma\alpha}\int_x^{\infty} t^{\rho-1} \ln t\, (t-x)^{\alpha-1}\, t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{x}{t}\right) \times I_{p_i,q_i;r}^{m,n}\left[ at \left| \begin{matrix} (a_j,A_j)_{1,n}; (a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m}; (b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] dt =$

$x^{\rho-\beta-1}\left[ \ln x\, I_{p_i+2,q_i+2;r}^{m+2,n}\left[ ax \left| \begin{matrix} (a_j,A_j)_{1,n}; (a_{ji},A_{ji})_{n+1,p_i}, (1-\rho,1)(1-\rho+\alpha+\beta+\eta,1) \\ (1-\rho+\beta,1)(1-\rho+\eta,1), (b_j,B_j)_{1,m}; (b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] \right. +$

$h=1mv=0\infty-1vaxs\mathcal{X}s\Gamma 1-\rho+\eta-s\Gamma 1-\rho+\beta-sv!$
$Bh\Gamma 1-\rho+\alpha+\beta+\eta-s\Gamma 1-\rho-s\psi 1-\rho+\beta-s+\psi 1-\rho+\eta-s-\psi 1-\rho-s-\psi 1-\rho+\eta+\sigma-s$
(29)

where $s=(b_h+v)/B_h$ .

The R. H. S. of the (29) will be designated by $J(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\alpha,\beta,\eta,a)$.

In term of Saigo operator, (29) can be written as

$$J_{x,\infty}^{\alpha,\beta,\eta}\left(x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)\right) = J(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\eta,\sigma,a) \qquad (30)$$

On the account of the properties of analyticity and continuity at $\alpha=0,\beta=0$ and $\eta=0$, we interchanging the role of $\eta$ by $\eta-\alpha$, $\beta$ by $-\beta$ and $\alpha$ by $-\alpha$ . Hence for the differentiation of

$$x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)$$

to an arbitrary order, we find that

$J_{x,\infty}^{-\alpha,-\beta,\eta-\alpha}\left(x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)\right) = J(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,-\alpha,-\beta,\eta-\alpha,a)$

$J_{x,\infty}^{-\alpha,-\beta,\eta-\alpha}\left(x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)\right) = x^{\rho+\beta-1}\left[ \ln x\, I_{p_i+2,q_i+2;r}^{m+2,n}\left[ ax \left| \begin{matrix} (a_j,A_j)_{1,n}; (a_{ji},A_{ji})_{n+1,p_i}, (1-\rho,1)(1-\rho-\beta+\eta,1) \\ (1-\rho-\beta,1)(1-\rho+\eta+\alpha,1), (b_j,B_j)_{1,m}; (b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] \right. +$

$h=1mv=0\infty-1vaxs\mathcal{X}s\Gamma 1-\rho+\eta+\alpha-s\Gamma 1-\rho-\beta-sv!$
$Bh\Gamma 1-\rho-\beta+\eta-s\Gamma 1-\rho-s\psi 1-\rho-\beta-s+\psi 1-\rho+\eta+\alpha-s-\psi 1-\rho-s-\psi 1-\rho-\beta+\eta-s$
(31)

where $s=(b_h+v)/B_h$.

Now we consider the following Volterra type integral:

$$\frac{1}{\Gamma\alpha}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta.-\eta;\alpha;1-\frac{x}{t}\right) f(t)dt = x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax) \qquad (32)$$

where $Re(\rho+\min b_j/B_j)>0, Re\ \sigma\ge 0, |\arg a|<(1/2)\pi\,\lambda_i,\ \lambda_i>0;\mu_i\le 0.$

Since (32) is of convolution type, it can be solved by applying Laplace transform. However, we use the technique of fractional integration operator to solve it, due to its elegance and simplicity.

On writing (32) in the operator form, we have

$$J_{x,\infty}^{\alpha,\beta,\eta}\left(x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)\right) = x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax) \qquad (33)$$

Operating on both sides of (33) with $J_{x,\infty}^{-\alpha,-\beta,\eta-\alpha}$ on both sides, we get

$$f(x) = J_{x,\infty}^{-\alpha,-\beta,\eta-\alpha}\left(x^{\rho-1}\ln x\, I_{p_i,q_i;r}^{m,n}(ax)\right) \qquad (34)$$

In view of (30) we can write the solution of the integral (31) as

$f(x) =$

$$x^{\rho+\beta-1}\left[\ln x\, I^{m+2,n}_{p_i+2,q_i+2;r}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho,1)(1-\rho-\beta+\eta,1)\\(1-\rho-\beta,1)(1-\rho+\eta+\alpha,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]+\right.$$

*h=1mv=0∞−1vaxsXsΓ1−ρ+η+α−sΓ1−ρ−β−sv!*
*BhΓ1−ρ−β+η−sΓ1−ρ−sψ1−ρ−β−s+ψ1−ρ+η+α−s−ψ1−ρ−s−ψ1−ρ−β+η−s*
(35)

To verified the solution, we substitutes (35) in to (32) in terms of argument $t$.

On writing

$$t = x + t - x = x\left(1 + \frac{t-x}{x}\right) = x\left(1 - \frac{x-t}{x}\right)$$

Where $x$ and $t$ are real and $x > 0$, so we obtain a series expansion of $\ln t$ in the form

$$\ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right) \qquad (36)$$

when $|(t-x)/x| < 1$, $\ln\big(1 + ((t-x)/x)\big)$ can be expanded in to a Taylor's series expansion. Thus

$$\ln t = \ln x - \sum_{k=1}^{\infty}\frac{(x-t)^k}{kx^k} \qquad (37)$$

with the interval of the convergence $0 < t \le 2x$ .

If we substitutes (35) and (37) in (32) and evaluate the corresponding beta type integrals, the desired result (28) is achieved.

### *Special Cases:*

Putting $l = 0, \beta = 0, \eta = 0$ in equation (28), then we get the functional relation by using Weyl integral operator [3]

$$\sum_{k=1}^{\infty}\frac{(-1)^k(\alpha)_k}{k}\times I^{m+1,n}_{p_i+1,q_i+1;r}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho,1)\\(1-\rho-k,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\mathcal{X}(s)}{v!\,B_h}\{\psi(1-\rho+\alpha-s) -$$

*ψ1−ρ−s* (38)

If we put $p_i = p$ , $q_i = q$ for all values of $i$ and $r = 1$ in equation (38), then we get the functional relation of H- function using Weyl integral operator [5]

$$\sum_{k=1}^{\infty}\frac{(-1)^k(\alpha)_k}{k}\times H^{m+1,n}_{p+1,q+1}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho,1)\\(1-\rho-k,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\mathcal{X}(s)}{v!\,B_h}\{\psi(1-\rho+\sigma-s) -$$

*ψ1−ρ−s* (39)

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