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AN EXACT PENALTYAPPROACHFOR MATHEMATICAL PROGRAM WITH EQUILIBRIUM CONSTRAINTS.

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Abstract: This paper presents an exact penalty approach to solve the mathematical problems with equilibrium constraints (MPECs). This work is based on the smoothing functions introduced in [3] but it does not need any complicate updating rule for the smoothing penalty parameters. Some numerical academic experiments are carried out to show the efficiency and robustness of this new approach. Two generic applications are also considered : the binary quadratic programs and simple number partitioning problems.

Keywords: Optimization; nonlinear programming; exact penalty function.

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Mathematical programs with equilibrium constraints (MPECs) represent an optimization problem including a set of parametric variational inequality or complementary constraints. In this paper, we consider optimization problems with complementary constraints, in the following form

$$(P) \quad \begin{cases} f^* = \min f(x,y) \\ < x.y >= 0 \\ (x,y) \in D \end{cases} \tag{1}$$

where $f : \Re^{2n} \to \Re$ is continuously differentiable and $D = [0,v]^{2n}$. $<.>$ denotes the inner product on $\Re^n$.

We made this choice for D only to simplify the exposition, one can consider any bounded set D.

Due to the presence of complementary constraints, there is no feasible point satisfying all inequality constraints strictly so that the usual nonlinear programming constraint qualification such as Mangasarian-Fromovitz constraint qualification (MFCQ) is violated at any feasible point.

Many smoothing and relaxation methods for solving (P) have already been proposed in the literature [2, 5, 7, 8, 10]. In this study, we propose a smoothing technique to regularize the complementary constraints based on [3], we replace each constraint

$$x_i y_i = 0$$

by

$$\theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) \le 1$$

where the parameterized function $\theta_\varepsilon : \Re_+ \to [0,1]$ is at least C^2 and satisfies :

$$\theta_\varepsilon(x) \begin{cases} \approx 1 & if \quad x \ne 0 \\ = 0 & if \quad x = 0 \end{cases}$$

Then we define a penalty scheme to solve the problem. To avoid the updating parameter problem, we will consider ε as some new optimization variable.

This paper is organized as follows. In section 2, we present some preliminaries and assumptions on the smoothing functions and introduce the penalty method. We prove under some mild assumptions and exact penalty property in section 3 and present numerical experiments concerning academic MPECs of small sizes in section 4. The last section presents a large set of numerical experiments considering binary quadratic programs and simple number partitioning problems.

2 Preliminaries

In this section, we present some preliminaries concerning the regularization and approximation process. We consider functions $\theta_\varepsilon(\varepsilon > 0)$ with the following properties:

1. θ_ε is nondecreasing, strictly concave and continuously differentiable,
2. $\forall \varepsilon > 0,\ \theta_\varepsilon(0) = 0,$
3. $\forall x > 0, \lim_{\varepsilon \to 0} \theta_\varepsilon(x) = 1,$
4. $\lim_{\varepsilon \to 0} \theta_\varepsilon^{'}(0) > 0,$
5. $\exists\, m > 0, \exists \varepsilon_0 > 0 \forall x \in [0,v], \forall \varepsilon \in]0,\varepsilon_0], |\,\partial_\varepsilon \theta_\varepsilon(x)\,| \le \dfrac{m}{\varepsilon^2}$

For $\varepsilon = 0$, we set $\theta_0(0) = 0$ and $\theta_0(x) = 1, \forall\ x \ne 0$.Examples of such functions are:

$$(\theta_\varepsilon^1): \quad \theta_\varepsilon(x) = \frac{x}{x+\varepsilon}$$

$$(\theta_\varepsilon^{w_1}): \quad \theta_\varepsilon(x) = (1-e^{-(\frac{x}{\varepsilon})})^k, \, for \quad k \le 1$$

$$(\theta_\varepsilon^{log}): \quad \theta_\varepsilon(x) = \frac{\log(1+x)}{\log(1+x+\varepsilon)}$$

Using function θ_ε , we obtain the relaxed following problem :

$$(P_\varepsilon) \qquad \begin{cases} f^* = \min f(x,y) \\ \theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) \le 1, \quad i = 1,\dots,n \\ (x,y) \in D \end{cases} \tag{2}$$

Remark 2.1 $< x.y >= 0 \Rightarrow \forall \varepsilon > 0, \theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) \le 1$. *Thus any feasible point for* (P_ε) *is also feasible for* (P) *and then* $\forall \varepsilon > 0, f_\varepsilon^* \le f^*$.

We first transform the inequality constraints into equality constraints, by introducing some slacks variables e_i :

$$\theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) + e_i - 1 = 0, \; e_i \ge 0 \quad i = 1,\dots,n. \tag{3}$$

The problem (P_ε) becomes:

$$(\tilde{P}_\varepsilon) \qquad \begin{cases} \min f(x,y) \\ \theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) + e_i - 1 = 0, \quad i = 1,\dots,n \\ (x,y,e) \in D \times [0,1]^n \end{cases} \tag{4}$$

Indeed each e_i can not exceed 1.

Remark 2.2 *The limit problem* $(\tilde{P}_\varepsilon)$ *for* $\varepsilon = 0$

$$(\tilde{P}_\varepsilon) \qquad \begin{cases} \min f(x,y) \\ \theta_\varepsilon(x_i) + \theta_\varepsilon(y_i) + e_i - 1 = 0, \quad i = 1,\dots,n \\ e_i \in [0,1] \;, i = 1,\dots,n \end{cases} \tag{5}$$

is equivalent to (P).

Until now, this relaxation process was introduced in [3]. To avoid the updating of parameters problem, we define the penalty functions f_σ on $D \times [0,1] \times [0,\bar{\varepsilon}]$:

$$f_\sigma(x,y,e,\varepsilon) = \begin{cases} f(x,y) & if \quad \varepsilon = \Delta(x,y,e,\varepsilon) = 0, \\ f(x,y) + \dfrac{1}{2\varepsilon}\Delta(x,y,e,\varepsilon) + \sigma\beta(\varepsilon) & if \quad \varepsilon > 0, \\ +\infty & if \quad \varepsilon = 0 \; and \; \Delta(x,y,e,\varepsilon) \ne 0 \end{cases}$$

where Δ measures the feasibility violation $\Delta(z,\varepsilon) = \|G_\varepsilon(z)\|^2$ where $(G_\varepsilon(z))_i = (\theta_\varepsilon(x) + \theta_\varepsilon(y) + e - 1)_i$ and

$z=(x,y,e)$.

The function $\beta:[0,\bar{\varepsilon}]\to[0,\infty)$ is continuously differentiable on $(0,\bar{\varepsilon}]$ with $\beta(0)=0$.

Remark 2.3 $\forall z\in D^{'},\Delta(z,0)=0\Leftrightarrow z$ *feasible for* $\widetilde{P}\Leftrightarrow(x,y)$ *feasible for* (P).

Then we consider the following problem:

$$(P_\sigma)\qquad\begin{cases}\min f_\sigma(x,y,e,\varepsilon)\\(x,y,e,\varepsilon)\in D\times[0,1]^n\times[0,\bar{\varepsilon}]\end{cases}\tag{6}$$

From now on, we will denote

$$D^{'}=D\times[0,1]^n\tag{7}$$

Definition 2.1 We say that the Mangasarian-Fromovitz condition [9] for P_σ holds at $z\in D^{'}$ if $G_\varepsilon^{'}(z)$ has full rank and there exists a vector $p\in\Re^n$ such that $G_\varepsilon^{'}(z)p=0$ and

$$p_i\quad\begin{cases}>0 & if\quad z_i=0\\<0 & if\quad z_i=w_i\end{cases}$$

with

$$w_i=\begin{cases}v & if\quad i\in\{1\ldots 2n\}\\1 & if\quad i\in\{2n+1\ldots 3n\}\end{cases}$$

Remark 2.4 This regularity condition can be replaced by one of those proposed in [11].

3 The smoothing technique

The following theorem yields a condition to find a solution for (P_σ). It also proves a direct link to (P):

Theorem 3.1 We suppose that $z\in D^{'}$ satisfies the Mangasarian-Fromovitz condition, and that

$$\beta^{'}(\varepsilon)\geq\beta_1>0\quad for\quad 0<\varepsilon<\bar{\varepsilon}.$$

i) If σ is sufficiently large, there is no KKT point of P_σ with $\varepsilon>0$.

ii) For σ sufficiently large, every local minimizer $(z^*,\varepsilon^*),(z^*=(x^*,y^*,e^*))$ of the problem (P_σ) has the form $(z^*,0)$, where (x^*,y^*) is a local minimizer of the problem (P).

Proof:

$i)$ Let (z,ε) be a Kuhn Tucker point of P_σ, then there exist λ and $\mu\in\Re^{3n+1}$ such that:

$$\begin{array}{ll}(i) & \nabla\ell(z,\varepsilon)=\nabla f_\sigma(z,\varepsilon)+\lambda-\mu=0,\\(ii) & \min(\lambda,z_i)=\min(\mu,w_i-z_i)=0,\quad i=1\ldots 3n\\(iii) & \mu_{3n+1}=\min(\lambda_{3n+1},\bar{\varepsilon}-\varepsilon)=0,\end{array}\tag{8}$$

where ∇f_σ is the gradient of f_σ with respect to (z,ε).

Assume that there exists a sequence of KKT points (z_k,ε_k) of P_{σ_k} with $\varepsilon_k\neq 0,\forall k$ and $\lim_{k\to+\infty}\sigma_k=+\infty$.Since $D^{'}$ is bounded and closed, up to a subsequence, we have

$$\lim_{k\to+\infty} \varepsilon_k = \varepsilon^*$$
$$\lim_{k\to+\infty} z_k = z^*$$

(8.i) yields to $\partial_\varepsilon f_{\sigma_k}(z_k,\varepsilon_k)+\lambda_{3n+1}-\mu_{3n+1}=0$. So that $\partial_\varepsilon f_{\sigma_k}(z_k,\varepsilon_k)\le 0$.

Then, if we denote $\Delta_k=\Delta(z_k,\varepsilon_k)$, we have

$$\begin{aligned}\partial_\varepsilon f_{\sigma_k} &= -\frac{1}{4\varepsilon_k^2}\Delta_k+\frac{1}{2\varepsilon_k}\partial_\varepsilon\Delta_k+\sigma_k\beta'(\varepsilon_k)\\ &= -\frac{1}{4\varepsilon_k^2}\Delta_k+\frac{1}{\varepsilon_k}(\theta_\varepsilon(x_k)+\theta_\varepsilon(y_k)+e_k+1)(\partial_\varepsilon\theta_\varepsilon(x_k)+\partial_\varepsilon\theta_\varepsilon(y_k))+\sigma_k\beta'(\varepsilon_k)\le 0\end{aligned}$$

Multiplying by $4\varepsilon^3$, we obtain

$$4\varepsilon_k^2(\theta_\varepsilon(x_k)+\theta_\varepsilon(y_k)+e_k-1)(\partial_\varepsilon\theta_\varepsilon(x_k)+\partial_\varepsilon\theta_\varepsilon(y_k))+4\varepsilon_k^3\sigma_k\beta'(\varepsilon_k)\le\varepsilon_k\Delta_k$$

Since $\Delta_k,\theta_\varepsilon$ and $\varepsilon^2\partial_\varepsilon\theta_\varepsilon$ are bounded (by definition (v)), $\sigma_k\to\infty$ when $k\to\infty$. We have $\varepsilon^*=0$.

(ii) Let σ sufficiently large and (z^*,ε^*) a local minimizer for (P_σ). If (z^*,ε^*) satisfies the Magasarian-Fromovitz condition, then (z^*,ε^*) is a Kuhn-Tucker points for f_σ. By (i), we conclude that $\varepsilon^*=0$.

Let ν be a neighborhood of $(z^*,0)$, for any z feasible for $\tilde{P}$ such that $(z,0)\in\nu$ we have

$$f_\sigma(z^*,0)\le f_\sigma(z,0)=f(x,y)<+\infty \qquad (9)$$

(since $\Delta(z,0)=0$).

We can conclude that $\Delta(z^*,0)=0$, otherwise $f_\sigma(z^*,0)$ would be $+\infty$. So that $<x^*,y^*>=0$ and (x^*,y^*) is a feasible point of (P).

Back to (9) $f(x^*,y^*)=f_\sigma(z^*,0)\le f_\sigma(z,0)=f(x,y)$ .Therefore (x^*,y^*) is a local minimizer for (P).

Remark 3.1 The previous theorem is still valid if we consider penalty functions of the form

$$f_\sigma(x,y,e,\sigma)=f(x,y)+\alpha(\varepsilon)\Delta(x,y,e,\varepsilon)+\sigma\beta(\varepsilon) \qquad (10)$$

with $\alpha(\varepsilon)>\frac{1}{2\varepsilon}$.

4 Numerical results

In this section we consider some preliminary results obtained with the approach described in the previous section. We used the SNOPT solver [6] for the solution on the AMPL optimization platform [1]. In all our tests, we use the same function β defined by $\beta(\varepsilon):=\sqrt{\varepsilon}$.

We consider various MPECs where the optimal value is know [4]. Tables 1 and 2 summarizes our different informations concerning the computational effort of the SNOPT, by using respectively θ^{w_1} and θ_1 function:

- Obj.value : is the optimal value

• it : correspond to the total number of iterations

• (Obj.) and (grad.) : correspond to the total number of objective function evaluations and objective function gradient evaluations

• (constr.) and (jac.) : give respectively the total number of constraints and constraints gradient evaluations.

Problem	*Data*	*Obj.val.*	*it*	*Obj.*	*grad*	*constr.*	*Jac*
*bard*1	*no*	17	331	193	192		
desilva	(0,0)	-1	892	655	656	655	656
	(2,2)	-1	448	416	415	416	415
*Df*1	*no*	$3.26e-12$	3	28	27	28	27
*Bilevel*1	(25,25)	5	470	214	213		
	(50,50)	5	295	168	169		
*Bilevel*2	(0,0,0,0)	-6600	232	55	54		
	(0,5,0,20)	-6600	180	56	55		
	(5,0,15,10)	-6600	-6599.9	331	97	96	
*flp*4	*flp*4 − 1.*dat*	$1.9e-29$	66	9	8		
	*flp*4 − 2.*dat*	$3.08e-29$	66	9	8		
	*flp*4 − 3.*dat*	$1.1e-28$	66	9	8		
gauvin	*no*	0	184	71	70		
*jr*1	*no*	0.5	1175	814	813		
*scholtes*1	1	2	12	10	9	10	9
*hs*044	*no*	14.97	375	101	100		
*nash*1	(0,0)	0	0	2	1		
	(5,5)	$1.72e-17$	13	13	12		
	(10,10)	$2.24e-12$	12	12	11		
	(10,0)	$4.29e-12$	12	11	10		
	(0,10)	$1.46e-13$	13	12	11		
*qpec*1	*no*	80	1249	443	442		
*liswet*1 − *inv*	*liswet*1 − 050	0.0929361	215	126	125		
*Stack*1	0	-3266.67	27	26			
	100	-3266.67	7	17	16		
	200	-3266.67	7	17	16		
Water − *net*	*Water* − *net.dat*	931.1	2070	886	885	886	885

Table 1: using the θ^{w_1} function

Problem	Data	Obj.val.	it	Obj.	grad	constr.	Jac
$bard1$	no	17	433	248	247		
$desilva$	(0,0)	-1	7	255	254	255	254
$Df1$	no	0	657	961	960	961	960
$gauvin$	no	$9.5e-05$	164	82	81	82	81
$Bilevel1$	(25,25)	5	401	190	198		
	(50,50)	5	391	183	182		
$Bilevel2$	(0,0,0,0)	-6600	2458	487	486	487	486
	(0,5,0,20)	-6600	2391	727	721		
	(5,0,15,10)	-6600	2391	727	721		
$hs044$	no	17.08	617	261	260	261	260
$jr1$	no	0.5	67	54	53		
$nash1$	(0,0)	$3.35e-13$	203	111	110		
	(5,5)	$6.7e-24$	146	71	70		
	(10,10)	$2.3e-17$	133	85	84		
	(10,0)	$8.1e-16$	379	238	237		
	(0,10)	$2.37e-18$	1228	848	847		
$qpec1$	no	80	1895	518	517		
$liswet1-inv$	$liswet1-050$	0.028	3559	462	461		
$scholtes1$	1	2	51	106	105	106	105
$Stack1$	0	-3266.67	64	58	57		
	100	-3266.67	30	32	31		
	200	-3266.67	30	32	31		
$Water-net$	$Water-net.dat$	931.369	919	282	281	282	281

Table 2: using the θ^1 function

We remark that by considering θ^{w_1} or θ^1 we obtain the optimal know value in almost all the considered test problems.

5 Application to simple partitioning problem and binary quadratic problems

In this section, we consider two real applications : the simple number partitioning and binary quadratic problems. These two classes of problems are know to be NP hard. We use here our approach as a simple heuristic to obtain local solutions.

5.1 Application to simple partitioning problem

The number partitioning problem can be stated as a quadratic binary problem. We model this problem as follows.

We consider a set of numbers $S=\{s_1,s_2,s_3,\dots,s_m\}$. The goal is to divide S into two subsets such that the subset sums are as close to each other as possible. Let $x_j=1$ if s_j is assigned to subset 1, 0 otherwise. Then sum$_1$, subset 1's sum, is sum$_1=\sum_{j=1}^{m}s_jx_j$ and the sum for subset 2 is sum$_2=\sum_{j=1}^{m}s_j-\sum_{j=1}^{m}s_jx_j$. The difference in the sums is then given by

$$diff=\sum_{j=1}^{m}s_j-2\sum_{j=1}^{m}s_jx_j=c-2\sum_{j=1}^{m}s_jx_j. \qquad (c=\sum_{j=1}^{m}s_j).$$

We will minimize the square of this difference

$$diff^2:=(c-2\sum_{j=1}^{m}s_jx_j)^2,$$

We can rewrite $diff^2$ as

$$diff^2 = c^2 + 4x^TQx,$$

where

$$q_{ii} = s_i(s_i - c), \quad q_{ij} = s_i s_j.$$

Dropping the additive and multiplicative constants, our optimization problem becomes simply

$$UQP \quad \begin{cases} \min x^TQx \\ x \in \{0,1\}^n \end{cases}$$

We rewrite the problem as the follows:

$$UQP \quad \begin{cases} \min x^TQx \\ x.(1-x) = 0 \end{cases}$$

With this formulation, the proposed algorithm can be used to get some local solutions for (UPQ).

We considered modest-sized random problems ($m = 25$ and $m = 75$). Five instances of each size were generated with the elements drawn randomly from the interval (50,100).

Table 3 summarizes Table the obtained results. For each instance, we used 100 different initial points generated randomly from the interval $[0,1]$:

- Best diff : corresponds to the best value of $\left|\sum_{i=1}^{100}(Q * round(x_i) - 0.5 * c\right|$
- Integrality measure : correspond to the $\max_i |round(x_i) - x_i|$
- Nb: correspond to the number of tests such that the best sum is satisfied.
- Nb_{10} : correspond to the number of tests such that the sum : $\left|\sum_{i=1}^{100}(Q * round(x_i) - 0.5 * c\right| \le 10$

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$Problem$	$Best\ diff$ (θ^1,θ^{w_1})	Nb (θ^1,θ^{w_1})	$Integrality\ measure$ (θ^1,θ^{w_1})	Nb_{10} (θ^1,θ^{w_1})
$NP25.1$	(1,0)	(1,2)	(0.011,0)	(15,15)
$NP25.2$	(0,0)	(2,2)	(0.0055,0.005)	(16,14)
$NP25.3$	(0,0)	(1,1)	(0,0)	(16,14)
$NP25.4$	(0,0)	(1,2)	(0,0)	(22,22)
$NP25.5$	(0,0)	(1,4)	(0.008,0.0045)	(11,10)
$NP75.1$	(0,0)	(1,2)	(0.003,0)	(14,14)
$NP75.2$	(0,0)	(2,1)	(0,0)	(15,15)
$NP75.3$	(0,0)	(1,1)	(0,0)	(17,17)
$NP75.4$	(0,0)	(2,2)	(0,0)	(18,18)
$NP75.5$	(0,1)	(1,1)	(0,0)	(17,17)

Table 3: Results on Partitioning Problem using the θ^1 and θ^{w_1} function

5.2 Application to binary quadratic problems

We consider some test problems from the Biq Mac Library [12]. These problems are written in the simple following formulation:

$$\min y^T Q y$$
$$y \in \{0,1\}^n$$

where Q is a symmetric matrix of order n.

We used ten instances with $n = 100$ for the Q matrix generated using the following restrictions:

- diagonal coefficients in the range $[-100,100]$,
- off-diagonal coefficients in the range $[-50,50]$,
- All coefficients are integers,
- seeds $1,2,\ldots,10$.

The third column (Nbop) of table 4 report the number of optimal realizations (times we obtained the known optimal value) with 100 different initial points generated randomly from the interval $[0,1]$. The fourth column precise the obtained value when it is different from the known optimal value.

Problem	Know.value	$Nbop(\theta^1,\theta^{w_1})$	$Foundvalue(\theta^1,\theta^{w_1})$
$be100.1$	-19412	$(17,14)$	
$be100.2$	-17290	$(14,12)$	
$be100.3$	-17565	$(9,13)$	
$be100.4$	-19125	$(9,14)$	
$be100.5$	-15868	$(2,2)$	
$be100.6$	-17368	$(31,31)$	
$be100.7$	-18629	$(0,0)$	$(-18473,-18475)$
$be100.8$	-18649	$(1,1)$	
$be100.9$	-13294	$(0,0)$	$(-13248,-13248)$
$be100.10$	-15352	$(11,4)$	

Table 4: Results on Biq Mac test problems using the θ^1 and θ^{w_1} functions

Using θ^{w_1} or θ^1 we obtain the optimal know value for almost all the instances. We obtain a local solutions for only two examples. For each instance, the algorithm found a solution and needs $<1s$ for the resolution.

6 conclusion

In this paper, we introduced an exact penalty approach to solve the mathematical program with equilibrium constraints. We proved a convergence result under suitable constraint qualification conditions and performed some numerical experiments. We initially tested our approach on some tests from the library MaMPEC. Then, we considered some examples from the Biq Mac Library and some randomly generated partitioning problems. We used two different smoothing functions and our limited numerical tests gave very promising results (almost the same result for each one).

References

[1] AMPL.: A Modeling Language for MathematicalProgramming. http://www.ampl.com

[2] Birbil, S.I., Fang, S-H., and Han, J. An entropicapproach for mathematical programs with equilibrium constraints. Computer and Operations Research, 31

[3] Haddou, M. 2009 A new class of smoothing methodsfor mathematical programs with equilibrium constraint. Pacific

Journal of Optimization, Vol. 5, pp. 87-95.

[4] MacMPEC.: Ampl collection of Mathematical Programswith Equilibrium Constraints.http://www.unix.mcs.anl.gov/leyffer/MacMPEC

[5] Facchinei, F., Jiang, H., and Qi, L. 1995 A smoothingmethod for mathematical programs with equilibrium constraints. Mathematical Programming, Vol. 85, pp. 81-106

[6] Gill, P., Murray, W., Saunders, M.: SNOPT, A large-scale smooth optimization problems having linear or nonlinear objectives and constraints.http://www.neos.mcs.anl.gov/neos/solvers

[7] Lin, G.H. and Fukushima, M. 2003. Some exact penalty results for nonlinear programs and mathematical programs with equilibrum constraints, J.Optimization theory and applications. Vol. 118, pp.67-80 .

[8] Liu, G., Ye, J.and Zhu, J. 2008 Partial exact penalty for mathematical programs with equilibrium constraints, J.Set-Valued Anal. Vol. 16, pp.785-804.

[9] Mangasarian, O.L.andFromovitz, S. 1967 The Fritz Johnnecessary optimality conditions in the presence of equality andinequality constraints, J. Math. Anal.Appl., 17, pp.37-47.

[10] Mangasarian, O.L. and Pang, J.S. Exact penalty functions for mathematical programs with linear complementary constraints, J. Optimization.

[11] Ralph, D. and S. J. Wright , 2000 Some properties ofregularization and penalization scheme for MPECS, Springer-Verlag, New York.

[12] Wiegele A. 2007 Biq Mac Library - A collection of Max-Cutand quadratic 0-1 programming instances of medium size.