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Statistical Determination Of The Constants Of The Atmospheric Refraction Formula

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Abstract. In the present paper an efficient algorithm based on the least squares method was developed for the determination of the constants A&B of the atmospheric refraction formula

The principles of atmospheric refraction and the least squares method together with its error analysis were first developed and summarized together with its error analysis. The equations of condition for the problem were then established using N polar stars at their upper and lower culminations. Analytical formulae for the least squares solution of the equations of condition are given. Analytical formulae of the errors estimate are also established ,of these are: the standard error of the fit , the standard errors for the least squares solutions, the probable errors ,finally, the average squared distance between the exact solutions and the least squares solutions.

Keywords: Spherical astronomy; atmospheric refraction; least squares.

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1. Introduction

Atmospheric refraction was mentioned as early as the first century A.D. by Cleomedes

independently by Ptolemy (discussed in his *Optics*),ca. A.D.150..

The refraction is the bending of light while passing from transparent homogenous medium to another transparent homogenous medium whose density is different from the first medium. The refraction follows some basic roles ,of these are : the incident ray,the refracted ray ,and the normal to the surface separating two media at the point O(say) ,all lie in same plane. The relation between the incident and the refracted angles ψ and φ respectively ,is given by :

$$\frac{\sin\psi}{\sin\varphi} = \mu ,$$

where μ is called the *refractive index* for the two media, μ is a constant quantity depending on the optical properties of the two media and can be determined by laboratory experiment. The value of μ changes for the same ray that is the refractive index for the blue light is different refractive index for the red light. This phenomenon is known in optics as light s*cattering* and it is not important in studying the effect of the refraction on astronomical observations. In the vacuum of space $\mu = 1$. The value of μ_{air} depends on wavelength , temperature, and pressure as well .For example $\mu = 1.000277$ is for green light $\approx 5500A^o$,for dry air for the conditions $T = 15^o C$ and pressure $P = 1.013\times 10^5$ Pa. The value of μ for the atmosphere at the Earth's surface, at temperature 0^oC ,and at atmospheric pressure 760 mm Hg is around 1.0002927 for the yellow light where the human eye has maximum sensitivity . Atmospheric refraction plays important roles in many applications of spherical astronomy ,of these as for example, in topocentric phenomena ,such as the time of rising and setting of the Sun and Moon, and in the prediction of local circumstances of eclipses. Also for observational reductions the effect of refraction on the equatorial coordinates of a star must be included **.**

In the present paper an efficient algorithm based on the least squares method was developed for the determination of the constants A&B of the atmospheric refraction formula Although the least-squares method is the most powerful techniques that has been devised for the problems of astronomy it is at the same time exceedingly critical. This is because the least-squares method suffers from the deficiency that, its estimation procedure does not have detecting and controlling techniques for the sensitivity of the solution to the optimization criterion of the variance σ^2 is minimum. As a result, there may exist a situation in which there are many significantly different solutions that reduce the variance σ^2 to an acceptable small value.

At this stage we should point out that (1) the accuracy of the estimators and the accuracy of the fitted curve are two distinct problems; and (2) an accurate estimator will always produce small variance, but small variance does not guarantee an accurate estimator. This could be seen from Equation (3) by noting that the lower bounds for the average square distance between the exact and the least-squares values is σ^2/λ_{min} which may be large even if σ^2 is very small ,depending on the magnitude of the minimum Eigen value , λ_{min} ,of the coefficient matrix of the least-squares of the normal equations. Unless detecting and controlling this situation, it is not possible to make a well-defined decision about the results obtained from the applications of the least squares method. Consequently, we include error analysis to control as much as we can the accuracy of the solutions

2.Linear least squares fit

Let y be represented by the general linear expression of the form:

$$\sum_{i=1}^{n} c_i\varphi_i(x) ,$$

where φ's are linear independent functions of x. Let $\mathbf{c}$ be the vector of the exact values of the c's coefficients and $\hat{\mathbf{c}}$ the least squares estimators of **c** obtained from the solution of the normal equations of the form $\mathbf{G}\hat{\mathbf{c}} = \mathbf{b}$.The coefficients matrix $\mathbf{G}(n\times n)$ is symmetric positive definite, that is ,all its eigen values $\lambda_i; i = 1,2,\cdots,n$ are positive. Let $E(z)$ denotes the expectation of z and σ^2 the variance of the fit, defined as:

$$\sigma^2 = \frac{q_n}{(N-n)} \quad , \qquad (1.1)$$

where

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$$q_n = (\mathbf{y} - \mathbf{\Phi}^T \hat{\mathbf{c}})^T (\mathbf{y} - \mathbf{\Phi}^T \hat{\mathbf{c}}), \tag{1.2}$$

N is the number of observations, **y** is the vector with elements y_k and $\mathbf{\Phi}(n \times N)$ has elements $\Phi_{ik} = \Phi_i(x_k)$. The transpose of a vector or a matrix is indicated by the superscript "T".

According to the least squares criterion ,it could be shown that (Kopal and Sharaf 1980):

1-The estimators $\hat{\mathbf{c}}$ obtained by the least squares method gives the minimum of q_n.

2- The estimators $\hat{\mathbf{c}}$ of the coefficients $\mathbf{c}$, obtained by the least squares method, are unbiased; i.e. $E(\hat{\mathbf{c}}) = \mathbf{c}$.

3-The variance-covariance matrix $\mathrm{Var}(\hat{\mathbf{c}})$ of the unbiased estimators $\hat{\mathbf{c}}$ is given by:

$$\mathrm{Var}(\hat{\mathbf{c}}) = \sigma^2 \mathbf{G}^{-1}, \tag{2}$$

where $\mathbf{G}^{-1}$ is the inverse of the matrix $\mathbf{G}$.

4-The average squared distance between $\hat{\mathbf{c}}$ and $\mathbf{c}$ is:

$$E(L^2) = \sigma^2 \sum_{i=1}^{n} \frac{1}{\lambda_i}. \tag{3}$$

Also it should be noted that, if the precision is measured by probable error e, then :

$$e = 0.6745\sigma.$$

3. Atmospheric refraction

The starlight moves in straight line until it meets the outer surface of the atmosphere ,then it suffers through its passage in the Earth's atmosphere series of refractions called *astronomical refraction*. Since the air density of the upper layers of the Earth's atmosphere is very rarer, then its effect on the refraction is small compared with that of total refraction. Consequently, the effective layers on the refraction are those at few tens of kilometers from the Earth's surface

and in these lyres , the ray is bending till it reaches the observer. So the star is observed in a direction not parallel to its true direction.

For illustrating the above points, consider Fig.1 where z is the zenith , O is an observer on the Earth's surface and:

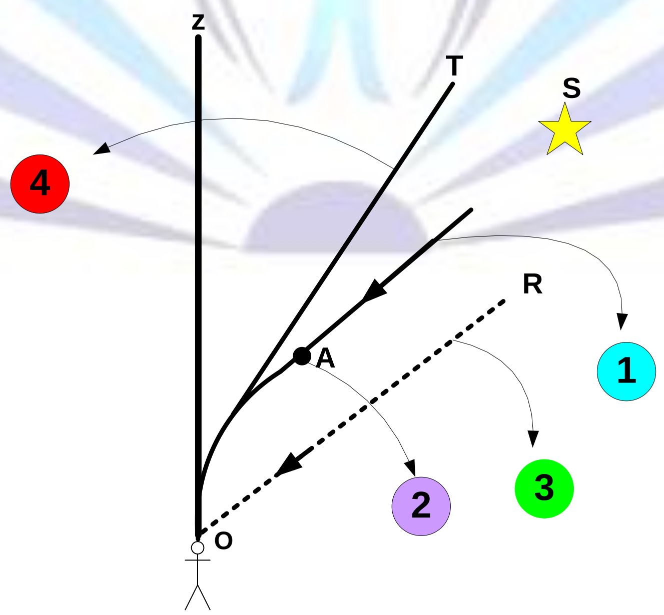

Fig.1:The refraction of the starlight in the Earth's atmosphere

1- The straight pass from the star S (say) till it enter the effective region of the Earth's atmosphere at the point A.

2-After the entrance at the point A , the ray continuously bending till it reaches the observer at the point O.

3- The parallel line to SA is the position of the star if there were no atmosperical refraction, and this is the true direction of the star.

4-The observer will see the star in the direction TO which is the direction of the tangent

for the curve of the refracted ray at the point O and this is the apparent (observed) direction of the star.

From the above ,it is clear that the refraction changes the true poison of the zenith of the star, that is ; the zenith distance decreases due to the refraction.

3.1 General theorem for the atmospheric refraction

Assumptions of the theorem

i-The Earth's is regarded as sphere .

ii-The atmosphere made up of a large number of thin spherical layers, concentric with

the Earth's center.

iii- Each layer has its own optical properties and ,in particular, its own refractive index

The formula

The general formula for the atmospheric refraction (Meeus 2000) is :

$$R = A\tan\xi + B\tan^3\xi. , \tag{4}$$

where ξ is the observed zenith distance.

The present paper is devoted for establishing an algorithm for the determination of the constants A&B and their error analysis.

Clearly formula (4) is not valid when the observed zenith distance equal to 90^o . Also the formulae is insufficient when the zenith distance exceeds 75^o .It appears that, at high altitudes, the refraction is proportional to the tangent of the zenith distance.

4. Equations of condition

The values of the constants A&B of Equation (4) could be determined by the least

squares method. For the application of the method we need first to find the equations of condition

In Fig 2

® X : The true position of a polar star at the upper culmination.

® Y: The true position of the same star at the lower culmination.

® X_1 : The observed position due to the atmospheric refraction of the star at the upper culmination.

® Y_1: The observed position due to the atmospheric refraction of the star at the lower culmination.

® z: The true zenith distance of the star at the upper culmination.

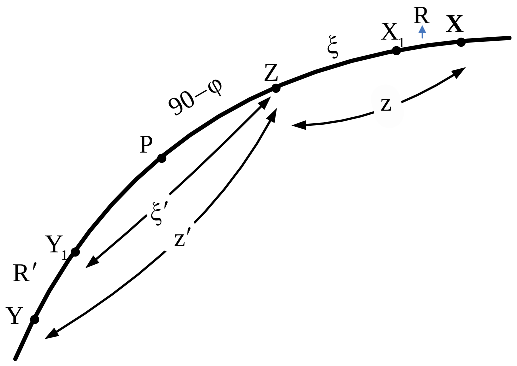

Fig.2: Determinations of the constants A&B

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® ξ : The observed zenith distance of the star at the upper culmination (obtained from observations).

® z' :The true zenith distance of the star at the lower culmination.

® ξ' : The observed zenith distance of the star at the lower culmination(obtained from observations).

® ϕ :The latitude of the observer

® δ :The declination of the star

® R: The value of refraction corresponding to ξ

® R':: The value of refraction corresponding to ξ'

Since

$$z = XZ = PX - PZ = (90-\delta) - (90-\varphi) = \varphi - \delta, \quad z = \xi + R,$$

$$z' = YZ = ZP + PY = (90-\varphi) + (90-\delta) = 180 - \varphi - \delta, \; z' = \xi' + R'.$$

Consequently,

$$\varphi - \delta = \xi + A\tan\xi + B\tan^3\xi, \tag{5}$$

$$180 - \varphi - \delta = \xi' + A\tan\xi' + B\tan^3\xi'. \tag{6}$$

Adding Equations (5) and(6) we get :

$$180 - 2\delta = \xi + \xi' + A(\tan\xi + \tan\xi') + B(\tan^3\xi + \tan^3\xi'). \tag{7}$$

Noting that, all the quantities in Equation (7) are known except A&B.

For N of polar stars (like the star X),Equation (7) becomes

$$Q_i = A\beta_i + B\gamma_i ; i = 1,2,\cdots,N, \tag{8}$$

where the quantities γ's, β's, Q's are known for the N stars and are computed from

$$Q_i = 180 - 2\delta_i + \xi_i + \xi'_i, \tag{9}$$

$$\beta_i = \tan\xi_i + \tan\xi'_i, \tag{10}$$

$$\gamma_i = \tan^3\xi_i + \tan^3\xi'_i \tag{11}$$

Equations (8) are the required equations of condition.

5. Least squares for solving the system (8)

According to the least squares criterion of Section 4 we have:

$$A = (T_5T_2 - T_3T_4)/\Delta, \tag{12}$$

$$B = (T_3T_2 - T_5T_1)/\Delta, \tag{13}$$

where

$$\Delta = T_2^2 - T_4T_1, \tag{14}$$

also

$$T_1 = \sum_{i=1}^{N}\beta_i^2 , \quad T_2 = \sum_{i=1}^{N}\beta_i\gamma_i, \quad T_3 = \sum_{i=1}^{N}\beta_i Q_i, \tag{15}$$

$$T_4 = \sum_{i=1}^{N}\gamma_i^2, \quad T_5 = \sum_{i=1}^{N}\gamma_i Q_i, \quad T_6 = \sum_{i=1}^{N}Q_i^2. \tag{16}$$

(Note that T_6 is not used in Equations(12),(13),(14) but it will be used latter)

6. Errors estimate

According to Section 4 we deduce the following errors estimate

6.1 The standard error of the fit

$$\sigma = [\frac{1}{N-2}\sum_{i=1}^{N}(Q_i - A\beta_i -_i B\gamma_i)^2]^{1/2}$$

Expanding, then using Equations(15) and (16) we get:

$$\sigma = [\frac{1}{N-2}\{T_6 - A^2T_1 - B^2T_4 - 2ABT_2\}]^{1/2}, \quad (17)$$

If the precision is measured by probable error e, then :

$$e = 0.6745\sigma. \quad (18)$$

6.2 The standard errors for the least squares solutions

The standard errors for the least squares solutions A&B are:

$$\sigma_A = \sigma\sqrt{g_{11}} \quad ; \quad \sigma_B = \sigma\sqrt{g_{22}},$$

Where g_{22}, g_{11} are the diagonal elements of the matrix $\mathbf{G}^{-1}$,where G is the matrix of the coefficients of the system (8)

$$\mathbf{G} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_4 \end{bmatrix},$$

since $g_{22} = -T_1/\Delta$; $g_{11} = -T_1/\Delta$,then

$$\sigma_A = \sigma\sqrt{-T_4/\Delta}, \quad (19.1)$$

$$\sigma_B = \sigma\sqrt{-T_1/\Delta}. \quad (19.2)$$

6.3The probable errors

The corresponding probable errors are:

$$e_A = 0.6745\sigma_A, \quad (20.1)$$

$$e_B = 0.6745\sigma_B. \quad (20.2)$$

6.4 The average squared distance between $\hat{c}$ and c

According to Equation (3), the average squared distance between the exact solutions and the least squares solutions is

$$E(L^2) = \sigma^2\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}, \quad (21)$$

where λ_1, λ_2 are the Eigen values of the matrix G. and its characteristic equation is

$$\lambda^2 - \lambda(T_1 + T_4) + (T_1T_4 - T_4^2) = 0$$

then

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$$\lambda_1 + \lambda_2 = T_1 + T_4$$

$$\lambda_1 \lambda_2 = T_1 T_4 - T_4^2 = -\Delta$$

Consequently $E(L^2)$ for the system (8) is:

$$E(L^2) = \sigma_A^2 + \sigma_B^2 . \tag{22}$$

In concluding the present paper we stress that, an efficient algorithm based on the least squares method was developed for the determination of the constants A&B of the atmospheric refraction formula

The principles of atmospheric refraction and the least squares method together with its error analysis were first developed and summarized together with its error analysis. The equations of condition for the problem were then established using N polar stars at their upper and lower culminations. Analytical formulae for the least squares solution of the equations of condition are given. Analytical formulae of the errors estimate are also established ,of these are: the standard error of the fit , the standard errors for the least squares solutions, the probable errors ,finally, the average squared distance between the exact solutions and the least squares solutions..

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