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CONTINUED FRACTION SOLUTION OF THE GENERALIZEDBARKER EQUATION OF PARABOLIC ORBITAL MOTION

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Abstract In this present paper, simple and accurate algorithm was established for the solution ofgeneralized Barker's equation of parabolic orbital motion.The algorithm based on the continued fractionexpansion theory. Numerical applications of the algorithm are alsogiven

Keywords Parabolic Orbital Motion; Barker equation; Continued fractions

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1. Introduction

Parabolic orbit is a Kepler orbit with the eccentricity equal to 1. It is also called escape orbit,Many instances ofparabolic orbits occur in the solar system and recently among the space missions.

The relation between the *true anomaly* f and the time in parabolic orbits is called Barker's equation and is given as:

$$\tan^3\frac{f}{2}+3\tan\frac{f}{2}=2W, \qquad (1)$$

where

$$W=3\sqrt{\frac{\mu}{p^3}}(t-\tau),$$

μ the*gravitational parameter* ,pthe *parameter* of the orbit$=$2q, qthe*pericenter distance* , t the time, and τ is the time of *passage through pericenter*

We can write Equation (1) as

$$G(f)=\tan^3\frac{f}{2}+3\tan\frac{f}{2}-2W$$

we get the following cases

1-If $t-\tau\succ 0$, the function G is negative when $f=0$,and thatit increases continually with f until it equals infinity for $f=\pi$. Therefore ,there is but one real solution of Equation (1) for $\tan f/2$, and it is positive.

2-If $t-\tau\prec 0$ it is seen in a similar manner that there is one real negative solution.

In this present paper, simple and accurate algorithm was established for the solution of generalized Barker's equation of parabolic orbital motion.The algorithm based on the continued fractionexpansion theory. Numerical applications of the algorithm are alsogiven

2 Generalized Barker's equati

The generalized form of Barker's equation for the two epochs $t_n, t_\ell, \ell\succ n$ is given as

$$6\sqrt{\mu}\left(t_\ell-t_n\right)=\chi^3_{\ell,n}+3\sigma_n\chi^2_{\ell,n}+6r_n\chi_{\ell,n}, \qquad (2)$$

$$\chi_{\ell,n}=\sigma_\ell-\sigma_n;\quad \sigma_\ell=\frac{1}{\sqrt{\mu}}<\mathbf{r}_\ell,\mathbf{v}_\ell>\ ;\quad \sigma_n=\frac{1}{\sqrt{\mu}}<\mathbf{r}_n,\mathbf{v}_n>. \qquad (3)$$

where $(\mathbf{r}_n,\mathbf{v}_n)$ are the initial position and velocity vectors at the initial time t_n ,while

$(\mathbf{r}_\ell,\mathbf{v}_\ell)$ are the corresponding vectors at the finial time $t_\ell\succ t_n$. The distance r_ℓ could be

obtained in terms of $\chi_{\ell,n}$ as

$$r_\ell=r_n+\sigma_n\chi_{\ell,n}+\frac{1}{2}\ \chi^2_{\ell,n}.. \qquad (4)$$

The solution of the generalized form of Barker's Equation (1) is

$$\chi_{\ell.n}=\sqrt{p}\,z-\sigma_n, \qquad (5)$$

where z is the solution of

$$z^3+3z=2B \qquad (6)$$

and

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$$B=\frac{1}{p^{3/2}}\left[\sigma_n\left(r_n+p\right)+3\sqrt{\mu}\left(t_\ell-t_n\right)\right]. \qquad (7).$$

Finally if f the true anomaly, then

$$z=\tan\frac{1}{2}f. \qquad (8)$$

So ,to solve the generalized Barker Equation (2), it is sufficient to solve Equation 6) for z ,then $\chi_{\ell.n}$ from Equation (5).

3. Solution of Equation (6) by continued fraction

From the known Cardin'smethod (e.g.Battin 1999) ,the real root of Equation (6) is given as:

$$z=\tan\frac{1}{2}f=\left(B+\sqrt{B^2+1}\right)^{1/3}-\left(B+\sqrt{B^2+1}\right)^{-1/3}$$

Let

$$z=Q-Q^{-1}$$

where

$$Q=\left(B+\sqrt{B^2+1}\right)^{1/3}$$

Let

$$S=\sinh^{-1}B$$

then

$$S=\ln(B+\sqrt{B^2+1})$$

Since $\ln Q=\frac{1}{3}S$,then

$$z=2\sinh\left(\frac{1}{3}\sinh^{-1}B\right). \qquad (9)$$

From Equation (9) we can write

$$y=2\sinh\frac{1}{3}x \text{ and } B=\sinh x,$$

From the known continued fraction expansion of $\sinh\frac{x}{3}/\sinh x$ (Wall 1948) we deduce that

$$z=\tan\frac{f}{2}=\cfrac{n_1}{d_1+\cfrac{n_2}{d_2+\cfrac{n_3}{d_3+\ddots}}} \qquad (10)$$

where

$$n_1=\frac{2}{3}B,$$

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$$n_i = \begin{cases} \frac{B^2}{9}\frac{(3i-1)(3i-4)}{(2i-3)(2i-1)} & i \ \text{odd} \succ 1, \\ \frac{B^2}{9}\frac{(3i-2)(3i-5)}{(2i-3)(2i-1)}. & i \ \text{even} \ge 2. \end{cases} \quad (11)$$

$$d_j = 1 \ \forall j \ge 1.$$

4. Continued fraction evaluation

In fact, continued fraction expansions are, generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series.

4.1 Top- down continued fractionevaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method is, obviously, having to decide far down the fraction to being in order to ensure convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint .

Gautschi[1967] proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as in Equation (9) ,then initialize the following parameters

$$a_1 = 1,$$
$$b_1 = n_1/d_1,$$
$$c_1 = n_1/d_1$$

and iterate (k=1,2,…) according to :

$$a_{k+1} = \frac{1}{1+\left[\frac{n_{k+1}}{d_k d_{k+1}}\right]a_k}$$

$$b_{k+1} = [a_{k+1} - 1]b_k,$$

$$c_{k+1} = c_k + b_{k+1}.$$

In the limit, the c sequence converges to the value of the continued fraction.

5. Computational developments

Consider random values of $B \in [-5,5]$, the applications of Gautschi's algorithm of Section 4 for the solution z of Barker's equation with these values of B,n's and d's of Equation (11) yield z as listed in Table 1. The accuracy of the computed values are checked by the condition that:

$$CH = z^3 + 3z - 2B,$$

such that, The smaller the value of CH the more accurate solution will be.

Table 1:Continued fraction solution of Barker's equations

No	B	z	CH	No	B	z	CH
1	2.86599	1.25375	$\square 4.44089 \square 10^{\square 15}$	16	□4.23471	□1.55932	0.
2	3.48339	1.40256	$\square 1.77636 \square 10^{\square 15}$	17	□3.59692	□1.42777	$1.77636 \square 10^{\square 15}$
3	1.12827	0.657455	$\square 4.44089 \square 10^{\square 16}$	18	□0.118218	□0.0786497	0.
4	□1.01962	□0.605684	$8.88178 \square 10^{\square 16}$	19	□1.56947	□0.845113	$8.88178 \square 10^{\square 16}$
5	□2.61988	□1.18787	0.	20	1.82399	0.939539	$\square 8.88178 \square 10^{\square 16}$
6	0.172316	0.114378	0.	21	0.349696	0.229121	$\square 1.11022 \square 10^{\square 16}$
7	□2.61113	□1.18545	$8.88178 \square 10^{\square 16}$	22	3.76144	1.46327	$\square 1.77636 \square 10^{\square 15}$
8	□3.05601	□1.30186	$2.66454 \square 10^{\square 15}$	23	1.56888	0.844884	$4.44089 \square 10^{\square 16}$
9	3.93263	1.499	$\square 3.55271 \square 10^{\square 15}$	24	4.00733	1.51423	$3.55271 \square 10^{\square 15}$
10	4.81672	1.667	$3.55271 \square 10^{\square 15}$	25	2.48371	1.1495	0.
11	1.74856	0.912467	0.	26	□4.72196	□1.65016	$5.32907 \square 10^{\square 15}$
12	□1.45155	□0.798188	0.	27	□4.55939	□1.62067	0.
13	4.53934	1.61697	0.	28	0.026959	0.0179707	$\square 6.93889 \square 10^{\square 18}$
14	3.46544	1.39852	$5.32907 \square 10^{\square 15}$	29	0.103592	0.0689518	0.
15	4.31578	1.57496	$1.77636 \square 10^{\square 14}$	30	0.105728	0.0703692	0.

In concluding the present paper, simple and accurate algorithmwas established for the solution of Barker's equation ofparabolic orbital motion.The algorithm based on the continued fractionexpansion theory. Numerical applications of the algorithm are alsogiven

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