# AN EXTENSION OF SOME RESULTS DUE TO JARDEN

Anthony G. Shannon[a], Charles K. Cook[b], Rebecca A. Hillman[c]

[a]Faculty of Engineering & IT, University of Technology, Sydney, NSW 2007, Australia
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Campion College, PO Box 3052, Toongabbie East, NSW 2146, Australia
t.shannon@campion.edu.au, Anthony.Shannon@uts.edu.au
[b]Emeritus, University of South Carolina Sumter, Sumter, SC 29150, USA
charliecook29150@aim.com
[c]Associate Professor of Mathematics, University of South Carolina Sumter, Sumter, SC 29150, USA
hillmanr@uscsumter.edu

## ABSTRACT

This paper defines some generalized Fibonacci and Lucas sequences which satisfy arbitrary order linear recurrence relations and which answer a problem posed by Jarden in 1966 about generalizing an elegant result for a connection between even and odd subscripted Fibonacci and Lucas numbers.

## Indexing terms/Keywords

product sums, shift operators, basic sequences, primordial sequences, fundamental sequences, Fibonacci sequences, Lucas sequences, characteristic equations, Cauchy calculus

## Academic Discipline And Sub-Disciplines

Mathematics: Number Theory

## SUBJECT CLASSIFICATION

11B75, 11Z05, 11B65

## TYPE (METHOD/APPROACH)

Properties of characteristic equations of arbitrary order difference equations are used to solve a long-standing question about generalizing a result which connects the second oreder recursive sequences whose elemenata are the Fibonacci and Lcas numbers

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Jarden [7, pp.87] extended the well-known relation between the second order Fibonacci and Lucas numbers, $F_{2n} = F_n L_n$, to the third order case. In doing so he showed "that third order sequences have another arithmetic than that of Lucas' sequences" [7, p.88]. This paper explores some arbitrary order extensions of Jarden's results.

In order to generalize the sequences of Horadam[14] and Jarden[7, pp.30,114] we define $r$ basic sequences of order $r$, $\{U_{s,n}^{(r)}\}$, by the recurrence relation which we define formally after the initial terms for $s = 1,2,...,r$:

$$U_{s,n}^{(r)} = \overset{r}{\underset{j=1}{\overset{\circ}{a}}} U_{s,n-j}^{(r)}, \ n > r,$$

(1.1)

in which the $r$ initial terms, $n = 0,1,2,...,r-1$, are given by $U_{s,n}^{(r)} = \delta_{s,n}$, the Kronecker delta. We also consider $f(x)$ with zeros, $\alpha_i$, assumed distinct, to be the associated $r^{th}$ order characteristic polynomial. Examples of (1.1) appear in a later table. We also define the special sequence with $s = 0$ $\{U_{0,n}^{(r)}\}$ [12] to satisfy (1.1) but with the $r$ initial terms defined by

$$U_{0,n}^{(r)} = \sum_{i=1}^{r} \alpha_i^n, \ 0 \le n < r$$

(1.2)

so that the first few values are, for example, $U_{0,0}^{(r)} = r$, $U_{0,1}^{(r)} = 1$, $U_{0,2}^{(r)} = 3$. The $\{U_{0,n}^{(r)}\}$ and $\{U_{r,n}^{(r)}\}$ correspond to Lucas' "primordial" and "fundamental" sequences respectively [14], while the other $\{U_{s,n}^{(r)}\}$ are "basic" sequences [8]. For notational convenience, $\{U_{s,n}^{(r)}\}$, $s \ne 0$, $U_{s,n}^{(r)} = 0$, $n \le 0$.

## 2. A PROBLEM POSED BY JARDEN

Jarden[7, p.88] posed the problem to determine the values of $F_{2n} - F_n L_n = 0$ "for appropriate recurring sequences of higher order", or in the notation of this paper:

$$U_{s,2n}^{(r)} - U_{s,n}^{(r)} U_{0,n}^{(r)}$$

(2.1)

since $U_{2,2n}^{(2)} - U_{2,n}^{(2)} U_{0,n}^{(2)}$ "is of great importance for the arithmetic of Lucas' second order recurring sequences".

In order to do this we consider the sequence $\{U_{s,2n}^{(r)}\}$ as a bisection of the original sequence with auxiliary equation

$$h(x) = \prod_{i=1}^{r} \left( x - \alpha_i^2 \right).$$

This can be illustrated with the bisection of the ordinary sequence of Fibonacci numbers with the auxiliary polynomial

$$\left( x - \alpha_1^2 \right)\left( x - \alpha_2^2 \right) = x^2 - \left( \alpha_1^2 + \alpha_2^2 \right)x + \left( \alpha_1 \alpha_2 \right)^2$$
$$= x^2 - 3x + 1,$$

which, in effect, is related to the second order recurrence relation

$$W_n^{(2)} = 3W_{n-1}^{(2)} - W_{n-2}^{(2)},$$

which, in turn, can generate the sequence {1,3,8,21,55,...}. Similarly, with the bisection of the Tribonacci sequence [2]: let $A = \left( 19 + 3\sqrt{33} \right)^{1/3}$, $B = \left( 19 - 3\sqrt{33} \right)^{1/3}$, so that

$$\alpha_1 = \frac{1}{3}(A + B + 1)$$

$$\alpha_2 = \frac{1}{6}\left(2 - A - B + \sqrt{3}i(A - B)\right)$$

$$\alpha_3 = \frac{1}{6}\left(2 - A - B - \sqrt{3}i(A - B)\right)$$

are the characteristic roots of $x^3 - x^2 - x - 1$ [15], and so

$$(x - \alpha_1^2)(x - \alpha_2^2)(x - \alpha_3^2) = x^3 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)x^2 + (\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2)x - (\alpha_1\alpha_2\alpha_3)^2$$
$$= x^3 - 3x^2 - x - 1,$$

which, in effect, is related to the third order recurrence relation

$$W_n^{(3)} = 3W_{n-1}^{(3)} + W_{n-2}^{(3)} + W_{n-3}^{(3)},$$

which, in turn, can generate the sequence {1,7,21,71,241,...}, a bisection of the Tribonacci sequence as seen in the table which follows. In more general terms, then

$$U_{s,n}^{(r)} = \sum_{i=1}^{r} A_{s,i}\alpha_i^n, \; s = 0,1,2,...,r,$$

where the $A_{s,i}$ depend on the initial values, $U_{s,1}^{(r)}, U_{s,2}^{(r)},..., U_{s,r}^{(r)}$, ($A_{0,r}$=1). There are $r$ in each recurrence relation for each $U_{s,n}^{(r)}, s = 0,1,2,...,r$, so the number of terms in $U_{s,2n}^{(r)} - U_{s,n}^{(r)}U_{0,n}^{(r)}$ is $r^2$-$r$ or $r(r-1)$, of which $r$ can be paired as coefficients of each $\alpha_i\alpha_j, i \neq j$; that is, the number of terms in $U_{s,2n}^{(r)} - U_{s,n}^{(r)}U_{0,n}^{(r)}$ is ½$r(r-1) = \binom{r}{2}$. Thus,

$$U_{s,2n}^{(r)} - U_{s,n}^{(r)}U_{0,n}^{(r)} = \sum_{i=1}^{r} A_{s,i}\alpha_i^{2n} - \sum_{i=1}^{r} A_{s,i}\alpha_i^n \sum_{m=1}^{r}\alpha_m^n$$

$$= \sum_{i=1}^{r} A_{s,i}\alpha_i^{2n} - \sum_{i=1}^{r} A_{s,i}\alpha_i^{2n} - \sum_{i<j}(A_{s,i} + A_{s,j})\alpha_i^n\alpha_j^n$$

$$= -\sum_{\substack{i,j=1 \\ i<j}}^{r}(A_{s,i} + A_{s,j})\alpha_i^n\alpha_j^n$$

$$= \sum_{k} B_{s,k}\beta_k^n$$

$$= Y_{s,n}^{\left(\binom{r}{2}\right)}.$$

When $r = 2$, the *Y*-sequence will be a first order sequence as expected, and when $r = 3$, it will also be a 3rd order sequence. For example, consider

$$U_{0,n}^{(3)} = \alpha_1^n + \alpha_2^n + \alpha_3^n$$

$$U_{3,n}^{(3)} = A_{3,1}\alpha_1^n + A_{3,2}\alpha_2^n + A_{3,3}\alpha_3^n$$

$$U_{3,2n}^{(3)} = A_{3,1}\alpha_1^{2n} + A_{3,2}\alpha_2^{2n} + A_{3,3}\alpha_3^{2n}$$

so that

$$
\begin{aligned}
U_{3,2n}^{(3)} - U_{3,n}^{(3)}U_{0,n}^{(3)} &= A_{3,1}\alpha_1^{2n} + A_{3,2}\alpha_2^{2n} + A_{3,3}\alpha_3^{2n} - (A_{3,1}\alpha_1^{2n} + A_{3,2}\alpha_2^{2n} + A_{3,3}\alpha_3^{2n} \\
&\quad + A_{3,2}\alpha_1^n\alpha_2^n + A_{3,3}\alpha_1^n\alpha_3^n + A_{3,1}\alpha_1^n\alpha_2^n + A_{3,3}\alpha_2^n\alpha_3^n + A_{3,1}\alpha_1^n\alpha_3^n + A_{3,2}\alpha_2^n\alpha_3^n) \\
&= -\left\{ (A_{3,1} + A_{3,2})\alpha_1^n\alpha_2^n + (A_{3,2} + A_{3,3})\alpha_2^n\alpha_3^n + (A_{3,3} + A_{3,1})\alpha_3^n\alpha_1^n \right\} \\
&= B_{3,1}\beta_1^n + B_{3,2}\beta_2^n + B_{3,3}\beta_3^n \\
&= Y_{3,n}^{\left(\binom{3}{2}\right)}.
\end{aligned}
$$

The table below illustrates the foregoing, for the *U*-sequences *r* = 3, *s* = 0, (see Fielder [3]) and for *r* =3, *s* = 1,2,3 (see Emerson [1]). Our choice of the initial conditions, while it emphasises the "basic" nature of these sequences, shifts the elements of the sequences from their normal subscript identifiers as we can see in the table. Thus, for instance, the usual form of the Fibonacci sequence is obtained in our notation from

$$\left\{ U_n^{(2)} \right\} \equiv \left\{ \left\{ U_{1,n}^{(2)} \right\} + \left\{ U_{2,n}^{(2)} \right\} \right\},$$

and the original form of the Tribonacci sequence was [3]

$$\left\{ U_n^{(3)} \right\} \equiv \left\{ \left\{ U_{1,n}^{(3)} \right\} + \left\{ U_{2,n}^{(3)} \right\} + \left\{ U_{3,n}^{(3)} \right\} \right\}$$

which is similar to Jarden's third order sequences, {$U_n$} and {$V_n$}, with unit coefficients and initial terms $U_0 = U_1 = 0, U_2 = 1, V_0 = 3, V_1 = 1, V_2 = 3,$ respectively [7, p.86]. Note that Jarden's subscripts also differ by one from those which we use. This is why we have introduced the more familiar $\left\{ U_n^{(r)} \right\}, r = 2,3,$ at this stage for notational convenience though they generally obscure the roles of the underlying basic sequences.

**Table 1: Second and third order examples**

| *n* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $U_{0,n}^{(2)}$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 |
| $U_{1,n}^{(2)}$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| $U_{2,n}^{(2)}$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| $U_n^{(2)}$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| $Y_{2,n}^{(1)}$ | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| $U_{0,n}^{(3)}$ | 1 | 3 | 7 | 11 | 21 | 39 | 71 | 131 | 241 | 443 | 815 | 1,499 |
| $U_{1,n}^{(3)}$ | 1 | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 24 | 44 | 81 |
| $U_{2,n}^{(3)}$ | 0 | 1 | 0 | 1 | 2 | 3 | 6 | 11 | 20 | 37 | 68 | 125 |
| $U_{3,n}^{(3)}$ | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 24 | 44 | 81 | 149 |
| $U_n^{(3)}$ | 1 | 1 | 1 | 3 | 5 | 9 | 17 | 31 | 57 | 105 | 193 | 355 |
| $Y_{3,n}^{(3)}$ | 0 | 1 | -3 | 2 | 2 | -7 | 7 | 2 | -16 | 21 | -3 | -34 |

For the *Y*-sequences, each term is the homogeneous product sum of weight *n* of the roots of the auxiliary equation [9]; that is, the *Y*-sequences in the table satisfy the 1$^{st}$ and 3$^{rd}$ order recurrence relations respectively:

$$Y_{0,n}^{(1)} = -Y_{0,n-1}^{(1)}, \ n > 1, \tag{2.2}$$

and

$$Y_{0,n}^{(3)} = -Y_{0,n-1}^{(3)} - Y_{0,n-2}^{(3)} + Y_{0,n-3}^{(3)}, \ n > 3. \tag{2.3}$$

Jarden [7, p.87, Equation (20)] found in his notation that

$$U_{2n+1} - U_{n+1}V_n = a^n U_{-n+1} \tag{2.4}$$

which, because of the different initial values, is in our notation

$$U_{3,2n-1}^{(3)} - U_{3,n-1}^{(3)}U_{0,n-1}^{(3)} = U_{3,-n+1}^{(3)} \tag{2.5}$$

with his recurrence relation coefficients *a* = *b* = *c* = 1 in his search for "the counterpart of the formula $U_{2n} - U_nV_n = 0$". Jarden's (2.4) can be re-written as

$$U_{3,2n-1}^{(3)} - U_{3,n-1}^{(3)}U_{0,n-1}^{(3)} = Y_n^{(3)} + U_{3,-n+1}^{(3)} \tag{2.6}$$

which reduces to our

$$U_{3,2n}^{(3)} - U_{3,n}^{(3)}U_{0,n}^{(3)} = Y_n^{(3)} \tag{2.7}$$

because

$$U_{-n} - U_{3,-n}^{(3)} = Y_n^{(3)},$$

and (2.7) is a particular case of the previous result, namely,

$$U_{2,2n}^{(r)} - U_{2,n}^{(r)}U_{0,n}^{(r)} = Y_{2,n}^{\left(\binom{r}{2}\right)},$$

when *s* = *r* = 3. When *r* = *s* = 2, it can be similarly confirmed from the table that

$$\begin{aligned}
U_{2,2n}^{(2)}U_{0,n}^{(2)} &= \tfrac{1}{\sqrt{5}}\left(\alpha_1^{n-1} - \alpha_2^{n-1}\right)\left(\alpha_1^n + \alpha_2^n\right) \\
&= \tfrac{1}{\sqrt{5}}\left(\alpha_1^{2n-1} - \alpha_2^{2n-1} + \alpha_1^{n-1}\alpha_2^n - \alpha_1^n\alpha_2^{n-1}\right) \\
&= \tfrac{1}{\sqrt{5}}\left(\alpha_1^{2n-1} - \alpha_2^{2n-1} + (-1)^{n-1}\alpha_2 - (-1)^{n-1}\alpha_1\right) \\
&= U_{2,2n}^{(2)} - (-1)^{n-1};
\end{aligned}$$

that is,

$$U_{s,2n}^{(2)} - U_{s,n}^{(2)} U_{0,n}^{(2)} = Y_{s,n}^{(1)}$$

as in Table 1.

There are various other inter-relationships among these sequences in the table which the interested reader might like to unravel and then extend to the arbitrary order analogs. For example, the following capture the initial terms of the sequences:

$$U_{0,n}^{(2)} = U_{1,n}^{(2)} + 3U_{2,n}^{(2)}$$

and

$$U_{0,n}^{(3)} = U_{1,n}^{(3)} + 3U_{2,n}^{(3)} + 7U_{3,n}^{(3)}.$$

## 3. EXTENSIONS

We next show that the ordinary generating function for the sequence when $s = r$, namely $\left\{U_{r,n}^{(r)}\right\}$, can be expressed formally as an exponential by

$$\sum_{n=1}^{\infty} U_{r,n}^{(r)} x^n = x^r \exp\left(\sum_{m=1}^{\infty} U_{0,m}^{(r)} \frac{x^m}{m}\right).$$  (3.1)

*Proof:*

$$\sum_{n=1}^{\infty} U_{r,n}^{(r)} x^{n-r} = \prod_{j=1}^{r} \left(1 - \alpha_j x\right)^{-1}$$

$$\log\left(\sum_{n=1}^{\infty} U_{r,n}^{(r)} x^{n-r}\right) = \log \prod_{j=1}^{r} \left(1 - \alpha_j x\right)^{-1}$$

$$\left|\alpha_j x\right| < 1$$

$$= -\log \prod_{j=1}^{r} \left(1 - \alpha_j x\right)$$

$$= -\sum_{j=1}^{r} \log\left(1 - \alpha_j x\right)$$

$$= \sum_{j=1}^{r} \sum_{m=1}^{\infty} \alpha_j^m \frac{x^m}{m}$$

$$= \sum_{m=1}^{\infty} \left(\sum_{j=1}^{r} \alpha_j^m\right) \frac{x^m}{m}$$

$$= \sum_{m=1}^{\infty} U_{0,m}^{(r)} \frac{x^m}{m}$$

$$\sum_{n=1}^{\infty} U_{r,n}^{(r)} x^{n-r} = \exp\left(\sum_{m=1}^{\infty} U_{0,m}^{(r)} \frac{x^m}{m}\right). \qquad \blacksquare$$

If we take the derivatives of each side of (3.1) with respect to $x$, and use the double summation $\sum_{m}^{\infty} \sum_{n}^{\infty} A(n,m) = \sum_{m}^{\infty} \sum_{n}^{m} A(n, m-n)$ rewritten as $\sum_{m}^{\infty} \sum_{j}^{\infty} A(j, n-j)$ [10], we get

$$\frac{d}{dx}\left( \sum_{n=1}^{\infty} U_{r,n}^{(r)} x^{n-r} = \exp\left( \sum_{m=1}^{\infty} U_{0,m}^{(r)} \frac{x^m}{m} \right) \right)$$

so that

$$\sum_{n=r+1}^{\infty} (n-r) U_{r,n}^{(r)} x^{n-r-1} = \exp\left( \sum_{m=1}^{\infty} U_{0,m}^{(r)} \frac{x^m}{m} \right)\left( \sum_{n=1}^{\infty} U_{0,n}^{(r)} x^{n-1} \right)$$

and, since $U_{s,n}^{(r)} = 0$, $n < 0$ ($s \neq 0$),

$$
\begin{aligned}
\sum_{n=1}^{\infty} (n-1) U_{r,n+r-1}^{(r)} x^n &= \left( \sum_{m=1}^{\infty} U_{r,m}^{(r)} x^m \right)\left( \sum_{n=1}^{\infty} U_{0,n}^{(r)} x^{n-1} \right) \\
&= \left( U_{r,1}^{(r)} x + U_{r,2}^{(r)} x^2 + U_{r,3}^{(r)} x^3 + ... \right) \times \\
&\quad \left( U_{0,1}^{(r)} + U_{0,2}^{(r)} x + U_{0,3}^{(r)} x^2 + ... \right) \\
&= U_{0,1}^{(r)} U_{r,1}^{(r)} x + \left( U_{0,1}^{(r)} U_{r,2}^{(r)} + U_{0,2}^{(r)} U_{r,1}^{(r)} \right) x^2 + ... \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n} U_{0,j+1}^{(r)} U_{r,n-j}^{(r)} x^n
\end{aligned}
$$

(3.2)

in which we have also used the fact that $U_{r,0}^{(r)} = U_{r,1}^{(r)} = 0$ (which we use again in what follows). On equating coefficients of $x$, (3.2) becomes

$$(n-1) U_{r,n+r-1}^{(r)} = \sum_{j=0}^{n} U_{0,j+1}^{(r)} U_{r,n-j}^{(r)}$$

(3.3)

as in [13]. When $r = 2$, and after replacing $n$ by $(n + 1)$, this can be written as

$$n U_{2,n+2}^{(2)} = \sum_{j=0}^{n-1} U_{0,j+1}^{(2)} U_{2,n-j+1}^{(2)}$$

which, from the table, in turn can be reduced to the known result [6]:

$$n F_{n+1} = \sum_{j=0}^{n-1} L_{j+1} F_{n-j}.$$

(3.4)

For example, when $n = 6$, $n F_{n+1} = 6 \times 13$, and

$$\sum_{j=0}^{n-1} L_{j+1} F_{n-j} = 8 + 15 + 12 + 14 + 11 + 18$$

$$= 78.$$

## 4. CONCLUDING COMMENTS

The algebra associated with the ordinary generating function is sometimes known as the Cauchy calculus, whereas the algebra associated with the exponential generating function is known variously as the Blissard or umbral or symbolic calculus, depending on the context [4]. Further related research can similarly be built upon the relation in (4.1) between the ordinary and exponential generating functions for these arbitrary order sequences:

$$\sum_{n=0}^{\infty} U_{r,n}^{(r)} t^n = \int_0^{\infty} e^{-z} \sum_{n=0}^{\infty} U_{r,n}^{(r)} \frac{t^n}{n!} z^n dz, \; if \; \left|\alpha_j t\right| < 1. \tag{4.1}$$

The work can also be extended to generalizations of recurrence relations associated with some of the special functions [5,11].

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# Authors' biographies with Photos
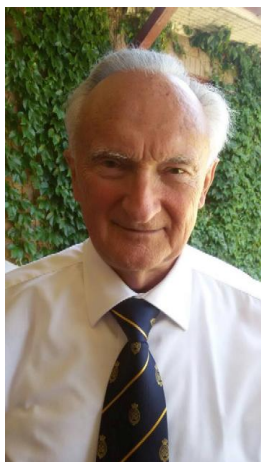


Professor A. G. (Tony) Shannon AM is an Emeritus Professor of the University of Technology, Sydney, where he was Foundation Dean of the UTS Graduate Research School and Professor of Applied Mathematics, and where he is currently doing research in the Centre for Health Technologies within the Faculty of Engineering and Information Technology.

He holds the doctoral degrees of Ph.D., Ed.D. and D.Sc. He is co-author of numerous books and articles in medicine, mathematics and education. His research interests are in the philosophy of education, number theory, and epidemiology, particularly through the application of generalized nets and intuitionistic fuzzy logic. He has taught and mentored at all levels from primary school to post-doctoral.

Professor Shannon is a Fellow of several professional societies. He is presently Registrar of Campion College, a liberal arts degree granting institution in Sydney. In June 1987 he was appointed a Member of the Order of Australia (AM) for services to education. He enjoys reading, walking, theatre, number theory, and thoroughbred racing.
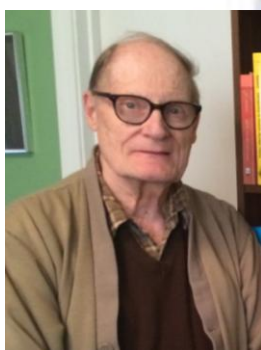


Professor Charles K. Cook is a distinguished emeritus professor of mathematics of the University of South Carolina Sumter, Sumter, SC. USA. His primary research interests lie in Fibonacci and related sequential studies.

He holds a BEd from Chicago Teacher's College, an MS in mathematics from De Paul University and a PhD from Purdue University. His professional organizations include membership in AMS, AMA and The Fibonacci Association. He enjoys theater and mysteries, particularly those involving Sherlock Holmes; and belongs to two Baker Street Irregulars scion societies.



Rebecca A. Hillman (Becky) is an Associate Professor of Mathematics at The University of South Carolina Sumter. She earned her Ph.D. in Mathematics from North Carolina State University in 2005. She is co-author of many articles in mathematics and worked on a grant in the field of Math Education. Her research interests are in number theory (especially Fibonacci numbers), math education, and algebraic topology.

Dr. Hillman has a passion for teaching undergraduate mathematics and offers many of the math courses for the Elementary Education Bachelor's Degree through USC Upstate at Sumter. She is dedicated to reducing the perceived fear of mathematics, which typically begins at a young age and continues well into ones college career. She believes math is fun; but also enjoys vacationing, the beach, professional and college football, and nascar racing.