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ON COMPLETENESS OF FUZZY NORMED SPACES

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Abstract:

In this paper, a new direction has been presented between the subject of domain theory and fuzzy normed spaces to introduce the so called fuzzy domain normed spaces and proved some results related to this subject concerning the completeness of such spaces.domain

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I. Introduction

Domain theory is one of modern and interesting topics of mathematic with basic and common main components that studies special kinds of partially ordered sets (posets) commonly called domains. A main features of the subject were suggested by D.Scott[8] as a tool of formulation some programming language. It studied and modified by Abramsky[1] and other authors in the nineties of the last century

Fuzzy sets are sets whose elements have degrees of some membershipfunctions; these were proposed by Zadeh [10] in 1965. Many contributions about fuzzy metric spaces were added after that, an important one was given by George and Veermani [5] in 1994, while in 2003, Bag and Samanta [2] construct the fuzzy normed space.

Edalat and Heckmann [3] established a new connection between domain theory with metric spaces in 1998. They gave the main results and common theorems about metric spaces in the point view of Domain theory. Eidi [4] introduced a fuzzy metric space in domain theory in 2010.

This paper discuss the fuzzy normed spaces in domain theory by ordering closed balls in any fuzzy normed space using the reverse inclusion , depending on the membership function of fuzzy sets Instead of perverting the course of action to crisp sets using α-level sets and give the sufficient condition that make this space complete.

II. preliminaries

For Completeness purpose, we recall briefly some basic concepts in domain theory, for more details about this subject, the reader may be referred to [1].

Definition (2.1), [1]:

A non-empty set P with a binary relation ⊑ is called partially ordered set or (poset) denoted by (P, ⊑), if the following holds for all x, y, z ∈ P:

$x \sqsubseteq x$ (Reflexivity.

$x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$ (Transitivity).

$x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$ (Antisymmetry).

If (P, ⊑) is a poset then (P, ⊒) is a poset too when ⊒ is the reverse relation of ⊑.

An element x ∈ P is said to be maximal element of P if for all y ∈ P such that x ≤ y, implies x = y. The dual concept for minimal element can be defined by the same way.

Let Y ⊆ P, then the greatest element of Y is an element z ∈ Y such that y ≤ z for all y ∈Y. A top element of P is a greatest element of P. An upper bond of Y is an element x ∈ P in which y ≤ x for all y ∈Y, the supremum of Y is an element x ∈ P, which is the least element of the set of upper bounds of P. If it is found and it will be denoted by ⊔Y. Also, Y will be defined as a directed set if each pair of elements of Y has supremum. An ordered set (P,⊑) is said to be directed complete partially ordered set or briefly dcpo if every directed subset of P has supremum. Many contributions added to define fuzzy lattice and and fuzzy poset actually zadeh [10] consider the fuzzy lattice that proposed implicitly in his work when he define the α-level sets and other authors [6] [9] are modified and formulated them with more details

This paper introduce some similar arguments proposed for the fuzzy partial ordered sets with some differentiations accrue according to define fuzzy sets by the membership function it will be illustrate later.

Definition(2.2)[6]:

Let P be a set. A function A : P × P → [0, 1] is called a fuzzy relation in P. The fuzzy relation A in P is reflexive iff A(x, x) = 1 for all x ∈ P, A is transitive iff A(x, z) $\geq \sup_{y \in X} \min(A(x, y), A(y, z))$,

and A is antisymmetriciff A(x, y> 0 and A(y, x) > 0 implies x = y. A fuzzy relation A is a fuzzy partial ordered

relation if A is reflexive, antisymmetric, and transitive. A fuzzy partial order relation A is a fuzzy total order relation iff A(x, y) > 0 or, A(y, x) > 0 for all x, y ∈ P. If A is a fuzzy partial order relation on a set P, then (P, A) is called a fuzzypartially ordered set or a fuzzy poset. If B is a fuzzy total order relation on a set P, then (P, B) is called a fuzzy totally ordered set or a fuzzy chain

.Concepts of fuzzy metric spaces and complete fuzzy metric spaces, introduced by George and Veermani in [5], as follows,

Definition (2.3), [5]:

The 3-tuple (X,M,*) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X2×(0,∞) satisfying the following conditions for all x,y,z∈ X and t,s> 0

i. $M(x, y, 0) > 0$.

ii. $M(x, y, t) = 1$ if and only if $x = y$, for all $t > 0$.

iii. $M(x, y, t) = M(y, x, t)$.

iv. $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$.

v. $M(x, y, \circ): (0, \infty) \longrightarrow [0,1]$ is continuous.

Definition (2.4), [5]:

A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is a Cauchy sequence if and only if for each $0 < \epsilon < 1$ and t > 0, there exists $n_0 \in \mathbb{N}$, such that for all n, m ≥ n_0

$M(x_n, x_m, t) > 1 - \epsilon$.

And a sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Convergent sequence to x ∈ X if and only if for each $\epsilon > 0$ and t > 0 there exists $n_0 \in \mathbb{N}$, such that for all n ≥ n_0

$M(x_n, x, t) > 1 - \epsilon$

A fuzzy metric space (X,M,*)is said to be complete if and only if every Cauchy sequence in X is convergent.

A fuzzy normed space defined for the first time by Bag and Samantha [2] in 2003, and some mathematicians defined fuzzy normed spaces in different ways, such as Saadati and Vaezpour [5].

Definition (2.5), [2]:

Let X be a linear space over a field of scalars F. A fuzzy subset N of X× $\mathbb{R}$ ($\mathbb{R}$ is set of real numbers) is called a fuzzy norm on X if and only if for all x,u∈ X and c ∈ F the following axioms are satisfied:

i. For all $t \in \mathbb{R}, t \leq 0, N(x, t) = 0$.

ii. For all $t \in \mathbb{R}$, $t \geq 0$, $N(x, t) = 1$ if and only if $x = 0$.

iii. For all $t \in \mathbb{R}$, $t \geq 0$, $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$, $c \neq 0$.

iv. $N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$.

v. $N(x, \circ)$ is anon decreasing function of $\mathbb{R}$, and $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X,N) will be referred to as a fuzzy normed linear space.

Remark:

The continuous t-norm $*$ will be defined as the minimum operator, and a fuzzy normed space will be denoted by $(X, N, *)$

Lemma (2.6), [7]:

Let $(X, N, *)$ be a fuzzy normed space. Define M(x,y,t) = N(x-y,t), for all x,y∈ X, and all t ≥ 0, then M is a fuzzy metric on X, which is called a fuzzy metric induced by the norm N.

Definition (2.7), [7]

Let $(X, N, *)$ be a fuzzy normed space, define the open and closed balls in $(X, N, *)$ with center $x \in X$, and radius $0 < r < 1$, for all $t > 0$, as follows:

$$B(x, \alpha, t) = \{y \in X | N(x - y, t) > 1 - \alpha\}$$

$$B[x, \alpha, t] = \{y \in X | N(x - y, t) \geq 1 - \alpha\}$$

The main results and interesting facts crowned by introducing topological sense for the partial ordered fuzzy normed space a follows:

.III The partially ordered Fuzzy Normed Space:

The set of formal balls defined on a fuzzy normed space $(X, N, *)$ is a subset of the product space $X \times [0,1] \times \mathbb{R}^+$where $\mathbb{R}^+$is the set of non-negative real numbers .Ordering the closed balls of $(X, N, *)$ by the reverse inclusion then an ordered triple $[x, \alpha, t]$(the elements of $X \times [0,1] \times \mathbb{R}^+$) represent the center and radius of some closed balls of $(X, N, *)$ the details will be explain below**:**

Notation

For each closed ball in $(X,N,*)B[x,\alpha,r]$we will mention an ordered triples $[x,\alpha,r]$ of $X\times[0,1]\times\mathbb{R}^{+}$ these will be ordered by the relation $\sqsubseteq$ however the against closed balls ordering by the ordinary reverse inclusion In other words $[x,\alpha,r]\sqsubseteq[y,\beta,s]$ if and only if $B[x,\alpha,r]\supseteq B[y,\beta,s]$ for every $x,y\in X, 0\le\alpha,\beta\le 1, r,s>0$.

Definition (3.1):

A formal ball in a fuzzy normed space $(X,N,*)$ is the the ordered triple $[x,\alpha,r]$ where $x\in X$, $0\le\alpha\le 1$ and $r>o$ in which for every $y\in X$, $0<\beta<\alpha<1$, $r>s>0$, then

$[x,\alpha,r]\sqsubseteq[y,\beta,s]$ if and only if $N(x-y,r-s)\ge 1-(\alpha-\beta)$

The set of formal balls will be denoted by BX.

Definition (3.2):

An ordered pair (BX,⊑) is said to be a fuzzy domain normed space of a fuzzy normed space $(X,N,*)$.

Remark

According to the reverse inclusion the maximal elements of BX can be considered to be contained in the set:

$UX=\{[x,0,r]|x\in X,r>0\}$, against the set of closed balls with center x and radius 0 for all $r>0$.

Actually UX is homeomorphic to $X\times(0,\infty)$ by defining $i:X\times(0,\infty)\to UX$, with $i(x,r)=[x,0,r]$, i is continuous, one to one, onto, and define i^{-1}: UX→ $X\times(0,\infty)$ with $i^{-1}([x,0,r])=(x,r)$ which is continuous.

Theorem (3.3):

For every fuzzy normed space $(X,N,*)$, then (BX,⊑) is a poset.

Proof:

For all x,y,z,∈ X, $0<\alpha<\beta<1$, and $r>s>0$, then:

1. Reflexivity: $[x,\alpha,r]\sqsubseteq[x,\alpha,r]$ is directly satisfied, since:

$N(x-x,t)=1$ for all $t>0$and$1-(\alpha-\alpha)=1$

2. Transitivity: suppose that $[x,\alpha,r]\sqsubseteq[y,\beta,s]$ and $[y,\beta,s]\sqsubseteq[z,\gamma,t]$ to prove that $[x,\alpha,r]\sqsubseteq[z,\gamma,t]$, i.e., to prove that $N(x-z,r-t\ge 1-(\alpha-\gamma)$.

$$N(x-z,r-t)=N(x-y+y-z,r-s+s-t)$$
$$\ge N(x-y,r-s)*N(y-z,s-t)$$
$$\ge 1-(\alpha-\beta)*1-(\beta-\gamma)$$
$$\ge 1-((\alpha-\beta)+(\beta-\gamma)=1-(\alpha-\gamma))$$

3. Antisymmetry: Assume that $[x,\alpha,r]\sqsubseteq[y,\beta,s], 0\le\beta\le\alpha\le 1, r\ge s\ge 0$ as well as $[y,\beta,s]\sqsubseteq[x,\alpha,r], 0\le\alpha\le\beta\le 1$, and $s\ge r\ge 0$, then for all $t\ge 0$

$N(x-y,t)=N(y-x,t)\ge 1-(\alpha-\beta)=1$. This implies $x=y$. ■

Definition (3.4):

Let (BX,⊑) be a fuzzy domain normed space. A sequence $\{[x_n,\alpha_n,r_n]\}$ is said to be Cauchy sequence in BX if and only if for all $\epsilon>0$, there exist $n_0\in\mathbb{N}$, such that:

$N(x_n-x_m,r_n-r_m)\ge 1-(\alpha_n-\alpha_m)>1-\epsilon$, ∀ $n,m\ge n_0$

And a sequence $[x_n,\alpha_n,r_n]$ is said to be convergent to $[x,\alpha,r]$ in BX if and only if for all $\epsilon>0$, there exist $n_0\in\mathbb{N}$, such that:

$N(x_n-x,r_n-r)\ge 1-(\alpha_n-\alpha)>1-\epsilon$, ∀ $n\ge n_0$

(BX,⊑) is said to be complete if and only if every Cauchy sequence in Bx is convergent. A complete fuzzy domain normed space is called A fuzzy domain Banach space.

The next example illustrate some of the above concepts:

Example (3.5)

Consider the 3-tuple $(\mathbb{R},N,*)$, which is a fuzzy domain normed space with standard fuzzy norm, $N(x,t)=\frac{t}{t+|x|}$, then $(B\mathbb{R},\sqsubseteq)$ is a fuzzy domain normed space.

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A closed ball in $(\mathbb{R}, N,*)$ with center $x \in \mathbb{R}$, and radius $0 \le \alpha \le 1$, for all $t \ge 0$, is given by:

$$B[x,\alpha,r] = \{y \in \mathbb{R} | N(x-y,r) \ge 1-\alpha\}$$

and for all $x, y, z \in \mathbb{R}$, $r \ge s \ge t \;\; 0$, and $0 \le \beta \le \alpha \le 1$, Then:

$$N(x-z, r-t) = \frac{r-t}{r-t+|x-z|}$$

$$= \frac{(r-s)+(s-t)}{(r-s)+(s-t)+|x-y+y-z|}$$

$$\ge \frac{(r-s)+(s-t)}{(r-s)+(s-t)+|x-y|+|y-z|}$$

$$\ge \min\left\{\frac{r-s}{r-s+|x-y|}, \frac{s-t}{s-t+|y-z|}\right\}$$

$$\ge \min\{1-(\alpha-\beta), 1-(\beta-\gamma)\}$$

$$\ge 1-\big((\alpha-\beta)+(\beta-\gamma)\big) = 1-(\alpha-\gamma)$$

Then $[x,\alpha,r] \sqsubseteq [z,\gamma,t]$. In fact $\sqsubseteq$ form a partially ordered relation, then all closed balls in $(\mathbb{R}, N,*)$ can be ordered by this relation.

Hence, $(B\mathbb{R}, \sqsubseteq)$ is a fuzzy domain normed space, actually it is a fuzzy domain Banach space.

Proposition (3.6):

Let BX be a fuzzy domain normed space. For every directed set D of BX there is an ascending sequence $\{[x_n, \alpha_n, r_n]\}$ of elements of D, which have the same upper bounds of D.

Proof:

Let $\beta = \inf\{\alpha | [x,\alpha,r] \in D\}$,

then for all $n \in \mathbb{N}$, $n > 1$, there exist $[y_n, \beta_n, s_n] \in D$, such that:

$\beta_n \;\le\; \beta + \frac{1}{n+1}$, $[x_1, \alpha_1, r_1] = [y_1, \beta_1, s_1]$, $\beta_1 \;\le\; \beta + \frac{1}{2}$ ....

$[x_n, \alpha_n, r_n] = [y_n, \beta_n, s_n]$, $\beta_n \;\le\; \beta + \frac{1}{n+1}$, with, $\beta_{n+1} \le \beta_n \le \cdots \le \beta_2 \le \beta_1$,

then:

$$[x_1, \alpha_1, r_1] \sqsubseteq [x_2, \alpha_2, r_2] \sqsubseteq \cdots$$

Suppose that $[x_n, \alpha_n, r_n]$ is an upper bound of all $[x_{n-1}, \alpha_n - 1, r_{n-1}]$ when, $[x_n, \alpha_n, r_n] = [y_n, \beta_n, s_n] \in D$, since D is directed set.

Let $[z,\gamma,t]$ be an upper bound of all $[x_n, \alpha_n, r_n]$ and $[a,\delta,u]$ be any element of D, to prove that $[z,\gamma,t]$ is an upper bound of $[a,\delta,u]$, that is to prove $[a,\delta,u] \sqsubseteq [z,\gamma,t]$.

Since D is directed set then both of $[x_n, \alpha_n, r_n]$ and $[a,\delta,u]$ have an upper bond say $[b,\varepsilon,v]$, i.e., $N(a-z, u-t) \ge 1-(\delta-\gamma)$. Suppose that $u \ge t$, then:

$N(a-z, u-t) = N(a-b+b-x_{n+}x_n-z, u-t)$

$$\ge N\left(a-b, \frac{u-t}{3}\right) * N\left(b-x_n, \frac{u-t}{3}\right) * N\left(x_n-z, \frac{u-t}{3}\right)$$

$$= N\left(a-b, \frac{u-t}{3}\right) * N\left(x_n-b, \frac{u-t}{3}\right) * N\left(x_n-z, \frac{u-t}{3}\right)$$

$$\ge 1-(\delta-\varepsilon) * 1-(\alpha_n-\varepsilon) * 1-(\alpha_n-\gamma)$$

and upon using the properties of the minimum operator, we get:

$$N(a-z, u-t) \ge 1-[(\delta-\varepsilon)+(\alpha_n-\varepsilon)+(\alpha_n-\gamma)]$$

$= 1-[(\delta-\gamma+2(\alpha_n-\varepsilon)]$ (since β is the smallest in (0,1), such that $[y,\beta,s] \in D$, $\forall\, y \in X, s > 0$),

$\ge 1-[(\delta-\gamma+2(\alpha_n-\beta)]$ (using the assumption $\alpha_n \le \beta + \frac{1}{n+1}$)

$\ge 1-[\,\delta-\gamma+2\left(\frac{1}{n+1}\right)]$ and as $n \to \infty$

Therefore:

$$N(a-z, u-t) \geq 1-(\delta-\gamma)$$

that completes the proof. ■

Lemma (3.7):

Let be (BX, ⊑) be a fuzzy domain normed space. If $\{[x_{n_k}, 2^{-k}, r_{nk}]\}$ is an ascending sequence in BX, then $\{\alpha_n\}$ is a convergent sequence in (0, 1] while $\{x_n\}$ is a Cauchy sequence in X.

Proof:

From the ascending sequence $\{[x_n, \alpha_n, r_n]\}$, we have $\alpha_{n+1} \leq \alpha_n$, that is $\{\alpha_n\}$ is a descending sequence which is bounded since $\alpha_n \in (0,1]$, $\forall$ n.

Hence $\{\alpha_n\}$ is convergent, so it is a Cauchy sequence, i.e., for all $\epsilon > 0, n_0 \in \mathbb{N}$ there exists $n, m \in \mathbb{N}$ with $\alpha_n \geq \alpha_m$, such that, $\alpha_n - \alpha_m > 1-\epsilon$, $\forall\, n, m \geq n_0$ and $\{[x_n, \alpha_n, r_n]\}$is an ascending sequence that means whenever $r_n \geq r_m$

Then $[x_n, \alpha_n, r_n] \sqsubseteq [x_m, \alpha_m, r_m]$, that yields to:

$N(x_n - x_m, r_n - r_m) \geq 1-(\alpha_n - \alpha_m) > 1-\epsilon\, \forall\, n, m \geq n_0$.

Then, $\{x_n\}$ is a Cauchy sequence in X. ■

Proposition (3.8):

Every Cauchy sequence $\{x_n\}$ in a fuzzy normed space (X,N,*) has a Cauchy subsequence $\{x_{n_k}\}$, $n, k \in \mathbb{N}$, such that $\{[x_{n_k}, 2^{-k}, r_{nk}]\}$, $r_{nk} \geq 0$, is a descending sequence in BX.

Proof:

Suppose that $\{x_n\}$ is a Cauchy sequence in X then for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ with $n_k \geq n_{k+1}$, such that:

$$N(x_i - x_j, r_i - r_j) \geq 1-2^{-(k+1)}, \qquad \forall\, i, j \geq n_k$$

that means that:

$$N(x_{nk} - x_{nk+1}, r_{nk} - r_{nk+1}) \geq 1-2^{-(k+1)}, \qquad \forall\, k \in \mathbb{N}$$

Then, $\{x_{n_k}\}$ is a Cauchy sequence in X, in fact:

$$N(x_{nk} - x_{nk+1}, r_{nk} - r_{nk+1}) \geq 1-\left(2^{-k} - 2^{-1}\right)$$
$$= 1-\left(2^{-k} - \left(1-2^{-1}\right)\right)$$
$$= 1-\left(2^{-k} - 2^{-(k+1)}\right)$$

Then $[x_{nk}, 2^{-k}, r_{nk}] \sqsubseteq [x_{nk+1}, 2^{-(k+1)}, r_{nk+1}]$ and hence $\{[x_{n_k}, 2^{-k}, r_{nk}]\}$ is a descending sequence. ■

Studying the least upper bounds of subsets of BX is equivalent to study their limits, as the following theorem shows:

Theorem (3.9):

Let (BX, ⊑) be a fuzzy domain normed space, and $\{[x_n, \alpha_n, r_n]\}$ be an ascending sequence in BX, $[y, \beta, s]$ is an element of BX, then the following statements are equivalent.

1. $[y, \beta, s]$ is a lest upper bound of $\{[x_n, \alpha_n, r_n]\}_{n\in\mathbb{N}}$.
2. $[y, \beta, s]$ is an upper bound of $([x_n, \alpha_n, r_n])_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} \alpha_n = \beta$.
3. $\lim_{n\to\infty} x_n = y$ and $\lim_{n\to\infty} \alpha_n = \beta$.

Proof:

(1⟹2). It is enough to show that $\lim_{n\to\infty} \alpha_n = \beta$, that is for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that $|\alpha_n - \beta| < \epsilon$, $\forall\, n > n_0$.

Since $[x_n, \alpha_n, r_n] \sqsubseteq [y, \beta, s]$, for all $n \in \mathbb{N}$, then $\alpha_n > \beta$.

Suppose the converse, that is there exists some $\epsilon > 0$, $n_0 \in \mathbb{N}$, in which for every $n \in \mathbb{N}$, $\alpha_n - \beta \geq \epsilon$ for some $n_0 > n$ implies $\alpha_{n_0} - \beta \geq \epsilon$.

Since $\{[x_n, \alpha_n, r_n]\}$ is an ascending sequence then by lemma (3.7), $\{x_n\}$ is a Cauchy sequence in X, that is for some $\epsilon > 0, n_0 \in \mathbb{N}$, there exists $n, m \in \mathbb{N}$, with $r_n \geq r_m$, such that:

$$N(x_n - x_m, r_n - r_m) > 1 - \epsilon \quad \forall\ n, m > n_0$$

and

$$N\left(x_n - x_{n_0}, r_n - r_{n_0}\right) \geq 1 - \left(\alpha_n - \left(\beta + \frac{\epsilon}{2}\right)\right)$$

which yields to $[x_n, \alpha_n, r_n] \sqsubseteq \left[x_{n_0}, \beta + \frac{\epsilon}{2}, r_{n_0}\right]$, in other wards $\left[x_{n_0}, \beta + \frac{\epsilon}{2}, r_{n_0}\right]$ is an upper bound for $\{[x_n, \alpha_n, r_n]\}$, i.e., $[y, \beta, s] \sqsubseteq \left[x_{n_0}, \beta + \frac{\epsilon}{2}, r_{n_0}\right]$

Hence $\beta + \frac{\epsilon}{2} \leq \beta$, which is a contradiction and hence $|\alpha_n - \beta| \leq \epsilon\ \forall,\ n \in \mathbb{N}$.

$(2 \Longrightarrow 3)$. It is enough to show that $x_n \longrightarrow y$ in X.

We know that $[y, \beta, s]$ is an upper bound of $\{[x_n, \alpha_n, r_n]\}$, then for all $n \in \mathbb{N}$ with $\alpha_n \geq \beta$ and $r_n \geq s$, we have:

$$N(x_n - y, r_n - s) \geq 1 - (\alpha_n - \beta)$$

but $\alpha_n \longrightarrow \beta$, which implies that:

$$N(x_n - y, r_n - s) \geq 1 \text{ as } n \longrightarrow \infty$$

Then $x_n \longrightarrow y$ as $n \longrightarrow \infty$.

$(3 \Longrightarrow 1)$. First, to show that $[y, \beta, s]$ is an upper bound.

Since $\{[x_n, \alpha_n, r_n]\}$ is an ascending sequence, then for all $n \geq m$ with $r_n \geq r_m \geq 0$, $0 \leq \alpha_m \leq \alpha_n \leq 1$; we have $[x_n, \alpha_n, r_n] \sqsubseteq [x_m, \alpha_m, r_m]$,

i.e., $N\left(x_n - x_m, r_n - r_m\right) \geq 1 - (\alpha_n - \alpha_m)$, and as $m \longrightarrow \infty$ then $x_m \longrightarrow y, \alpha_m \longrightarrow \beta$, which yields to $[x_n, \alpha_n, r_n] \sqsubseteq [y, \beta, s]$.

Now to show that $[y, \beta, s]$ is a least upper bound of $\{[x_n, \alpha_n, r_n]\}$

Suppose $[z, \gamma, t]$ is another upper bond of $\{[x_n, \alpha_n, r_n]\}$ with $r_n \geq t \geq 0$, $0 \leq \gamma \leq \alpha_n, \forall\, n \in \mathbb{N}$ then, $N(x_n - z, r_n - t) \geq 1 - (\alpha_n - \gamma)$.

By assumption $x_n \to y, \alpha_n \to \beta$, as $n \to \infty$ one has $N(y - z, s - t) \geq 1 - (\beta - \gamma), \forall\, s \geq t$.

Then, $[y, \beta, s] \sqsubseteq [z, \gamma, t]$ that is $[y, \beta, s]$ is a least upper bound of $\{[x_n, \alpha_n, r_n]\}$. ■

Theorem (3.10):

Let $(BX, \sqsubseteq)$ be a fuzzy domain normed space on a fuzzy normed space (X,N,*), then the following statements are equivalent:

1. (X,N,*) is complete.
2. Every ascending sequence in BX has a least upper bound.
3. BX is a dcpo.

Proof:

$(1 \Longrightarrow 2)$. Suppose (X,N,*) is complete let $\{[x_n, \alpha_n, r_n]\}$ be an ascending sequence in BX, by Lemma(3.7),$\{x_n\}$ is Cauchy sequence in X and $\{\alpha_n\}$ is convergent, but (X,N,*) is complete then $\{x_n\}$ is convergent in X then there exists some $y \in X$ and $\beta \in (0,1)$ with $x_n \longrightarrow y$ and $\alpha_n \longrightarrow \beta$, by theorem (3.9),$[y, \beta, s]$ with $r_n \geq s$, $\forall\ n \in \mathbb{N}$, is a least upper bound of $\{[x_n, \alpha_n, r_n]\}$, the reverse way can be obtained by considering $\{x_n\}$ as a Cauchy sequence in X since $\{[x_n, \alpha_n, r_n]\}$ is an ascending sequence in BX, and that $[y, \beta, s]$ is a least upper bound of $\{[x_n, \alpha_n, r_n]\}$ then by theorem (3.9), $x_n \longrightarrow y$ hence $\{x_n\}$ is convergent in X, then X is complete.

$(2 \Longrightarrow 3)$. Let D be a directed set of BX and $\{[x_n, \alpha_n, r_n]\}$ is an ascending sequence in BX then it has a least upper bound, (by assumption), say $[y, \beta, s]$ and by proposition (3.6), it is a least upper bound of D, then BX is adcpo.

(3 ⟹ 2). If BX is a dcpo, and $\{ [x_n, \alpha_{n,}, r_n] \}$ is an ascending sequence in BX, which is a directed set then it has a least upper bound then we done. ■

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