ISSN 2347-1921

The New Generalized of $\exp(-\phi(\xi))$ Expansion Method And Its Application To Some Complex Nonlinear Partial Differential Equations

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ABSTRACT

In this article, the generalized $\exp(-\phi(\xi))$ expansion method has been successfully implemented to seek traveling wave solutions of the Eckaus equation and the nonlinear Schrödinger equation.

The result reveals that the method together with the new ordinary differential equation is a very influential and effective tool for solving nonlinear partial differential equations in mathematical physics and engineering.

The obtained solutions have been articulated by the hyperbolic functions, trigonometric functions and rational functions with arbitrary constants.

Indexing terms/Keywords

Generalized $\exp(-\phi(\xi))$ Expansion Method, Exact Solutions; Eckaus Equation, Nonlinear Schrödinger Equation.

Academic Discipline And Sub-Disciplines

Mathematics;Differential Equations

SUBJECT CLASSIFICATION

E.g., Mathematics Subject Classification; Library of Congress Classification

ISSN 2347-1921

1. INTRODUCTION

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations (NLEEs). The study of exact solutions, especially traveling wave solutions, for NLPDEs plays a significant role in the study of nonlinear physical phenomena. These exact solutions when they exist can help to understand the dynamical processes that are modeled by the corresponding nonlinear evolution equations (NLEEs).

In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed.

A variety of powerful methods, such that, $\left(\frac{G'}{G}\right)$ expansion method[1-2], $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method[3], the $\exp(-\phi(\xi))$ expansion method[4-6], the modified tanh method [6], the $\coth_a(\xi)$ expansion method [7], sine-cosine method [8], Jacobi elliptic function expansion method [9], and the F expansion method [10]. The solution procedure this method, with the aid of Maple, is of utter simplicity and this method can easily extended to other kinds of nonlinear evolution equations. In this research, we use the new generalized of $\exp(-\phi(\xi))$ expansion method to obtain new solitary wave solutions for the Eckhaus [11,12] and Schrödinger equations[12,13].

2. Description of The Generalized of $\exp(-\phi(\xi))$ expansion method

Suppose that we have a nonlinear PDE in the following form :

$$F(u,u_t,u_x,u_{tt},u_{xt},u_{xx},u_{xxt},\ldots.)=0 \tag{2.1}$$

where $u=u(x,t)$ is an unknown function F is a polynomial in $u=u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

The main steps of this method are as follows

Step 1: Use the traveling wave transformation :

$$u(x,t)=u(\xi)\ ; \qquad \xi=k_1x+k_2t \tag{2.2}$$

where k_1,k_2 are a constants to be determined latter, permits us reducing (2.1) to an ODE for $u=u(\xi)$ in the form

$$P(u,k_1u_\xi,k_2u_\xi,k_1k_2u_{\xi\xi},\ldots.)=0 \tag{2.3}$$

where P is a polynomial of $u=u(\xi)$ and its total derivatives.

Step2 : Balancing the highest derivative term with the nonlinear terms in (2.3),we find the value of the positive integer (m). If the value (m) is noninteger one can transform the equation studied.

Step3 : Suppose that the solution of (2.3) can be expressed as follows :

$$u(\xi)=\sum_{i=0}^{m}\alpha_i\left[\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)\right]^i \tag{2.4}$$

where, $\alpha_i(i=0,1,..m)$ are constants to be determined, such that $\alpha_i\neq 0$ and $\phi(\xi)$ satisfies the following differential equation :

$$\phi'(\xi)=\frac{(A_3\phi(\xi)+A_4)^2}{(A_1A_4-A_2A_3)}\left(\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)+\mu\exp\left(\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)+\lambda\right) \tag{2.5}$$

where $(A_1A_4-A_2A_3)\neq 0$. Eq. (2.5) gives the following solutions:

Family 1: when

$$(A_1A_4-A_2A_3)\neq 0,\mu\neq 0,(\lambda^2-4\mu)>0,A_2=0$$

ISSN 2347-1921

$$\phi_1(\xi) = \left[\frac{A_4 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)}{A_1 - A_3 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)}\right] \quad (2.6)$$

Family 2: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) < 0, A_2 = 0$$

$$\phi_2(\xi) = \left[\frac{A_4 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)}{A_1 - A_3 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)\lambda}{2\mu}\right)}\right] \quad (2.7)$$

Family 3: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu = 0, \lambda \neq 0, (\lambda^2 - 4\mu) > 0$$

$$\phi_3(\xi) = \left[-\frac{A_2 + A_4 \ln\left(\frac{\lambda}{\exp\,(\lambda(\xi+c))-1}\right)}{A_1 + A_3 \ln\left(\frac{\lambda}{\exp\,(\lambda(\xi+c))-1}\right)}\right] \quad (2.8)$$

Family 4: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0, A_2 = 0$$

$$\phi_4(\xi) = \left[\frac{A_4 \ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}{A_1 - A_3 \ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}\right] \quad (2.9)$$

Family 5: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu = 0, \lambda = 0, (\lambda^2 - 4\mu) = 0$$

$$\phi_5(\xi) = \left[-\frac{A_2 - A_4 \ln(\xi + c)}{A_1 - A_3 \ln(\xi + c)}\right] \quad (2.10)$$

Family 6: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0, A_i \neq 0 (i = 1,2,3,4)$$

$$\phi_6(\xi) = \left[-\frac{A_2 - A_4 \ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}{A_1 - A_3 \ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}\right] \quad (2.11)$$

Family 7: when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) > 0, A_2 \neq 0$$

$$\phi_7(\xi) = \left[-\frac{2A_2 - 2A_4 \ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c)\right)-1\right)}\right)+A_4\ln(\lambda^2-4\mu)}{2A_1 - 2A_3 \ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c)\right)-1\right)}\right)+A_3\ln(\lambda^2-4\mu)}\right] \quad (2.12)$$

Step 4 : Substituting (2.4) into (2.3) and using (2.5), and then setting all the cofficients of $\left(\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)\right)^i$ of the resulting systems to zero, yields a system of algebraic equations for k_1, k_2, λ, μ and $\alpha_i (i = 0,1,2, \ldots m)$.

Step 5 : Suppose that the value of the constants k_1, k_2, λ, μ and $\alpha_i (i = 0,1,2, \ldots m)$ can be found by solving the algebraic equations which are obtained in step 4. Since the general solutions of (2.5) have been well known for us, substituting $k_1, k_2, \lambda, \mu, \alpha_i$ and the general solutions of (2.5) into (2.4), we have the exact solutions of the nonlinear PDEs (2.1).

3.ECKAUS EQUATION

In this section, we will apply the The Generalized of $\exp(-\phi(\xi))$ expansion method to find the exact solutions of the Eckaus equation. Let us consider Eckaus equation:

$$iW_t + W_{xx} + 2(|W|^2)_x W + |W|^4 W = 0 \quad (3.1)$$

We may choose the following traveling wave transformation

$$W(x,t) = u(\xi)\exp\big(i(\alpha x + \beta t)\big); \qquad \xi = k(x - 2\alpha t) \quad (3.2)$$

where k, α and β are costants to be determined later.

Eq. (3.1) becomes

$$k^2 u_{\xi\xi} - (\beta + \alpha^2)u + 4k u_\xi u^2 + u^5 = 0 \quad (3.3)$$

By balancing the height order derivative term $(u_{\xi\xi})$ with the nonlinear term $(u_\xi u^2)$ in (3.3), gives $\left(m = \frac{1}{2}\right)$. To obtain an analytic solution, (m) should be an integer. This requires the use of the transformation

$$u = v^{\left(\frac{1}{2}\right)} \quad (3.4)$$

That transforms (3.3) to

$$2k^2 v v_{\xi\xi} + 8kv^2 v_\xi - k^2 (v_\xi)^2 - 4(\beta + \alpha^2)v^2 + 4v^4 = 0 \tag{3.5}$$

By balancing the height order derivative term $(vv_{\xi\xi})$ with the nonlinear term (v^4) in (3.5), gives $(m = 1)$.

Therefore, the generalized of $\exp(-\phi(\xi))$ expansion method allows us to use the solution in the following form:

$$v(\xi) = \alpha_0 + \alpha_1 \exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right) \tag{3.6}$$

Substituting (3.6)and(2.5) into(3.5), the left-hand side is converted into polynomials in $\left(\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)\right)^j$,$(j = 0,1,2,....)$. We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations (for simplicity,which are not presented) for $\alpha_0, \alpha_1, k, \alpha, \beta, \lambda, \mu, A_1, A_2, A_3$and A_4. Solving these algebraic equations with the help of algebraic software Maple, we obtain

$$\alpha_0 = \frac{k\left(\lambda+\sqrt{\lambda^2-4\mu}\right)}{4}, \alpha_1 = \frac{k}{2}, \beta = \frac{k^2}{4}(\lambda^2 - 4\mu) - \alpha^2, \mu = \mu \tag{3.7}$$
$$A_1 = A_1, A_2 = A_2, A_3 = A_3, A_4 = A_4, k = k, \lambda = \lambda, \alpha = \alpha$$

Substituting (3.7) into(3.6), and use (3.2),(3.4) we have :

$$W(\xi) = \frac{\sqrt{k}}{2}\exp\left(i\left(\alpha x + \left(\frac{k^2}{4}(\lambda^2 - 4\mu) - \alpha^2\right)t\right)\right)\sqrt{\left(\lambda + \sqrt{\lambda^2 - 4\mu}\right) + 2\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)} \tag{3.8}$$

where

$$\xi = k(x - 2\alpha t)$$

Consequently,the exact solution of the Eckaus equation (3.1) with the help of Eq. (2.6) to Eq. (2.12) are obtained in the followin form:

Case (3-1): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) > 0, A_2 = 0$$

$$W_1^3(\xi) = \left[\frac{\sqrt{k}}{2} \exp\left(i\left(\alpha x + \left(\frac{k^2}{4}(\lambda^2 - 4\mu) - \alpha^2 \right) t \right) \right) \times \sqrt{\left[\left(\lambda + \sqrt{\lambda^2 - 4\mu}\right) + 2\exp\left(- \frac{A_1 \left(\frac{A_4 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)} \right)}{A_3 \left(\frac{A_4 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)} \right) + A_4} \right) \right]} \right]$$

$$\xi = k(x - 2\alpha t) \tag{3.9}$$

In particular setting

$$\mu = 4, A_1 = 1, A_3 = 1, A_4 = 1, k = 2, \lambda = 5, \alpha = 1, c = 0$$

we fined :

$$W_{1,0}^3(\xi) = \left[\exp\left(i(x + 8t) \right) \times \sqrt{\left(4 + \exp\left(- \frac{\left(\frac{\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right) - 5}{8} \right)}{1 - \ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right) - 4}{8} \right)} \right)}{\left(\frac{\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right) - 4}{8} \right)}{1 - \ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right) - 4}{8} \right)} \right) + 1} \right) \right)} \right] \tag{3.10}$$

$$\xi = 2x - 4t$$

See Figure (3 .1):

Case (3-2) : when

ISSN 2347-1921

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) < 0, A_2 = 0$$

$$W_2^3(\xi) = \left[\frac{\sqrt{k}}{2} \exp\left(i\left(\alpha x + \left(\frac{k^2}{4}(\lambda^2 - 4\mu) - \alpha^2 \right) t \right) \right) \times \sqrt{\left[\left(\lambda + \sqrt{\lambda^2 - 4\mu}\right) + 2\exp\left(- \frac{A_1 \left(\frac{A_4 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)} \right)}{A_3 \left(\frac{A_4 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c) \right) - \lambda}{2\mu} \right)} \right) + A_4} \right) \right]} \right]$$

$$\xi = k(x - 2\alpha t) \qquad (3.11)$$

In particular setting

$$\mu = 2, A_1 = -1, A_3 = 1, A_4 = 1, k = 4, \lambda = 2, \alpha = 1, c = 0$$

we fined :

$$W_{2,0}^3(\xi) = \left[\exp\left(i(x - 17t) \right) \times \sqrt{\left[(2 + 2i) + 2\exp\left(\frac{\left(-\frac{\ln\left(\frac{\tan(\xi) - 1}{2} \right)}{1 + \ln\left(\frac{\tan(\xi) - 1}{2} \right)} \right)}{\left(-\frac{\ln\left(\frac{\tan(\xi) - 1}{2} \right)}{1 + \ln\left(\frac{\tan(\xi) - 1}{2} \right)} \right) + 1} \right) \right]} \right] \qquad (3.12)$$

$$\xi = 4x - 8t$$

See Figure (3 .2)

Case (3-3) : when

$$(A_1A_4 - A_2A_3) \neq 0, \mu = 0, \lambda \neq 0, (\lambda^2 - 4\mu) > 0$$

$$W_3^3(\xi)=\left[\frac{\frac{\sqrt{k}}{2}\exp\left(i\left(\alpha x+\left(\frac{k^2}{4}\lambda^2-\alpha^2\right)t\right)\right)\times}{\sqrt{\left[(\lambda+|\lambda|)+2\exp\left(-\frac{A_1\left(-\frac{A_2+A_4\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}{A_1+A_3\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}\right)+A_2}{A_3\left(-\frac{A_2+A_4\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}{A_1+A_3\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}\right)+A_4}\right)\right]}}\right] \quad (3.13)$$

$$\xi=k(x-2\alpha t)$$

In particular setting

$$\mu=0,A_1=-1,A_2=-1,A_3=1,A_4=2,k=4,\lambda=1,\alpha=1,c=0$$

we fined :

$$W_{3,0}^3(\xi)=\exp(i(x+3t))\sqrt{\left[2+2\exp\left(\frac{\left(\frac{1-2\ln\left(\frac{1}{\exp(\xi)-1}\right)}{-1+\ln\left(\frac{1}{\exp(\xi)-1}\right)}\right)+1}{\left(\frac{1-2\ln\left(\frac{1}{\exp(\xi)-1}\right)}{-1+\ln\left(\frac{1}{\exp(\xi)-1}\right)}\right)+2}\right)\right]} \quad (3.14)$$

$$\xi=4x-8t$$

See Figure (3 .3)

Case (3-4): when

$$(A_1A_4-A_2A_3)\neq 0,\mu\neq 0,\lambda\neq 0,(\lambda^2-4\mu)=0,A_2=0$$

$$W_4^3(\xi)=\left[\frac{\frac{\sqrt{k}}{2}\exp\left(i(\alpha x-\alpha^2 t)\right)\times}{\sqrt{\left(\lambda+2\exp\left(-\frac{A_1\left(\frac{A_4\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}{A_1-A_3\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}\right)}{A_3\left(\frac{A_4\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}{A_1-A_3\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}\right)+A_4}\right)\right)}}\right] \quad (3.15)$$

where

$$\xi=k(x-2\alpha t)$$

In particular setting

$$\mu=1,A_1=-1,A_3=1,A_4=2,k=4,\lambda=2,\alpha=1,c=0$$

we fined :

$$W_{4,0}^{3}(\xi)=\exp\big(i(x-t)\big)\sqrt{\left(2+2\exp\left(\frac{\left(\frac{-2\ln\left(-\frac{\xi+1}{\xi}\right)}{1+\ln\left(-\frac{\xi+1}{\xi}\right)}\right)}{\left(\frac{-2\ln\left(-\frac{\xi+1}{\xi}\right)}{1+\ln\left(-\frac{\xi+1}{\xi}\right)}\right)+2}\right)\right)} \tag{3.16}$$

$$\xi=4x-8t$$

See Figure (3 .4)

Case (3-5): when

$$(A_1A_4-A_2A_3)\neq 0,\mu=0,\lambda=0,(\lambda^2-4\mu)=0$$

$$W_{5}^{3}(\xi)=\left[\frac{\sqrt{k}}{2}\exp\big(i(\alpha x-\alpha^2 t)\big)\times\sqrt{\left(2\exp\left(-\frac{A_1\left(-\frac{A_2-A_4\ln(\xi+c)}{A_1-A_3\ln(\xi+c)}\right)+A_2}{A_3\left(-\frac{A_2-A_4\ln(\xi+c)}{A_1-A_3\ln(\xi+c)}\right)+A_4}\right)\right)}\right] \tag{3.17}$$

$$\xi=k(x-2\alpha t)$$

In particular setting

$$A_1=-1,A_2=-1,A_3=1,A_4=2,k=4,\alpha=1,c=0$$

we fined :

$$W_{5,0}^{3}(\xi)=\sqrt{2}\exp\big(i(x-t)\big)\sqrt{\exp\left(\frac{\left(\frac{1+2\ln(\xi)}{1+\ln(\xi)}\right)-1}{\left(\frac{1+2\ln(\xi)}{1+\ln(\xi)}\right)-2}\right)} \tag{3.18}$$

$$\xi=4x-8t$$

See Figure (3 .5)

Case (3-6): when

$$(A_1A_4-A_2A_3)\neq 0,\mu\neq 0,\lambda\neq 0,(\lambda^2-4\mu)=0,A_i\neq 0(i=1,2,3,4)$$

$$W_{6}^{3}(\xi)=\left[\frac{\sqrt{k}}{2}\exp\big(i\alpha(x-\alpha t)\big)\times\sqrt{\left(\lambda+2\exp\left(-\frac{A_1\left(-\frac{A_2-A_4\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}{A_1-A_3\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}\right)+A_2}{A_3\left(-\frac{A_2-A_4\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}{A_1-A_3\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}\right)+A_4}\right)\right)}\right] \tag{3.19}$$

$$\xi=k(x-2\alpha t)$$

In particular setting

$$A_1=-1, A_2=-1, A_3=1, A_4=2, k=4, \alpha=1, c=0, \lambda=2, \mu=1$$

we fined :

$$W_{6,0}^3(\xi)=\left[\frac{\exp(i(x-t))\times}{\sqrt{\left(2+2\exp\left(\frac{\left(-\frac{1+2\ln\left(\frac{-\xi}{\xi-1}\right)}{1+\ln\left(\frac{-\xi}{\xi-1}\right)}\right)+1}{\left(-\frac{1+2\ln\left(\frac{-\xi}{\xi-1}\right)}{1+\ln\left(\frac{-\xi}{\xi-1}\right)}\right)+2}\right)\right)}}\right] \quad (3.20)$$

$$\xi=4x-8t$$

See Figure (3 .6)

Case (3-7): when

$$(A_1A_4-A_2A_3)\neq 0, \mu\neq 0, (\lambda^2-4\mu)>0, A_2\neq 0$$

$$W_7^3(\xi)=\left[\frac{\frac{\sqrt{k}}{2}\exp\left(i\left(\alpha x+\left(\frac{k^2}{4}(\lambda^2-4\mu)-\alpha^2\right)t\right)\right)\times}{\sqrt{\left(\left(\lambda+\sqrt{\lambda^2-4\mu}\right)+2\exp\left(-\frac{A_1\phi_7(\xi)+A_2}{A_3\phi_7(\xi)+A_4}\right)\right)}}\right] \quad (3.21)$$

where

$$\phi_7(\xi)=-\frac{2A_2-2A_4\ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)-1\right)}\right)+A_4\ln(\lambda^2-4\mu)}{2A_1-2A_3\ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)-1\right)}\right)+A_3\ln(\lambda^2-4\mu)}$$

$$\xi=k(x-2\alpha t)$$

In particular setting

$$A_1=-1, A_2=-1, A_3=1, A_4=2, k=4, \alpha=1, c=0, \lambda=4, \mu=3$$

we fined :

ISSN 2347-1921

$$W_{7,0}^{3}(\xi)=\left[\exp\left(i(x+15t)\right)\times\sqrt{6+2\exp\left(\frac{\left(-\frac{1+2\ln\left(\frac{6\exp(2\xi)+2}{3(\exp(2\xi)-1)}\right)-\ln(4)}{1+\ln\left(\frac{6\exp(2\xi)+2}{3(\exp(2\xi)-1)}\right)-\ln(2)}\right)+1}{\left(-\frac{1+2\ln\left(\frac{6\exp(2\xi)+2}{3(\exp(2\xi)-1)}\right)-\ln(4)}{1+\ln\left(\frac{6\exp(2\xi)+2}{3(\exp(2\xi)-1)}\right)-\ln(2)}\right)+2}\right)}\right] \quad (3.22)$$

$\xi=4x-8t$

See Figure (3 .7)

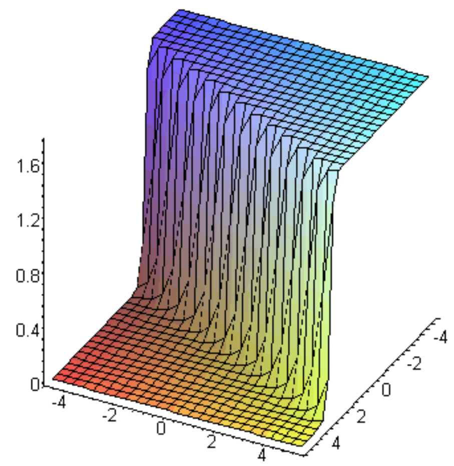

Figure (3.1)

3D plot of $|W_{1,0}^{3}|$

$\xi=2x-4t$

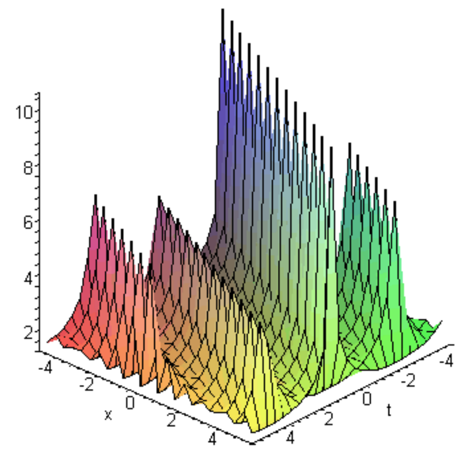

Figure (3.2)

3D plot of $|W_{2,0}^{3}|$

$\xi=4x-8t$

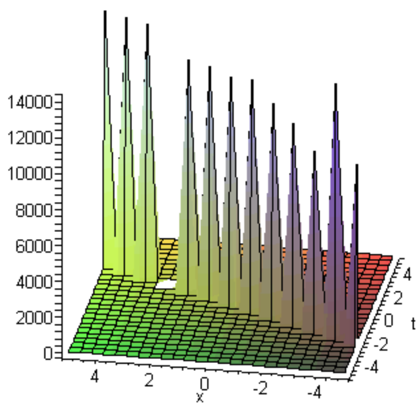

Figure (3.3)

3D plot of $|W_{3,0}^{3}|$

$\xi=4x-8t$

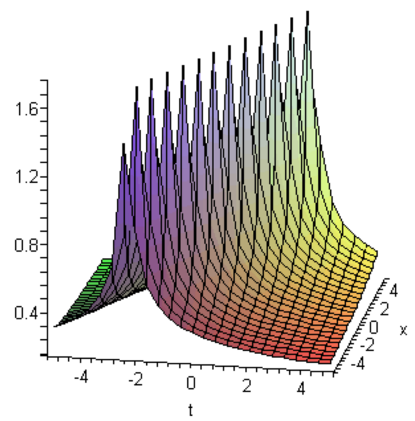

Figure (3.4)

3D plot of $|W_{4,0}^{3}|$

$\xi=4x-8t$

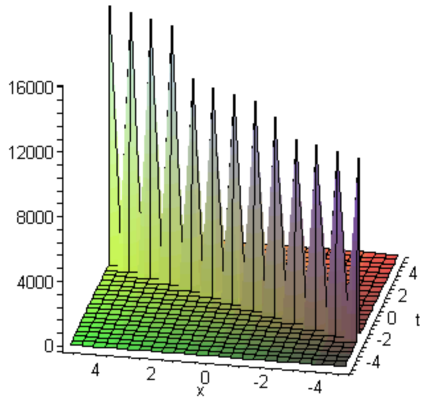

Figure (3.5)

3D plot of $\left|W_{5,0}^{3}\right|$

$\xi = 4x - 8t$

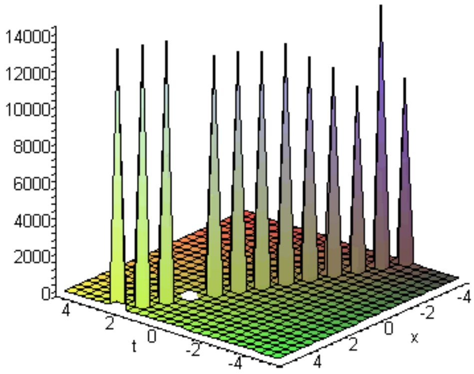

Figure (3.6)

3D plot of $\left|W_{6,0}^{3}\right|$

$\xi = 4x - 8t$

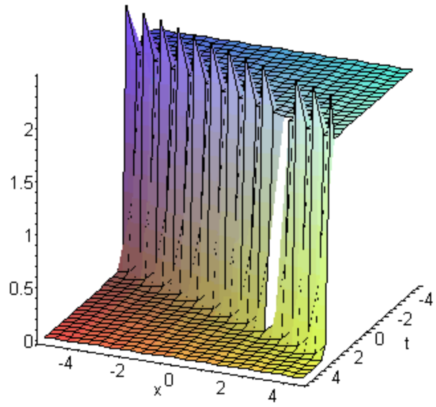

Figure (3.7)

3D plot of $\left|W_{7,0}^{3}\right|$

$\xi = 4x - 8t$

4.NONLINEAR SCHRÖDINGER EQUATION

In this section, we will apply the The Generalized of $\exp(-\phi(\xi))$ expansion method to find the exact solutions of the nonlinear Schrödinger equation. Let us consider nonlinear Schrödinger equation:

$$iW_t = -\frac{1}{2}W_{xx} + \delta|W|^2W \quad (4.1)$$

We may choose the following traveling wave transformation

$$W(x,t) = u(\xi)\exp(i(\alpha x + \beta t)); \qquad \xi = k(x - \alpha t) \quad (4.2)$$

where k, α and β are constants to be determined later.

Eq. (4.1) becomes

$$-(\alpha^2 + 2\beta)u + k^2 u_{\xi\xi} - 2\delta u^3 = 0 \quad (4.3)$$

By balancing the height order derivative term $(u_{\xi\xi})$ with the nonlinear term (u^3) in (4.3), gives $(m = 1)$. Therefore, the generalized of $\exp(-\phi(\xi))$ expansion method allows us to use the solution in the following form:

$$u(\xi) = \alpha_0 + \alpha_1 \exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right) \quad (4.4)$$

Substituting (4.4)and(2.5) into(4.3), the left-hand side is converted into polynomials in $\left(\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)\right)^j, (j = 0,1,2,\ldots.)$. We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations (for simplicity,which are not presented) for $\alpha_0, \alpha_1, k, \alpha, \beta, \lambda, \mu, A_1, A_2, A_3$ and A_4. Solving these algebraic equations with the help of algebraic software Maple, we obtain

$$\alpha_0 = \frac{\lambda k}{2\sqrt{\delta}}, \alpha_1 = \frac{k}{\sqrt{\delta}}, \beta = -\frac{1}{4}(k^2(\lambda^2 - 4\mu) + 2\alpha^2), \mu = \mu \quad (4.5)$$

$$A_1 = A_1, A_2 = A_2, A_3 = A_3, A_4 = A_4, k = k, \lambda = \lambda, \alpha = \alpha$$

Substituting (4.5) into(4.4), and use (4.2) we have :

$$W(\xi) = \exp\left(i\left(\alpha x - \frac{1}{4}(k^2(\lambda^2 - 4\mu) + 2\alpha^2)t\right)\right)\left[\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}}\exp\left(-\frac{A_1\phi(\xi)+A_2}{A_3\phi(\xi)+A_4}\right)\right] \quad (4.6)$$

where

$$\xi = k(x - \alpha t)$$

Consequently,the exact solution of the nonlinear Schrödinger equation (4.1) with the help of Eq. (2.6) to Eq. (2.12) are obtained in the followin form:

Case (4-1): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) > 0, A_2 = 0$$

$$W_1^4(\xi) = \left[\exp\left(i\left(\alpha x - \frac{1}{4}\left(k^2(\lambda^2 - 4\mu) + 2\alpha^2\right)t\right)\right) \times \left(\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}} \exp\left(- \frac{A_1 \left(\frac{A_4 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu} \right)} \right)}{A_3 \left(\frac{A_4 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu} \right)}{A_1 - A_3 \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)\right)-\lambda}{2\mu} \right)} \right) + A_4} \right) \right) \right] \quad (4.7)$$

$$\xi = k(x - \alpha t)$$

In particular setting

$$\mu = 4, A_1 = 1, A_3 = 1, A_4 = 1, k = 2, \lambda = 5, \alpha = 1, \delta = 1, c = 0$$

we fined :

$$W_{1,0}^4(\xi) = \left[\exp\left(i\left(x - \frac{19}{2}t\right)\right) \times \left(5 + 2\exp\left(- \frac{ \frac{\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right)-5}{8}\right)}{1-\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right)-5}{8}\right)} }{ \left(\frac{\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right)-5}{8}\right)}{1-\ln\left(\frac{-3\tanh\left(\frac{3}{2}\xi\right)-5}{8}\right)} \right) + 1 } \right) \right) \right] \quad (4.8)$$

$$\xi = 2(x - t)$$

See Figure (4 .1):

Case (4-2) : when

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$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2 - 4\mu) < 0, A_2 = 0$$

$$W_2^4(\xi) = \left[\exp\left(i\left(\alpha x - \frac{1}{4}\left(k^2(\lambda^2 - 4\mu) + 2\alpha^2\right)t\right)\right) \times \left(\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}} \exp\left(- \frac{A_1\left(\dfrac{A_4 \ln\left(\dfrac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)}{A_1 - A_3\ln\left(\dfrac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)} \right)}{A_3\left(\dfrac{A_4 \ln\left(\dfrac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)}{A_1 - A_3\ln\left(\dfrac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)\right)-\lambda}{2\mu}\right)} \right) + A_4} \right) \right) \right] \quad (4.9)$$

$$\xi = k(x - \alpha t)$$

In particular setting

$$\mu = 2, A_1 = -1, A_3 = 1, A_4 = 1, k = 4, \lambda = 2, \alpha = 1, c = 0, \delta = 1$$

we fined :

$$W_{2,0}^4(\xi) = \left[4\exp\left(i\left(x + \frac{31}{2}t\right)\right) \times \left(1 + \exp\left(\frac{\left(\frac{\ln(-\tan(\xi)-1)}{-1+\ln(-\tan(\xi)-1)}\right)}{\left(\frac{\ln(-\tan(\xi)-1)}{-1+\ln(-\tan(\xi)-1)}\right)-1} \right) \right) \right] \quad (4.10)$$

$$\xi = 4(x - t)$$

See Figure (4 .2)

Case (4-3) : when

$$(A_1A_4 - A_2A_3) \neq 0, \mu = 0, \lambda \neq 0, (\lambda^2 - 4\mu) > 0$$

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$$W_3^4(\xi) = \begin{bmatrix} \exp\left(i\left(\alpha x - \frac{1}{4}(k^2\lambda^2 + 2\alpha^2)t\right)\right) \times \\ \left(\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}}\exp\left(-\frac{A_1\left(-\frac{A_2 + A_4\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}{A_1 + A_3\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}\right) + A_2}{A_3\left(-\frac{A_2 + A_4\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}{A_1 + A_3\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c))-1}\right)}\right) + A_4}\right)\right) \end{bmatrix} \quad (4.11)$$

$$\xi = k(x - \alpha t)$$

In particular setting

$$\mu = 0, A_1 = -1, A_2 = -1, A_3 = 1, A_4 = 2, k = 4, \lambda = 1, \alpha = 1, c = 0, \delta = 4$$

we fined :

$$W_{3,0}^4(\xi) = \begin{bmatrix} \exp\left(i\left(x - \frac{9}{2}t\right)\right) \times \\ \left(1 + 2\exp\left(\frac{\left(\frac{1 - 2\ln\left(\frac{1}{\exp(\xi)-1}\right)}{1 - \ln\left(\frac{1}{\exp(\xi)-1}\right)}\right) - 1}{\left(\frac{1 - 2\ln\left(\frac{1}{\exp(\xi)-1}\right)}{1 - \ln\left(\frac{1}{\exp(\xi)-1}\right)}\right) - 2}\right)\right) \end{bmatrix} \quad (4.12)$$

$$\xi = 4(x - t)$$

See Figure (4 .3)

Case (4-4): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0, A_2 = 0$$

$$W_4^4(\xi) = \begin{bmatrix} \exp\left(i\left(\alpha x - \frac{1}{2}\alpha^2 t\right)\right) \times \\ \left(\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}}\exp\left(-\frac{A_1\left(\frac{A_4\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}{A_1 - A_3\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}\right)}{A_3\left(\frac{A_4\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}{A_1 - A_3\ln\left(\frac{-2\lambda(\xi+c)-4}{\lambda^2(\xi+c)}\right)}\right) + A_4}\right)\right) \end{bmatrix} \quad (4.13)$$

$$\xi = k(x - \alpha t)$$

In particular setting

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$$\mu = 1, A_1 = -1, A_3 = 1, A_4 = 2, k = 4, \lambda = 2, \alpha = 1, c = 0, \delta = 16$$

we fined :

$$W_{4,0}^4(\xi) = \left[\exp\left(i\left(x - \frac{1}{2}t\right)\right) \times \left(1 + \exp\left(\frac{\left(\frac{2\ln\left(-\frac{\xi+1}{\xi}\right)}{1-\ln\left(-\frac{\xi+1}{\xi}\right)}\right)}{\left(\frac{2\ln\left(-\frac{\xi+1}{\xi}\right)}{1-\ln\left(-\frac{\xi+1}{\xi}\right)}\right)+2}\right)\right)\right] \quad (4.14)$$

$$\xi = 4(x - t)$$

See Figure (4 .4)

Case (4-5): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu = 0, \lambda = 0, (\lambda^2 - 4\mu) = 0$$

$$W_5^4(\xi) = \left[\frac{k}{\sqrt{\delta}} \exp\left(i\left(\alpha x - \frac{1}{4}(2\alpha^2)t\right) - \frac{A_1\left(-\frac{A_2 - A_4\ln(\xi+c)}{A_1 - A_3\ln(\xi+c)}\right)+A_2}{A_3\left(-\frac{A_2 - A_4\ln(\xi+c)}{A_1 - A_3\ln(\xi+c)}\right)+A_4}\right)\right] \quad (4.15)$$

$$\xi = k(x - \alpha t)$$

In particular setting

$$A_1 = -1, A_2 = -1, A_3 = 1, A_4 = 2, k = 4, \alpha = 1, c = 0, \delta = 4$$

we fined :

$$W_{5,0}^4(\xi) = 2\exp\left(i\left(x - \frac{1}{2}t\right) + \frac{\left(\frac{1+2\ln(\xi)}{1+\ln(\xi)}\right)-1}{\left(\frac{1+2\ln(\xi)}{1+\ln(\xi)}\right)-2}\right) \quad (4.16)$$

$$\xi = 4(x - t)$$

See Figure (4 .5)

Case (4-6): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0, A_i \neq 0 (i = 1,2,3,4)$$

$$W_6^4(\xi) = \left[\exp\left(i\left(\alpha x - \frac{1}{2}\alpha^2 t\right)\right) \times \left(\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}} \exp\left(-\frac{A_1\left(-\frac{A_2 - A_4\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}{A_1 - A_3\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}\right)+A_2}{A_3\left(-\frac{A_2 - A_4\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}{A_1 - A_3\ln\left(-\frac{2(\xi+c)}{\lambda(\xi+c)-2}\right)}\right)+A_4}\right)\right)\right] \quad (4.17)$$

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$$\xi = k(x-\alpha t)$$

In particular setting

$$A_1=-1, A_2=-1, A_3=1, A_4=2, k=4, \alpha=1, c=0, \lambda=2, \mu=1, \delta=16$$

we fined :

$$W_{6,0}^4(\xi) = \left[\begin{array}{l}\exp\left(i\left(x-\frac{1}{2}t\right)\right)\times \\ \left(1+\exp\left(\frac{\left(\frac{1-2\ln\left(\frac{1-\xi}{\xi}\right)}{-1+\ln\left(\frac{1-\xi}{\xi}\right)}\right)+1}{\left(\frac{1-2\ln\left(\frac{1-\xi}{\xi}\right)}{-1+\ln\left(\frac{1-\xi}{\xi}\right)}\right)+2}\right)\right)\end{array}\right] \tag{4.18}$$

$$\xi = 4(x-t)$$

See Figure (4 .6)

Case (4-7): when

$$(A_1A_4 - A_2A_3) \neq 0, \mu \neq 0, (\lambda^2-4\mu) > 0, A_2 \neq 0$$

$$W_7^4(\xi) = \exp\left(i\left(\alpha x - \frac{1}{4}(k^2(\lambda^2-4\mu)+2\alpha^2)t\right)\right)\left[\frac{\lambda k}{2\sqrt{\delta}} + \frac{k}{\sqrt{\delta}}\exp\left(-\frac{A_1\phi_7(\xi)+A_2}{A_3\phi_7(\xi)+A_4}\right)\right] \tag{4.19}$$

where

$$\phi_7(\xi) = -\frac{2A_2-2A_4\ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)-1\right)}\right)+A_4\ln(\lambda^2-4\mu)}{2A_1-2A_3\ln\left(\frac{\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)\left(\lambda^2-4\mu+\lambda\sqrt{(\lambda^2-4\mu)}\right)-\left(\lambda^2-4\mu-\lambda\sqrt{(\lambda^2-4\mu)}\right)}{2\mu\left(\exp\left(\sqrt{(\lambda^2-4\mu)}(\xi+c_1)\right)-1\right)}\right)+A_3\ln(\lambda^2-4\mu)}$$

$$\xi = k(x-\alpha t)$$

In particular setting

$$A_1=-1, A_2=-1, A_3=1, A_4=2, k=2, \alpha=1, c=0, \lambda=4, \mu=3, \delta=4$$

we fined :

$$W_{7,0}^{4}(\xi)=\left[\begin{array}{l}\exp\left(i\left(x-\frac{9}{2}t\right)\right)\times \\ \left(2+\exp\left(\frac{\left(-\frac{2+4\ln\left(\frac{-6\exp\,(2\xi)+2}{3(\exp\,(2\xi)-1)}\right)-4\ln\,(2)}{2+2\ln\left(\frac{-6\exp\,(2\xi)+2}{3(\exp\,(2\xi)-1)}\right)-2\ln\,(2)}+1\right)}{\left(-\frac{2+4\ln\left(\frac{-6\exp\,(2\xi)+2}{3(\exp\,(2\xi)-1)}\right)-4\ln\,(2)}{2+2\ln\left(\frac{-6\exp\,(2\xi)+2}{3(\exp\,(2\xi)-1)}\right)-2\ln\,(2)}+2\right)}\right)\right)\end{array}\right] \qquad (4.20)$$

$\xi = 2(x-t)$

See Figure (4 .7)

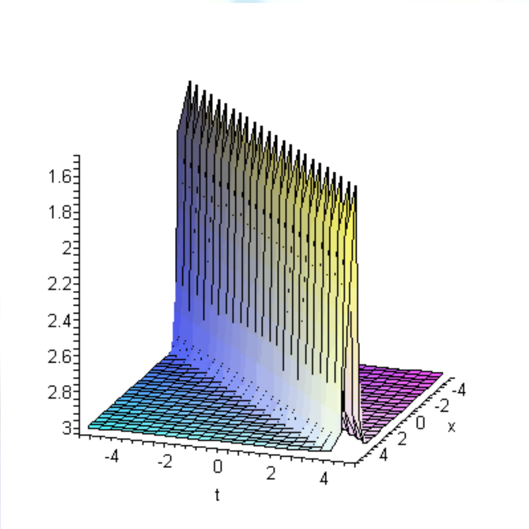

Figure (4.1)

3D plot of $\left|W_{1,0}^{4}\right|$

$\xi = 2x - 2t$

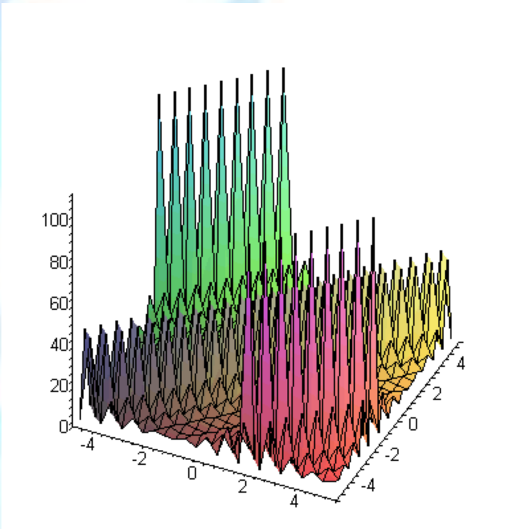

Figure (4.2)

3D plot of $\left|W_{2,0}^{4}\right|$

$\xi = 4x - 4t$

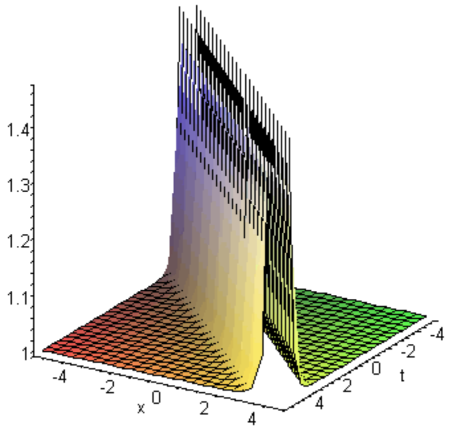

Figure (4.3)

3D plot of $\left|W_{3,0}^{4}\right|$

$\xi = 4x - 4t$

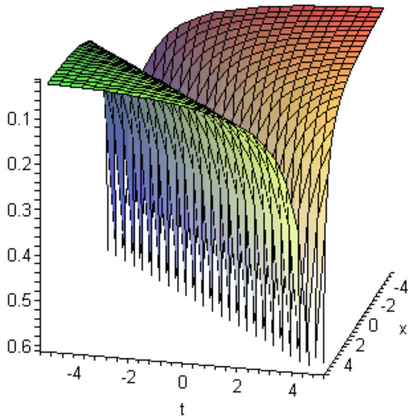

Figure (4.4)

3D plot of $\left|W_{4,0}^{4}\right|$

$\xi = 4x - 4t$

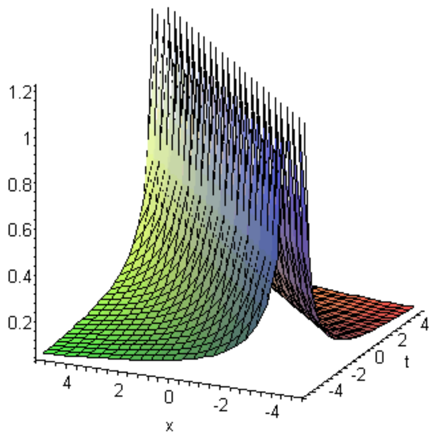

Figure (4.5)

3D plot of $\left|W_{5,0}^{4}\right|$

$\xi = 4x - 4t$

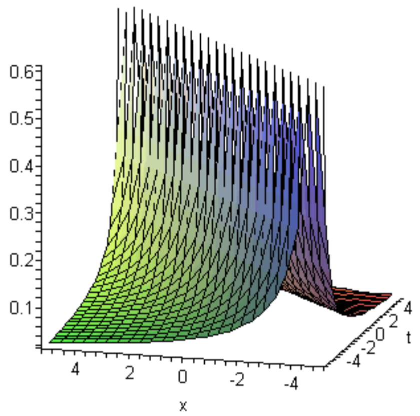

Figure (4.6)

3D plot of $\left|W_{6,0}^{4}\right|$

$\xi = 4x - 4t$

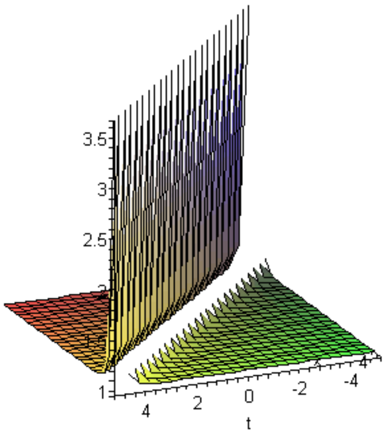

Figure (4.7)

D plot of $\left|W_{7,0}^{4}\right|$

$\xi = 2x - 2t$

5.Conclusion

In this artical, we considered two complex equations and the generalized $\exp(-\phi(\xi))$ expansion method has been successfully implemented to obtain new generalized traveling wave solutions of the Eckaus equation and the nonlinear Schrödinger equation. these solutions have rich local structures, It may be important to explain some physical phenomena . This work shows that, the new approach of $\exp(-\phi(\xi))$ expansion method is direct, effective and can be used for many other NLPDEs in mathematical physics.

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