ISSN 2347-1921

COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN HILBERT SPACE

Geeta Modi[1],
1. Prof. and Head Department of Mathematics, Govt. M.V.M. Bhopal (M.P.) India.
modi.geeta@gmail.com
R.N.Gupta[2]
2.Asst. Prof., Department of Mathematics, Jai Narain College of Technology,
Bhopal (M. P.), India.
ramnareshgupta@gmail. com

ABSTRACT

In this paper we prove a common fixed point theorem for weakly compatible mappings satisfies certain contractive condition in non- empty closed subset of a separable Hilbert Space. Our results generalize and extend the result Chauhan [7].

Keywords

Common fixed point, random operators, weakly compatible. Hilbert Space.

.Academic Discipline And Sub-Disciplines

Mathematics.

AMS Subject classification (2000): *47H10, 54H25*.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol.9, No 8
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1.INTRODUCTION

The study of random fixed point theory is started by Prague school of Probabilists in 1950 [8, 11]. Bharucha-Reid [5] has attracted much attention of many mathematicians by his survey article in this literature. Bharucha-Reid and Reagan [5, 10] obtain the solution of non linear random system by using random fixed point theory.

The structure of common random fixed point and random coincidence points for a pair of compatible random operators in Polish space studied by Beg [1, 2] and Beg and Shahzad [3, 4].Chouhan [7] has proved a fixed point theorem for four random operators in Separable Hilbert space.

In this paper we will prove a common fixed point theorem for weakly compatible random operators by using contractive condition in separable Hilbert spaces. For this we construct a sequence of measurable function of random fixed point to the four random operators.

2.PRELIMINARY NOTES

Let C be a closed subset of Separable Hilbert space H and (Ω, Σ) a measurable space.

Definition 2.1: A function $f: \Omega \to C$ is called measurable if $f^{-1}(B \cap C) \in \Sigma$ for each Borel subset B of H.

Definition 2.2: A function $F: \Omega \times C \to C$ is called random operator if $F(.,x): \Omega \to C$ is measurable for all $x \in C$.

Definition 2.3: A measurable function $y: \Omega \to C$ is called a random fixed point to the random operator $F: \Omega \times C \to C$ if $F\big(t, y(t)\big) = y(t)$ for all $t \in \Omega$.

Definition 2.4: A random operator $F: \Omega \times C \to C$ is called continuous if for fixed $t \in \Omega$, if $F(t,.): C \to C$ is continuous.

Definition 2.5: Two random operators $E, F: \Omega \times C \to C$ are called compatible if $E(t, .)$ and $F(t, .)$are compatible for all $t \in \Omega$.

Definition 2.6: Two random operators $E, F: \Omega \times C \to C$ are called weakly compatible if $E(t, y(t)) = F(t, y(t))$ for some measurable mapping compatible $y: \Omega \to C$

$$E\Big(t, F\big(t, y(t)\big)\Big) = F\Big(t, E\big(t, y(t)\big)\Big), \text{ For all } t \in \Omega.$$

3. MAIN RESULTS

Theorem 3.1: Let C be a non-empty closed subset of a Separable Hilbert space H. Let $E, F,$ S and T be four continuous random operators defined on C such that for $t \in \Omega, E(t,.), F(t,.), S(t,.), T(t,.): C \to C$ satisfy the following Conditions

$$(1)\ \ \|Ex - Fy\|^2 \le r\,\{\|Ex - Fy\|^2 + \|Tx - Sy\|^2 + \|Tx - Ey\|^2 + \|Sx - Fx\|^2\}$$

Where $\frac{1}{6} \le r < \frac{1}{3}$.

(2) The pair (E, T) and (F, S) are weakly compatible.

Then E, F, S and T have unique common random fixed point in C.

Proof: Let $y_0: \Omega \to C$ be an arbitrary measurable mapping for all $t \in \Omega$.

We construct a sequence of mappings $\{y_n(t)\}$.

Suppose that $\{y'_n(t)\}, \{y''_n(t)\}$ are two sequences such that

$y''_{2n}(t)$=$E(t,y'_{2n}(t)) = T(t,y'_{2n+1}(t))$,

$y''_{2n+1}(t)$=$F(t,y'_{2n+1}(t)) = S(t,y'_{2n+2}(t))$.

Firstly we show that $\{y''_n(t)\}$ is a Cauchy sequence.

If $y''_{2n}(t) = y'_{2n}(t) = y'_{2n+1}(t)$ and

$y''_{2n+1}(t) = y'_{2n+1}(t) = y'_{2n+2}(t) = y''_{2n}(t)$.

Then $y''_{2n}(t)$=$E(t,y''_{2n}(t)) = T\left(t,y''_{2n}(t)\right) = F(t,y''_{2n}(t)) = S(t,y''_{2n}(t))$.

Therefore $y''_{2n}(t)$ is a common random fixed point of E,F,S and T.

Now let the sequence $\{y''_n(t)\}$ and $\{y'_n(t)\}$ have no two consecutive terms equal at the same order.

For all $t \in \Omega$ and $n = 1,2, \dots \dots \dots$

$\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2$= $\|E(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2$

$\leq r\,\{\|E(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2 + \|T(t,y'_{2n+2}(t)) - S(t,y'_{2n+1}(t))\|^2$

$+\,\|T(t,y'_{2n+2}(t)) - E(t,y'_{2n+1}(t))\|^2 + \|S(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2\}$

$= r\,\{\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t) - y''_{2n}(t)\|^2$

$+\,\|y''_{2n+1}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t)) - y''_{2n+1}(t)\|^2\}$

$= r\,\{2\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t) - y''_{2n}(t)\|^2\}$

$\Rightarrow (1-2r)\left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\|^2 \leq r\|y''_{2n+1}(t) - y''_{2n}(t)\|^2$

$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\|^2 \leq \{\frac{r}{(1-2r)}\}\|y''_{2n+1}(t) - y''_{2n}(t)\|^2$

$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\| \leq \left\{\frac{r}{(1-2r)}\right\}^{\left(\frac{1}{2}\right)} \|y''_{2n+1}(t) - y''_{2n}(t)\|$

$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\| \leq q\|y''_{2n+1}(t) - y''_{2n}(t)\|$

Where $\left\{\frac{r}{(1-2r)}\right\}^{\left(\frac{1}{2}\right)} = q$.

So in general for all $t \in \Omega$ we have,

$\left\|y''_{n+1}(t) - y''_n(t)\right\| \leq q^n\|y''_1(t) - y''_0(t)\|$

Taking $n \to \infty$ we get $\left\|y''_{n+1}(t) - y''_n(t)\right\| \to 0$

Thus for all $t \in \Omega$, $\{y''_n(t)\}$ is a Cauchy sequence.

Hence $\{y''_n(t)\}$ is convergent in Separable Hilbert space.

Suppose that $\{y''_n(t)\} \to y''(t)$ as $n \to \infty$ for $t \in \Omega$

Since C is closed and y'' is a function from C to C.

Now we shall show that $y''_{2n}(t)$ is a common random fixed point of E,F,S and T.

For $t \in \Omega$,

$$\left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 = \left\|y''(t) - y''_{2n+1}(t) + y''_{2n+1}(t) - E\big(t, y''(t)\big)\right\|^2$$

$$\leq 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2\left\|y''_{2n+1}(t) - E\big(t, y''(t)\big)\right\|^2$$

$$= 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2\left\|F\big(t, y'_{2n+1}(t)\big) - E\big(t, y''(t)\big)\right\|^2$$

$$= 2\|y''^{(t)} - y''_{2n+1}(t)\|^2 + 2\left\|E\big(t, y''(t)\big) - F\big(t, y'_{2n+1}(t)\big)\right\|^2$$

$$\leq 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2r\left\{\left\|E\big(t, y''^{(t)}\big) - F\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|T\big(t, y''^{(t)}\big) - S\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|T\big(t, y''^{(t)}\big) - E\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|S\big(t, y''^{(t)}\big) - F\left(t, y''^{(t)}\right)\right\|^2\right\}$$

$$\leq 2\left\|y''^{(t)} - y''_{2n+1}(t)\right\|^2 + 2r\left\{\left\|E(t, y''(t)) - y''_{2n+1}(t)\right\|^2 + \left\|T(t, y''(t)) - y''_{2n}(t)\right\|^2 + \left\|T(t, y''(t)) - y''_{2n+1}(t)\right\|^2 + \left\|S\big(t, y''^{(t)}\big) - F\big(t, y''^{(t)}\big)\right\|^2\right\}$$

Letting $n \to \infty$, we have $\{y''_{2n}(t)\}, \{y''_{2n+1}(t)\} \to \{y''(t)\}$ because $\{y''_{2n}(t)\}, \{y''_{2n+1}(t)\}$ are subsequence of $\{y''_n(t)\}$.

$$\Rightarrow \left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 \leq 2\left\|y''^{(t)} - y''(t)\right\|^2 + 2r\left\{\left\|E\big(t, y''^{(t)}\big) - y''^{(t)}\right\|^2 + \left\|T\big(t, y''^{(t)}\big) - y''(t)\right\|^2 + \left\|T\big(t, y''^{(t)}\big) - y''(t)\right\|^2 + \left\|S\big(t, y''^{(t)}\big) - F\big(t, y''^{(t)}\big)\right\|^2\right\}$$

$$\Rightarrow (1 - 2r)\left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 \leq 2r\left\{\|T(t, y''(t)) - y''(t)\|^2 + \|T(t, y''(t)) - y''(t)\|^2 + \|S(t, y''(t)) - F(t, y''(t))\|^2\right\}$$

Since (E, T) and (F, S) are weakly compatible.

Therefore

$$(1 - 2r)\left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 \leq 2r\left\{2\|E(t, y''(t)) - y''(t)\|^2\right\}$$

$$\Rightarrow (1 - 6r)\left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 \leq 0$$

$$\Rightarrow (6r - 1)\left\|y''(t) - E\big(t, y''(t)\big)\right\|^2 \geq 0$$

$y''(t) = E\big(t, y''(t)\big)$ For all $t \in \Omega$, since $r \geq \frac{1}{6}$.

Similarly we can prove that for all $t \in \Omega$. $y''(t) = F(t, y''(t))$,

$y''(t) = S(t, y''(t))$, $\quad y''(t) = T(t, y''(t))$

Himmelberg [9] had proved if $G: \Omega \times C \to C$ is a continuous random operator on closed subset C then for any measurable function $f: \Omega \to C$ the function $f(t) = G(t, f(t))$, is also measurable function.

Thus $\{y''_n(t)\}$ is a sequence of measurable function. And hence $y''(t)$ is also a measurable function.

This implies that $y''(t)$ is a common random fixed point of E, F, S and T.

Uniqueness

Suppose that $g''(t): \Omega \to C$ be the another common random fixed point of E, F, S and T.

Therefore for all $t \in \Omega$,

$$E(t, g''(t)) = g''(t),\ F(t, g''(t)) = g''(t)$$

$$S(t, g''(t)) = g''(t),\ T(t, g''(t)) = g''(t)$$

Now

$\|y''(t) - g''(t)\|^2 = \|E(t, y''(t)) - F(t, g''(t))\|^2$

$\leq r\ \{\|E(t, y''(t)) - F(t, g''(t))\|^2 + \|T(t, y''(t)) - S(t, g''(t))\|^2$

$+ \|T(t, y''(t)) - E(t, g''(t))\|^2 + \|S(t, y''(t)) - F(t, y''(t))\|^2\}$

$< 2r\{\|y''(t) - g''(t)\|^2 + \|y''(t) - g''(t)\|^2 + \|y''(t) - g''(t)\|^2 + \|y''(t) - y''(t)\|^2\}$

$= 2r\{3\|y''(t) - g''(t)\|^2\}$

$\Rightarrow \|y''(t) - g''(t)\|^2 \leq 6r\|y''(t) - g''(t)\|^2$

$\Rightarrow (1 - 6r)\|y''(t) - g''(t)\|^2 \leq 0$

$\Rightarrow \|y''(t) - g''(t)\|^2 = 0$, Since $\frac{1}{6} \leq r$.

$y''(t) = g''(t)$

Hence E, F, S and T have a common unique random fixed point in C.

Example

Suppose that $E, F, S, T: C \to C$ define as $Ex = 1 + x, Fx = 2 + x, Sx = 3 + x$ and $Tx = 3 + x$.

Let $\{x_n\}$ and $\{y_n\}$ are two sequence such that $x_n = 1 + \frac{1}{n^2}$ and $y_n = 1 + \frac{1}{n^2}$

Now since $ETx = E(3 + x) = 4 + x$ and $TEx = T(1 + x) = 4 + x$;

$FSx = F(3 + x) = 5 + x$ And $SFx = S(2 + x) = 5 + x$.

Then clearly (E, T) and (F, S) are weakly compatible.

Now

$\|Ex_n - Fy_n\|^2 \leq r\ \{\|Ex_n - Fy_n\|^2 + \|Tx_n - Sy_n\|^2 + \|Tx_n - Ey_n\|^2 + \|Sx_n - Fx_n\|^2\}$

$\Rightarrow \|2 - 3\|^2 \leq r\ \{\|2 - 3\|^2 + \|4 - 4\|^2 + \|4 - 2\|^2 + \|4 - 3\|^2\}$

$\Rightarrow 1 \leq r\ \{1 + 4 + 1\}$

ISSN 2347-1921

$$\Rightarrow r \geq \frac{1}{6}$$

Hence theorem is verified with condition (1) and (2).

REFERENCES

[1] Beg, I. 1997. Random fixed point of random operators satisfying semi-contractive conditions, Mahtmatica Japonica, 46, No.1, 151-155.

[2] Beg, I. 2002. Approximation of random fixed point in normed space, Nonlinear Analysis 51, No.8, 1363-1372.

[3] Beg, I. and Shahzad, N.1993. Random fixed points of random multivalued operators on Polish spaces, Non-linear Analysis Theory Method and Applications, 20, No.7, 835-847.

[4] Beg, I. and Shahzad, N. 1995. Common random fixed points of random multivalued operators on Metric spaces, Bollet-tino Della Unione Mathematica Italiana, Series VII.A 9, No.3, 493-503.

[5] Bharucha-Reid, A.T. 1976. Fixed point theorems in Probabilistic analysis, Bulletin of the American Mathematical Society, 82, No.5, 641-657.

[6] Bharucha-Reid, A.T. 1972. Random Integral Equation, Academic press, New York.

[7] Chauhan, S.S. 2013. Common fixed point theorem for four random operators in Hilbert space, Advance in Applied Science Research, 4(2), 219-222.

[8] Han's, O.1957. Reduzierende Zufllige transformation, Czechoslovak Mathematical Journal, 7 (82),154-158.

[9] Himmelberg, C. J. 1975. Measurable relations, Fund. Math., 87, 53-72.

[10] Regan, D.O. 1998. Fixed points and random fixed points for weakly inward approximable maps, Proceedings of the American Mathematical Society, 126, No.10, 3045-3053.

[11] Spacek, A. 1955. Zufallige Gleichungen, Czechoshvak Mathematical Journal, 5(80), 462-466.

Author' biography with Photo

Dr. Geeta Modi has been awarded Ph.D., in 1990. She has 31 years of teaching experience. She is currently working as a Professor & Head of Department of Mathematics, Govt. MVM Bhopal. She is presently The Chairman of Board of Studies (Mathematics) Barkatullah University Bhopal, and associated with the Member of central Board of Studies (Mathematics) Government of Madhya Pradesh. She is V.C. nominee member of Board of Studies IEHE Bhopal Madhya Pradesh, India. She has published more than 45 articles in national and international journals. 08 Research scholar awarded Ph.D. under her supervision and 06 research scholar registered.

R. N. Gupta is a research scholar in the Department of Mathematics, Govt. MVM Bhopal, Madhya Pradesh,India. His field of research in Fixed Point theorey and has 15 years of teaching experience.