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The operator $G(a,b;D_q)$ for the polynomials $W_n(x,y,a,b;q)$

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Abstract. We give an identity which can be regarded as a basic result for this paper. We give special values to the parameters in this identity to get some wellknown identities such as Euler identity and Cauchy identity. Inspired by this identity we introduce an operator $G(a,b;D_q)$. The exponential operator $R(bD_q)$ defined by Saad and Sukhi [11] can be considered as a special case of the operator $G(a,b;D_q)$ for $a=0$. Also we introduce a polynomials $W_n(x,y,a,b;q)$. Al-Salam-Carlitz polynomials $U_n(x,y,b;q)$ [4] is a special case of $W_n(x,y,a,b;q)$ for $a=0$. So all the identities for the polynomials $W_n(x,y,a,b;q)$ are extensions of formulas for the Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$. We give a transformation formula of ${}_2\phi_2$ series. We give an operator proof for the generating function, the Rogers formula and the Mehler's formula for $W_n(x,y,a,b;q)$. Rogers formula leads to the inverse linearization formula. We give another Rogers-type formula for the polynomials $W_n(x,y,a,b;q)$.

Keywords: the q-exponential operator; Al-Salam-Carlitz polynomials; Rogers formula; Mehler's formula.

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1. Introduction

We list some definitions and terminology for the basic hypergeometric series [6]. We assume that $|q|<1$. The q-shifted factorial is defined as

$$(a;q)_n = \begin{cases} 1 & if \quad n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & if \quad n = 1,2,3,\ldots \end{cases}$$

We also define

$$(a,q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

We shall use the following notation for the multiple q-shifted factorials:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n.$$

$$(a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The generalized basic hypergeometric series is defined by

$${}_r\phi_s(a_1,a_2,\cdots,a_r,b_1,b_2,\cdots,b_s;q,x) = {}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s \end{matrix};q,x\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $q \neq 0$ when $r > s+1$. Note that

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,x\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_r;q)_n} x^n.$$

The q-binomial coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

One of the most important identities is the Cauchy identity

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.1}$$

Euler found the following special case of Cauchy identity (1.1):

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \tag{1.2}$$

The finite form of the Euler's identity (1.2) is (see [3])

$$(x;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^k. \tag{1.3}$$

The q-analog of the Chu-Vandermonde summation is

$$\sum_{k=0}^{h}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} n+m \\ h \end{bmatrix}. \tag{1.4}$$

In this paper, we will frequently use the following identities [6]:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.5}$$

$$(a;q)_{n+k} = (a;q)_n (aq^n;q)_k. \tag{1.6}$$

$$(q/a;q)_k = (-a)^{-k} q^{\binom{k+1}{2}} (aq^{-k};q)_\infty / (a;q)_\infty. \tag{1.7}$$

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn. \tag{1.8}$$

The Cauchy polynomials is defined by

$$P_k(x;y) = (x-y)(x-qy)\cdots(x-yq^{k-1}) = (y/x;q)_k x^k,$$

which has the generating function

$$\sum_{k=0}^{\infty} P_k(x,y)\frac{t^k}{(q,q)_k} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.9}$$

The Cauchy polynomials can be written in the following form [7, 8]:

$$P_n(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} y^k x^{n-k}.$$

In 1965, Al-Salam and Carlitz [2] defined the following polynomials, also see [9]:

$$u_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, qx/a\right).$$

The polynomials $u_n^{(a)}(x;q)$ can be rewritten as

$$u_n^{(a)}(x;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x,1).$$

In 2010, Chen *et al.* [4] extended the definition of Al-Salam-Carlitz polynomials as follows:

$$U_n(x,y,a;q) = (-1)^n a^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, y/x \\ 0 \end{matrix}; q, qx/a\right),$$

which have the generating function

$$\sum_{n=0}^{\infty} U_n(x,y,a;q)\frac{t^n}{(q;q)_n} = \frac{(at,yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.10}$$

The polynomials $U_n(x,y,a;q)$ can be rewritten as

$$U_n(x,y,a;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x,y).$$

They gave the following idetities for $U_n(x,y,a;q)$:

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The Rogers-type formula for $U_n(x,y,a;q)$:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}-nm} U_{m+n}(x,y,a;q)\frac{s^m}{(q;q)_m}\frac{t^n}{(q;q)_n}=\frac{(as;q)_\infty}{(at;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y/x, s/t \\ q/at \end{matrix};q,qx/a\right), \tag{1.11}$$

provided that $y/x=q^{-r}$ for a non-negative integer r and $\max\{|atq^{-r}|,|qx/a|\}<1$.

The inverse linearization formula for $U_n(x,y,a;q)$:

$$U_{n+m}(x,y,a;q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(aq^m)^k P_{n-k}(x,y)U_m(x,yq^{n-k},a;q). \tag{1.12}$$

The Mehler's formula for $U_n(x,y,a;q)$:

$$\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}U_n(x,y,a;q)U_n(u,v,b;q)\frac{t^n}{(q;q)_n}=\frac{(abt,ybt,avt;q)_\infty}{(xbt,aut;q)_\infty}$$

$$\times\,{}_3\phi_2\left(\begin{matrix} y/x, v/u, q/abt \\ q/aut, q/xbt \end{matrix};q,q\right), \tag{1.13}$$

where $y/x=q^{-r}$ or $v/u=q^{-r}$ for a non-negative integer r and $\max\{|xtbq^{-r}|,|autq^{-r}|\}<1$.

In 2010, Abdul Hussein [1] obtained the following formulas for $P_n(x,y)$:

$$P_n(x,y)P_m(x,y)=\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k y^k P_{n+m-k}(x,y). \tag{1.14}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}y^k}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}P_{n+m-k}(x,y)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}=\frac{(yt,ys;q)_\infty}{(xt,xs;q)_\infty}. \tag{1.15}$$

In 2013, Saad and Sukhi [11] obtained Mehler's formula for $P_n(x,y)$

$$\sum_{n=0}^{\infty}P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} w/z \\ xwt \end{matrix};q,yzt\right),\quad |xzt|<1. \tag{1.16}$$

The q-differential operator is defined by:

$$D_q\{f(a)\}=\frac{f(a)-f(aq)}{a}. \tag{1.17}$$

The Leibniz rule for D_q is (see [10])

$$D_q^n\{f(a)g(a)\}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}q^{k(k-n)}D_q^k\{f(a)\}D_q^{n-k}\{g(aq^k)\}. \tag{1.18}$$

Let k be a nonnegative integer. Then the following identities are easy to verify [5, 12, 13]:

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$$D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{t^k}{(xt;q)_\infty}. \tag{1.19}$$

$$D_q^k\{P_n(x,y)\}=\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y). \tag{1.20}$$

$$D_q^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}=t^k(v/t;q)_k\frac{(xvq^k;q)_\infty}{(xt;q)_\infty}. \tag{1.21}$$

2. The operator $G(a,b;D_q)$ and the polynomials $W_n(x,y,a,b;q)$

In this section we give an identity which is a basic result for this paper. We give special values to the parameters in this identity to get some wellknown identities. Inspired by this identity we introduce an operator $G(a,b;D_q)$. Also, we introduce a polynomials $W_n(x,y,a,b;q)$.

Theorem 2.1. We have

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}x^k=(x;q)_\infty\sum_{i=0}^{\infty}\frac{q^{i^2-i}(ax)^i}{(q,x;q)_i}. \tag{2.1}$$

Proof. By using (1.3), we get

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}x^k=\sum_{k=0}^{\infty}\frac{\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}(-1)^i q^{\binom{i}{2}}a^i}{(q;q)_k}(-1)^k q^{\binom{k}{2}}x^k$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}a^i}{(q;q)_i(q;q)_k}(-1)^{k+i}q^{\binom{k+i}{2}}x^{k+i}$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}a^i}{(q;q)_i(q;q)_k}(-1)^{k+i}q^{\binom{k}{2}+\binom{i}{2}+ki}x^{k+i} \qquad \text{(by using (1.8))}$$

$$=\sum_{i=0}^{\infty}\frac{q^{i^2-i}(ax)^i}{(q;q)_i}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(q^i x)^k}{(q;q)_k}$$

$$=(x;q)_\infty\sum_{i=0}^{\infty}\frac{q^{i^2-i}(ax)^i}{(q,x;q)_i}. \qquad \text{(by using (1.2) and (1.5))}$$

We give special values to the parameters in (2.1) to get some wellknown identities such as Euler identity and Cauchy identity.

- Setting $a=0$ in equation (2.1), we get Euler's identity (1.2).
- Setting $a=1$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2-n}x^n}{(q,x;q)_n}=\frac{1}{(x;q)_\infty}.$$

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The above equation is due to Cauchy [3].

- Setting $a = q$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} x^n = (x;q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2} x^k}{(q,x;q)_k}.$$

- Setting $x = -q$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(-1)^n q^{\binom{n}{2}}(-q)^n = (-q;q)_\infty \sum_{k=0}^{\infty}\frac{q^{k^2-k}(-aq)^k}{(q,-q;q)_k}$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = (-q;q)_\infty \sum_{k=0}^{\infty}\frac{(-1)^k (q^2)^{\binom{k}{2}}(aq)^k}{(q^2;q^2)_k}$$

$$= (-q;q)_\infty (aq;q^2)_\infty. \quad \text{(by using (1.2))}$$

Now, let D_q be defined as in (1.17). Inspired by the identity (2.1), we introduce the following operator:

$$\mathrm{G}(a,b;D_q) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}(bD_q)^k. \tag{2.2}$$

Setting $a = 0$ in (2.2), we are led to the operator

$$R(bD_q) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(bD_q)^k}{(q;q)_k}$$

defined by Saad and Sukhi [11]. This means that the operator $R(bD_q)$ is a special case of the operator $\mathrm{G}(a,b;D_q)$. We introduce the following polynomials:

$$W_n(x,y,a,b;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} b^k (a,q)_k P_{n-k}(x,y). \tag{2.3}$$

Setting $a = 0$ in $W_n(x,y,a,b;q)$, we get Al-Salam-Carlitz polynomials $U_n(x,y,b;q)$. This means that $U_n(x,y,b;q)$ is a special case of $W_n(x,y,a,b;q)$.

The polynomials $W_n(x,y,a,b;q)$ can be written in terms of a ${}_2\phi_2$ series as:

$$W_n(x,y,a,b;q) = (-1)^n q^{\binom{n}{2}} b^n (a;q)_n \; {}_2\phi_2\left(\begin{matrix} q^{-n}, y/x \\ 0, q^{1-n}/a \end{matrix}; q, q^{2-n}x/ab\right). \tag{2.4}$$

3. The generating function for $W_n(x,y,a,b;q)$

In this section, we use the operator $\mathrm{G}(a,b;D_q)$ to represent the polynomials $W_n(x,y,a,b;q)$. By using this representation we give operator proof for the generating function for the polynomials $W_n(x,y,a,b;q)$. We give a new formula for the polynomials $W_n(x,y,a,b;q)$. Also we give a transformation formula of ${}_2\phi_2$ series.

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From (1.20) and (2.2), we obtain the following representation for the polynomials $W_n(x,y,a,b;q)$:

$$\mathrm{G}(a,b;D_q)\{P_n(x,y)\}=W_n(x,y,a,b;q). \qquad (3.1)$$

Lemma 3.1. *Let* $\mathrm{G}(a,b;D_q)$ *be defined as in* (2.2). *Then*

$$\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{(bt;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(abt)^n}{(q,bt;q)_n}, \quad |xt|<1. \qquad (3.2)$$

Proof.

$$\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xt;q)_\infty}\right\}=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k\frac{t^k}{(xt;q)_\infty} \quad \text{(by using (1.19))}$$

$$=\frac{(bt;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(abt)^n}{(q,bt;q)_n}. \quad \text{(by using (2.1))}$$

Theorem 3.2 (The generating function for $W_n(x,y,a,b;q)$). *We have*

$$\sum_{n=0}^{\infty}W_n(x,y,a,b;q)\frac{t^n}{(q;q)_n}=\frac{(bt,yt;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(abt)^n}{(q,bt;q)_n}, \quad |xt|<1. \qquad (3.3)$$

Proof.

$$\sum_{n=0}^{\infty}W_n(x,y,a,b;q)\frac{t^n}{(q;q)_n}=\sum_{n=0}^{\infty}\mathrm{G}(a,b;D_q)\{P_n(x,y)\}\frac{t^n}{(q;q)_n} \quad \text{(by using (3.1))}$$

$$=\mathrm{G}(a,b;D_q)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\right\}$$

$$=(yt;q)_\infty\,\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xt;q)_\infty}\right\}, \quad |xt|<1 \quad \text{(by using (1.9))}$$

$$=\frac{(bt,yt;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(abt)^n}{(q,bt;q)_n}. \quad \text{(by using (3.2))}$$

Setting $a=0$ in the generating function for the polynomials $W_n(x,y,a,b;q)$ (3.3), we obtain the generating function for Al-Salam-Carlitz polynomials $U_n(x,y,b;q)$ (1.10).

Lemma 3.3. Let $\mathrm{G}(a,b;D_q)$ be defined as in (2.2). Then

$$\mathrm{G}(a,b;D_q)\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}=\frac{(xv;q)_\infty}{(xt;q)_\infty}{}_2\phi_2\left(\begin{matrix}a,v/t\\ xv,0\end{matrix};q,bt\right). \qquad (3.4)$$

Proof. $\mathrm{G}(a,b;D_q)\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k D_q^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k t^k (v/t;q)_k \frac{(xvq^k;q)_\infty}{(xt;q)_\infty} \quad \text{(by using (1.21))}$$

$$=\frac{(xv;q)_\infty}{(xt;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}a,v/t\\ xv,0\end{matrix};q,bt\right).$$

The polynomials $W_n(x,y,a,b;q)$ can be rewritten as:

Corollary 3.3.1. We have

$$W_n(x,y,a,b;q)=(y;q)_n\sum_{k=0}^{n}\frac{(q^{-n},x;q)_k}{(q,y;q)_k}q^k\ {}_2\phi_2\left(\begin{matrix}q^{-k},a\\ 0,x\end{matrix};q,bq^k\right). \tag{3.5}$$

Proof. The q-Vandermonde convolution formula [6]

$${}_2\phi_1\left(\begin{matrix}q^{-k},x\\ y\end{matrix};q,q\right)=\frac{(y/x;q)_n}{(y;q)_n}x^n. \tag{3.6}$$

Rewrite (3.6) as

$$P_n(x,y)=(y;q)_n\sum_{k=0}^{n}\frac{(q^{-n},x;q)_k}{(q,y;q)_k}q^k. \tag{3.7}$$

Applying the operator $\mathrm{G}(a,b;D_q)$ to both sides of (3.7) with respect to x, we get

$$\mathrm{G}(a,b;D_q)\{P_n(x,y)\}=(y;q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q,y;q)_k}q^k\mathrm{G}(a,b;D_q)\left\{\frac{(x;q)_\infty}{(xq^k;q)_\infty}\right\}.$$

By using (3.1) and (3.4), we get

$$W_n(x,y,a,b;q)=(y;q)_n\sum_{k=0}^{n}\frac{(q^{-n},x;q)_k}{(q,y;q)_k}q^k\ {}_2\phi_2\left(\begin{matrix}q^{-k},a\\ 0,x\end{matrix};q,bq^k\right).$$

∎

From equations (2.4) and (3.5), we get the following transformation formula of ${}_2\phi_2$ series:

$${}_2\phi_2\left(\begin{matrix}q^{-n},y/x\\ 0,q^{1-n}/a\end{matrix};q,q^{2-n}x/ab\right)=\frac{(-1)^n q^{-\binom{n}{2}}(y,q)_n}{b^n(a,q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},x;q)_k}{(q,y;q)_k}q^k$$
$$\times\ {}_2\phi_2\left(\begin{matrix}q^{-k},a\\ 0,x\end{matrix};q,bq^k\right).$$

4. The Rogers formula for $W_n(x,y,a,b;q)$

In this section, we give operator proof for the Rogers formula for the polynomials $W_n(x,y,a,b;q)$. From Rogers formula, we obtain the inverse linearization formula of $W_n(x,y,a,b;q)$. We also find another Rogers-type formula for the polynomials $W_n(x,y,a,b;q)$.

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Lemma 4.1. Let $\mathrm{G}(a,b;D_q)$ be defined as in (2.2). Then

$$\mathrm{G}(a,b;D_q)\left\{\frac{P_n(x,y)}{(xs;q)_\infty}\right\}=\frac{(bs;q)_\infty}{(xs;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^m q^{\binom{m}{2}} b^m \frac{(a,xs;q)_m}{(bs;q)_m} P_{n-m}(x,y)$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q;bsq^m;q)_i}. \qquad (4.1)$$

Proof. By using equation (2.2), we have

$$\mathrm{G}(a,b;D_q)\left\{\frac{P_n(x,y)}{(xs;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}} b^k D_q^k\left\{\frac{P_n(x,y)}{(xs;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}} b^k \sum_{m=0}^{k}\begin{bmatrix} k \\ m \end{bmatrix} q^{m(m-k)} D_q^m\{P_n(x,y)\} D_q^{k-m}\left\{\frac{1}{(xsq^m;q)_\infty}\right\} \qquad \text{(by using (1.18))}$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{k+m}}{(q;q)_m(q;q)_k}(-1)^{k+m} q^{\binom{k+m}{2}} b^{k+m} q^{-mk} D_q^m\{P_n(x,y)\} D_q^{k}\left\{\frac{1}{(xsq^m;q)_\infty}\right\}$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_m(aq^m;q)_k}{(q;q)_m(q;q)_k}(-1)^{k+m} q^{\binom{m}{2}+\binom{k}{2}+mk} b^{k+m} q^{-mk}\frac{(q;q)_n}{(q;q)_{n-m}} P_{n-m}(x,y)\frac{(sq^m)^k}{(xsq^m;q)_\infty}$$

(by using (1.6), (1.8), (1.20), (1.19))

$$=\frac{1}{(xs;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^m q^{\binom{m}{2}} b^m (a,xs;q)_m P_{n-m}(x,y)\sum_{k=0}^{\infty}\frac{(aq^m;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}(bsq^m)^k$$

$$=\frac{(bs;q)_\infty}{(xs;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^m q^{\binom{m}{2}} b^m \frac{(a,xs;q)_m}{(bs;q)_m} P_{n-m}(x,y)\sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q;bsq^m;q)_i}. \qquad \text{(by using (2.1))}$$

∎

Theorem 4.2 (The Rogers formula for $W_n(x,y,a,b;q)$). We have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(bs,ys;q)_\infty}{(xs;q)_\infty}\sum_{m=0}^{\infty}\frac{(a,xs;q)_m}{(q,bs,ys;q)_m}(-1)^m q^{\binom{m}{2}}(bt)^m$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q,bsq^m;q)_i}\,{}_2\phi_1\left(\begin{matrix} y/x,0 \\ ysq^m \end{matrix};q,xt\right), \qquad (4.2)$$

provided that $\max\{|xs|,|xt|\}<1$.

Proof.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

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$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{G}(a,b;D_q)\{P_{n+m}(x,y)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \qquad \text{(by using (3.1))}$$

$$= \mathrm{G}(a,b;D_q)\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty} P_m(x,yq^n)\frac{s^m}{(q;q)_m}\right\}$$

$$= (ys;q)_\infty \sum_{n=0}^{\infty}\frac{t^n}{(q,ys;q)_n}\mathrm{G}(a,b;D_q)\left\{\frac{P_n(x,y)}{(xs;q)_\infty}\right\}. \qquad \text{(by using (1.9))}$$

In view of (4.1), the above equation equals

$$\frac{(bs,ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q,ys;q)_n}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^m q^{\binom{m}{2}} b^m \frac{(a,xs;q)_m}{(bs;q)_m} P_{n-m}(x,y)\sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q,bsq^m;q)_i}$$

$$= \frac{(bs,ys;q)_\infty}{(xs;q)_\infty}\sum_{m=0}^{\infty}\frac{(a,xs;q)_m}{(q,bs,ys;q)_m}(-1)^m q^{\binom{m}{2}}(bt)^m \sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q,bsq^m;q)_i}\sum_{n=0}^{\infty}\frac{(y/x;q)_n}{(q,ysq^m;q)_n}(xt)^n$$

$$= \frac{(bs,ys;q)_\infty}{(xs;q)_\infty}\sum_{m=0}^{\infty}\frac{(a,xs;q)_m}{(q,bs,ys;q)_m}(-1)^m q^{\binom{m}{2}}(bt)^m \sum_{i=0}^{\infty}\frac{q^{i^2-i}(absq^{2m})^i}{(q,bsq^m;q)_i}\ {}_2\phi_1\left(\begin{matrix} y/x,0 \\ ysq^m \end{matrix};q,xt\right).$$

■

Setting $a=0$ and then $b=a$ in Rogers formula for $W_n(x,y,a,b;q)$ (4.2), we get Rogers formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ (1.11).

By rewriting Rogers formula (4.2), we obtain the following inverse linearization formula of $W_n(x,y,a,b;q)$:

Lemma 4.3. *For $n,m \geq 0$, we have*

$$W_{n+m}(x,y,a,b;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(a;q)_k (bq^m)^k P_{n-k}(x,y) W_m(x,yq^{n-k},aq^k,b;q). \qquad (4.3)$$

Proof. Rewrite Rogers formula (4.2) as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} W_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$= \frac{(bs,ys;q)_\infty}{(xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,xs;q)_k}{(q,bs,ys;q)_k}(-1)^k q^{\binom{k}{2}}(bt)^k\sum_{m=0}^{\infty}\frac{q^{m^2-m}(absq^{2k})^m}{(q,bsq^k;q)_m}\sum_{n=0}^{\infty}\frac{(y/x;q)_n}{(q,ysq^k;q)_n}(xt)^n$$

$$= \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}(bt)^k\sum_{m=0}^{\infty}\frac{q^{m^2-m}(absq^{2k})^m}{(q,bsq^k;q)_m}\sum_{n=0}^{\infty}\frac{(y/x;q)_n}{(q;q)_n}(xt)^n\frac{(bsq^k,ysq^{n+k};q)_\infty}{(xsq^k;q)_\infty}$$

(by using (1.5), (1.6))

$$= \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}(bt)^k\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\frac{(bsq^k,ysq^{n+k};q)_\infty}{(xsq^k;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{m^2-m}(absq^{2k})^m}{(q,bsq^k;q)_m}$$

$$= \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} b^k \sum_{n=0}^{\infty} P_n(x,y) \frac{t^{n+k}}{(q;q)_n} \sum_{m=0}^{\infty} W_m(x, yq^n, aq^k, b; q) \frac{(sq^k)^m}{(q;q)_m}.$$

(by using (3.3))

Equating the coefficients of $t^n s^m$ on both-hand sides we get

$$W_{n+m}(x,y,a,b;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (a;q)_k (bq^m)^k P_{n-k}(x,y) W_m(x, yq^{n-k}, aq^k, b; q).$$

■

Setting $a=0$ and then $b=a$ in (4.3), we get the inverse linearization formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ (1.12).

Employing the Leibniz formula (1.18), we may derive the following lemma:

Lemma 4.4. *Let* $\mathrm{G}(a,b;D_q)$ *be defined as in* (2.2). *Then*

$$\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xs,xt;q)_\infty}\right\} = \frac{(bt,q)_\infty}{(xs,xt;q)_\infty} \sum_{m=0}^{\infty} \frac{(xt,a;q)_m}{(q,bt;q)_m} (-bs)^m q^{\binom{m}{2}} \sum_{k=0}^{\infty} \frac{q^{k^2-k}(abtq^{2m})^k}{(q,btq^m;q)_k}. \qquad (4.4)$$

Proof.

$$\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xs,xt;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} b^k D_q^k \left\{\frac{1}{(xs,xt;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} b^k \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} q^{m(m-k)} D_q^m \left\{\frac{1}{(xs;q)_\infty}\right\} D_q^{k-m} \left\{\frac{1}{(xtq^m;q)_\infty}\right\}$$

(by using (1.18))

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a;q)_{k+m}}{(q;q)_m (q;q)_k} (-1)^{k+m} q^{\binom{k+m}{2}} b^{k+m} q^{-km} D_q^m \left\{\frac{1}{(xs;q)_\infty}\right\} D_q^k \left\{\frac{1}{(xtq^m;q)_\infty}\right\}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a;q)_m (aq^m;q)_k}{(q;q)_m (q;q)_k} (-1)^{k+m} q^{\binom{k}{2}+\binom{m}{2}+km} b^{k+m} q^{-km} \frac{s^m}{(xs;q)_\infty} \frac{(tq^m)^k}{(xtq^m;q)_\infty}$$

(by using (1.6), (1.8), (1.19)

$$= \frac{1}{(xs,xt;q)_\infty} \sum_{m=0}^{\infty} \frac{(xt,a;q)_m}{(q;q)_m} (-1)^m q^{\binom{m}{2}} (bs)^m (btq^m;q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2-k}(abtq^{2m})^k}{(q,btq^m;q)_k}$$

(by using (2.1))

$$= \frac{(bt,q)_\infty}{(xs,xt;q)_\infty} \sum_{m=0}^{\infty} \frac{(xt,a;q)_m}{(q,bt;q)_m} (-1)^m q^{\binom{m}{2}} (bs)^m \sum_{k=0}^{\infty} \frac{q^{k^2-k}(abtq^{2m})^k}{(q,btq^m;q)_k}. \qquad \text{(by using (1.5))}$$

∎

The following another Rogers-type formula for the polynomials $W_n(x,y,a,b;q)$:

Lemma 4.5. We have

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} y^k}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}W_{n+m-k}(x,y,a,b;q)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}$$
$$=\frac{(yt,ys,bt;q)_\infty}{(xs,xt;q)_\infty}\sum_{m=0}^{\infty}\frac{(xt,a;q)_m}{(q,bt;q)_m}(-1)^m q^{\binom{m}{2}}(bs)^m\sum_{k=0}^{\infty}\frac{q^{k^2-k}(abtq^{2m})^k}{(q,btq^m;q)_k}. \qquad (4.5)$$

Proof.

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} y^k}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}W_{n+m-k}(x,y,a,b;q)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}$$
$$=\mathrm{G}(a,b;D_q)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} y^k}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}P_{n+m-k}(x,y)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}\right\} \quad \text{(by using (3.1))}$$
$$=(yt,ys;q)_\infty\mathrm{G}(a,b;D_q)\left\{\frac{1}{(xs,xt;q)_\infty}\right\} \quad \text{(by using (1.15))}$$
$$=\frac{(yt,ys,bt;q)_\infty}{(xs,xt;q)_\infty}\sum_{m=0}^{\infty}\frac{(xt,a;q)_m}{(q,bt;q)_m}(-1)^m q^{\binom{m}{2}}(bs)^m\sum_{k=0}^{\infty}\frac{q^{k^2-k}(abtq^{2m})^k}{(q,btq^m;q)_k}. \quad \text{(by using (4.4))}$$

∎

Setting $a=0$ and then $b=a$ in (4.5), then the above formula reduced to the following formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$:

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} y^k}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}U_{n+m-k}(x,y,a;q)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}$$
$$=\frac{(yt,ys,at;q)_\infty}{(xs,xt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}xt\\at\end{matrix};q,as\right).$$

By induction on k, we can prove the following proposition:

Proposition 4.6. *Let k be a nonnegative integer, then*

$$D_q^k\{P_n(xq^i,y)\}=q^{ik}\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(xq^i,y). \qquad (4.6)$$

Proposition 4.7. We have

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$$\mathrm{G}(a,b;D_q)\{P_n(x,y)P_m(x,y)\}=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_{n-i}(x,y)W_m(xq^i,y,aq^i,bq^i;q). \quad (4.7)$$

Proof.

$$\mathrm{G}(a,b;D_q)\{P_n(x,y)P_m(x,y)\}$$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k D_q^k\{P_n(x,y)P_m(x,y)\}$$

$$=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}q^{i(i-k)}D_q^i\{P_n(x,y)\}D_q^{k-i}\{P_m(xq^i,y)\} \quad \text{(by using (1.18))}$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{k+i}}{(q;q)_i(q;q)_k}(-1)^{k+i}q^{\binom{k+i}{2}}b^{k+i}q^{-ik}D_q^i\{P_n(x,y)\}D_q^k\{P_m(xq^i,y)\}$$

$$=\sum_{i=0}^{\infty}\frac{(a;q)_i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}b^i\frac{(q;q)_n}{(q;q)_{n-i}}P_{n-i}(x,y)\sum_{k=0}^{\infty}\frac{(aq^i;q)_k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}b^k$$

$$\times q^{ik}\frac{(q;q)_m}{(q;q)_{m-k}}P_{m-k}(xq^i,y) \quad \text{(by using (1.6), (1.8), (1.20), (4.6))}$$

$$=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_{n-i}(x,y)\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(aq^i;q)_k(bq^i)^k P_{m-k}(xq^i,y)$$

$$=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_{n-i}(x,y)W_m(xq^i,y,aq^i,bq^i;q). \quad \text{(by using (2.3))}$$

■

Lemma 4.8. We have

$$\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k y^k W_{n+m-k}(x,y,a,b;q)$$

$$=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_{n-i}(x,y)W_m(xq^i,y,aq^i,bq^i;q). \quad (4.8)$$

Proof.

$$\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k y^k W_{n+m-k}(x,y,a,b;q)$$

$$=\mathrm{G}(a,b;D_q)\left\{\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k y^k P_{n+m-k}(x,y)\right\} \quad \text{(by using (3.1))}$$

$$=\mathrm{G}(a,b;D_q)\{P_n(x,y)P_m(x,y)\} \quad \text{(by using (1.14))}$$

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$$= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} (a;q)_i b^i P_{n-i}(x,y) W_m(xq^i, y, aq^i, bq^i; q). \quad \text{(by using (4.7))}$$

■

Setting $a=0$ and then $b=a$ in (4.8), we are led to the following formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$:

$$\sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (q;q)_k y^k U_{n+m-k}(x,y,a;q)$$

$$= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} a^i P_{n-i}(x,y) U_m(xq^i, y, aq^i; q).$$

5. The Mehler formula for $W_n(x,y,a,b;q)$

In this section, we give an operator proof to the Mehler's formula for the polynomials $W_n(x,y,a,b;q)$.

Theorem 5.1 (The Mehler formula for $W_n(x,y,a,b;q)$). *We have*

$$\sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} W_n(x,y,a,b;q) W_n(u,v,c,d;q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(ydt, bdt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(-but)^n q^{-\binom{n}{2}} (y/x, v/u, q/bdt;q)_n}{(q, q/xdt;q)_n} \sum_{m=0}^{\infty} (a;q)_m P_m(u, vq^n) \frac{(tbq^{-n})^m}{(q;q)_m}$$

$$\times \sum_{k=0}^{\infty} \frac{(bcdtq^{-n})^k q^{k^2-k} (xdtq^{-n}, aq^m;q)_k}{(q, ydt, dbtq^{-n};q)_k} \sum_{j=0}^{\infty} \frac{(-xcdtq^{k-n})^j q^{\binom{j}{2}} (yq^n/x;q)_j}{(q, ydtq^k;q)_j}$$

$$\times \sum_{i=0}^{\infty} \frac{(abdtq^{-n+m+2k})^i q^{i^2-i}}{(q, bdtq^{k-n};q)_i}, \tag{5.1}$$

where $y/x = q^{-\beta}$ or $v/u = q^{-\beta}$ for a non-negative integer β and $\max\{|xdtq^{-\beta}|, |ubtq^{-\beta}|\} < 1$.

Proof. By using (3.1) and (2.3), we get

$$\sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} W_n(x,y,a,b;q) W_n(u,v,c,d;q) \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \mathrm{G}(a,b;D_q)\{P_n(x,y)\} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} d^k (c;q)_k P_{n-k}(u,v) \frac{t^n}{(q;q)_n}$$

$$= \mathrm{G}(a,b;D_q) \left\{ \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (-1)^k q^{\binom{k}{2}} d^k (c;q)_k P_n(x,y) P_{n-k}(u,v) \frac{(-1)^n q^{-\binom{n}{2}} t^n}{(q;q)_k (q;q)_{n-k}} \right\}$$

$$= \mathrm{G}(a,b;D_q)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^k q^{\binom{k}{2}} d^k (c;q)_k P_{n+k}(x,y)P_n(u,v)\frac{(-1)^{n+k} q^{-\binom{n+k}{2}} t^{n+k}}{(q;q)_k (q;q)_n}\right\}$$

$$= \mathrm{G}(a,b;D_q)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}P_n(x,y)P_k(x,yq^n)P_n(u,v)\frac{q^{\binom{k}{2}} d^k (c;q)_k t^{n+k}(-1)^n q^{-\binom{n}{2}-\binom{k}{2}-nk}}{(q;q)_k (q;q)_n}\right\}$$

$$= \mathrm{G}(a,b;D_q)\left\{\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}} P_n(x,y)P_n(u,v)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty}(c;q)_k P_k(x,yq^n)\frac{(dtq^{-n})^k}{(q;q)_k}\right\}$$

$$= \mathrm{G}(a,b;D_q)\left\{\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}} P_n(x,y)P_n(u,v)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(1,c)P_k(x,yq^n)\frac{(dtq^{-n})^k}{(q;q)_k}\right\}.$$

The first sum in the above equation is finite under the terminating condition $y/x = q^{-\beta}$ or $v/u = q^{-\beta}$.

$$\mathrm{G}(a,b;D_q)\left\{\sum_{n=0}^{\infty}\frac{(-t)^n q^{-\binom{n}{2}} P_n(x,y)P_n(u,v)}{(q;q)_n}\frac{(ydt;q)_\infty}{(xdtq^{-n};q)_\infty}\sum_{j=0}^{\infty}\frac{(yq^n/x;q)_j q^{\binom{j}{2}}(-xcdtq^{-n})^j}{(q,ydt;q)_j}\right\}$$

(by using (1.16))

$$= (ydt;q)_\infty \sum_{n=0}^{\infty}\frac{(-t)^n q^{-\binom{n}{2}} P_n(u,v)}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\mathrm{G}(a,b;D_q)\left\{\frac{P_n(x,y)(yq^n/x;q)_j x^j}{(xdtq^{-n};q)_\infty}\right\}$$

$$= (ydt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\mathrm{G}(a,b;D_q)\left\{\frac{P_n(x,y)P_j(x,yq^n)}{(xdtq^{-n};q)_\infty}\right\}$$

$$= (ydt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\mathrm{G}(a,b;D_q)\left\{\frac{P_{n+j}(x,y)}{(xdtq^{-n};q)_\infty}\right\}.$$

By using (4.1), the above equation equals

$$(ydt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\frac{(bdtq^{-n};q)_\infty}{(xdtq^{-n};q)_\infty}$$

$$\times\sum_{m=0}^{n+j}\begin{bmatrix} n+j \\ m \end{bmatrix}(-b)^m q^{\binom{m}{2}}(a;q)_m P_{n+j-m}(x,y)\frac{(xdtq^{-n};q)_m}{(bdtq^{-n};q)_m}\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+2m})^i}{(q,bdtq^{-n+m};q)_i}$$

$$= (ydt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\frac{(bdtq^{-n};q)_\infty}{(xdtq^{-n};q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\begin{bmatrix} j \\ k \end{bmatrix}$$

$$\times\begin{bmatrix} n \\ m-k \end{bmatrix} q^{(j-k)(m-k)}(-b)^m q^{\binom{m}{2}}(a;q)_m P_{n+j-m}(x,y)\frac{(xdtq^{-n};q)_m}{(bdtq^{-n};q)_m}$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+2m})^i}{(q,bdtq^{-n+m};q)_i} \qquad \text{(by using (1.4))}$$

$$=(ydt;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(-cdtq^{-n})^j}{(q,ydt;q)_j}\frac{(bdtq^{-n};q)_\infty}{(xdtq^{-n};q)_\infty}\sum_{k=0}^{j}\sum_{m=0}^{n}\begin{bmatrix} j \\ k \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix}$$

$$\times q^{(j-k)m}(-b)^{m+k} q^{\binom{m}{2}+\binom{k}{2}+mk}\frac{(a;q)_{m+k}(xdtq^{-n};q)_{m+k}P_{n+j-m-k}(x,y)}{(bdtq^{-n};q)_{m+k}}$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+2(m+k)})^i}{(q,bdtq^{-n+m+k};q)_i}$$

$$=(ydt;q)_\infty\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_{n+m}(u,v)\frac{q^{-\binom{n}{2}-\binom{m}{2}-mn}(-t)^{n+m}q^{\binom{j}{2}+\binom{k}{2}+jk}(-cdtq^{-n-m})^{j+k}}{(ydt;q)_{j+k}}$$

$$\times\frac{(bdtq^{-n-m};q)_\infty}{(xdtq^{-n-m};q)_\infty}\frac{q^{jm+mk+\binom{m}{2}+\binom{k}{2}}(-b)^{m+k}(a;q)_{m+k}P_{n+j}(x,y)}{(q;q)_k(q;q)_j(q;q)_m(q;q)_n}\frac{(xdtq^{-n-m};q)_{m+k}}{(bdtq^{-n-m};q)_{m+k}}$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+m+2k})^i}{(q,bdtq^{-n+k};q)_i}$$

$$=(ydt;q)_\infty\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}P_n(x,y)P_n(u,v)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty}P_m(u,vq^n)\frac{(a;q)_m(tbq^{-n})^m}{(q;q)_m}$$

$$\times\sum_{k=0}^{\infty}\frac{(aq^m;q)_k(bcdtq^{-n})^k q^{k^2-k}(bdtq^{-n+k};q)_\infty}{(q,ydt;q)_k(xdtq^{-n+k};q)_\infty}\sum_{j=0}^{\infty}\frac{P_j(x,yq^n)q^{\binom{j}{2}}(-cdtq^{-n+k})^j}{(q,ydtq^k;q)_j}$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+m+2k})^i}{(q,bdtq^{-n+k};q)_i} \qquad \text{(by using (1.6))}$$

$$=\frac{(ydt,bdt;q)_\infty}{(xdt;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}\frac{(y/x,v/u,q/bdt;q)_n}{(q,q/xdt;q)_n}(but)^n\sum_{m=0}^{\infty}P_m(u,vq^n)\frac{(a;q)_m(tbq^{-n})^m}{(q;q)_m}$$

$$\times\sum_{k=0}^{\infty}\frac{(xdtq^{-n},aq^m;q)_k(bcdtq^{-n})^k q^{k^2-k}}{(q,ydt,bdtq^{-n};q)_k}\sum_{j=0}^{\infty}\frac{(yq^n/x;q)_j(-cxdtq^{-n+k})^j q^{\binom{j}{2}}}{(q,ydtq^k;q)_j}$$

$$\times\sum_{i=0}^{\infty}\frac{q^{i^2-i}(abdtq^{-n+m+2k})^i}{(q,bdtq^{-n+k};q)_i}. \qquad \text{(by using (1.5) and (1.7))}$$

■

Setting $a = 0$, $c = 0$, then $b = a$ and then $d = b$ in Mehler's formula for the polynomials $W_n(x, y, a, b; q)$ (5.1), we get Mehler's formula for Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ (1.13).

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