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MEAN CURVATURE FLOW OF SUBMANIFOLDS WITH SMALL TRACELESS SECOND FUNDAMENTAL FORM

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ABSTRACT

Consider a family of smooth immersions $F(\cdot,t): M^n \to \mathbb{R}^{n+k}$ of submanifolds in $\mathbb{R}^{n+k}$ moving by mean curvature flow $\frac{\partial F}{\partial t} = \vec{H}$, where $\vec{H}$ is the mean curvature vector for the evolving submanifold. We prove that for any $n \geq 2$ and $k \geq 1$, the flow starting from a closed submanifold with small L^2-norm of the traceless second fundamental form contracts to a round point in finite time, and the corresponding normalized flow converges exponentially in the C^∞-topology, to an n-sphere in some subspace $\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+k}$.

Keywords

Mean curvature vector; traceless second fundamental form; normalized flow, blow up.

SUBJECT CLASSIFICATION

2010 Mathernatics Subject Classification. 53C44, 53C40.

1. INTRODUCTION

Let M^n be an n-dimensional compact manifold without boundary, and let $F_0 : M^n \to \mathbb{R}^{n+k}$ be a smooth immersion of M^n into $\mathbb{R}^{n+k}$. Consider a smooth one-parameter family of immersions $F(\cdot,t) : M^n \to \mathbb{R}^{n+k}$ satisfying

(1.1) $$\frac{\partial F}{\partial t} = \vec{H}; F(\cdot,0) = F_0(\cdot),$$

where $\vec{H}$ is the mean curvature vector of the submanifold $M_t = F(\cdot,t)$.

Denote by $g = (g_{ij})$ the induced metric, and $A = (h_{ij}^{\alpha})$ the second fundamental form of M_t, where we make use of the convention on indices that $1 \le i, j, \cdots, \le n$ and $n+1 \le \alpha, \beta, \cdots, \le n+k$. Sometimes, the second fundamental form and the mean curvature vector are also written as $A = (h^{\alpha})$ and $\vec{H} = (H^{\alpha})$ respectively.

It is known that, without any special assumptions on M_0, the mean curvature flow (1.1) will in general develop singularities in finite time, characterized by a blow up of the second fundamental form $A(\cdot,t)$. For example, Huisken [5] and Andrews-Baker [1] proved that $\sup_{M_t} |A|(\cdot,t) \to \infty$ as $t \to T$ if $T < \infty$ is the first singular time for hypersurfaces and submanifolds, respectively. Moreover the mean curvature must blow up near a singularity for mean convex hypersurfaces [6] or star shaped hypersurfaces [14]. The mean curvature also needs to blow up for type I singularities [8]. It is still an open question whether the mean curvature needs to blow up at the first singular time for general compact hypersurfaces.

While for mean curvature flow of higher codimension, Liu-xu-ye-zhao [12] proved that if the L^p-norm of the traceless second fundamental form of the initial submanifold is small enough, the mean curvature flow of smooth closed sumbanifolds of dimension $n > 2$ has a maximal solution in finite time, and the corresponding rescaled solution converges to a round sphere. This in fact answers the above question under the assumption that the initial submanifold is close enough to a sphere for mean curvature flow with higher codimensional case. But it is unknown for submanifols with dimension 2. Recently, Lin-Sesum [11] also give a partial answer to the above question only for hypersurfaces but including the dimension $n = 2$ case. In this paper we shall extend results of Lin-Sesum [11] to mean curvature flow of closed submanifolds with dimension greater than or equal to 2 and any codimension. As in the hypersurface case, the traceless second fundamental form is defined by $\mathring{A} = A - \frac{\vec{H}}{n} g$, whose squared norm is given by

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{n}|\vec{H}|^2 .$$

The traceless second fundamental form measures the roundness of a submanifold, the smaller it is in a considered norm the closer we are to a sphere in that norm.More precisely, if a submanifold M satisfies $\int_M |\mathring{A}|^2 d\mu < \varepsilon$, we say that M is ε-close to a sphere in the L^2 sense. Our main result proves that the mean curvature flow contracts to a round point in finite time, and the corresponding normalized flow converges exponentially in the C^∞-topology, to a round n-sphere under the assumption that the initial submanifold is ε-close to a sphere in the L^2 sense.

Theorem 1.1. Let $M_t^n \subset \mathrm{R}^{n+k} (n \ge 2, k \ge 1)$ be a smooth compact solution to the mean curvature flow (1.1) for $t \ge 0$. There exists an $\varepsilon > 0$ depending only on n, k, the area of M_0, $\max_{M_0} |A|$ and the bound on $\int_{M_0} |\nabla^m A|^2 d\mu$ (for all $m \in [1, \hat{m}]$ for some fixed $\hat{m} \gg 1$) such that if

$$\int_{M_0} |\mathring{A}|^2 d\mu < \varepsilon,$$

then the flow (1.1) contracts uniformly to a round point in finite time. Moreover the normalized mean curvature flow (3.1) exists for all time and converges exponentially to a round n-sphere in some subspace R^{n+1} of R^{n+k}.

As a consequence, this gives a partial answer to the aforementioned question for mean curvature flow of submanifolds with any dimension and codimension. More precisely we have the following

Corollary 1.2. Let $M_t^n \subset \mathrm{R}^{n+k} (n \ge 2, k \ge 1)$ be a smooth compact solution to the mean curvature flow (1.1) for $t \in [0,T)$ with $T < \infty$. Assume there exists a constant c_0 so that $\sup_{M_t} |\vec{H}|(\cdot,t) \le c_0$ for all $t \in [0,T)$. There exists an $\varepsilon > 0$ depending only on n, k, c_0, the area of M_0, $\max_{M_0} |A|$ and the bound on $\int_{M_0} |\nabla^m A|^2 d\mu$ (for all $m \in [1, \hat{m}]$ for some fixed $\hat{m} \gg 1$) such that if

$$\int_{M_0} |\mathring{A}|^2 \, d\mu < \varepsilon,$$

then the flow (1.1) can be smoothly extended past time T .

We remark that the quantity $|A||\vec{H}|$ needs to blow up at the singularity for mean curvature flow of compact submanifolds [4]. An integral bound of the second fundamental form or the mean curvature on space and time is also enough to extend the mean curvature flow past some finite time (see [9, 16]).

The organization of the paper is as follows. In Section 2 we collect some known facts for later use. In Section 3 we prove the main results.

2. PRELIMINARIES

In this section, we collect some necessary preliminary results for latter use. We begin with the following evolution equations of several geometric quantities from Andrews-Baker in [1].

Lemma 2.1. We have the evolution equations for g , $|\vec{H}|^2$ and $|A|^2$

$$(i)\frac{\partial}{\partial t} g_{ij} = -2H^{\alpha} h_{ij}^{\alpha},$$

$$(ii)\frac{\partial}{\partial t}|\vec{H}|^2 = \Delta |\vec{H}|^2 - 2|\nabla \vec{H}|^2 + 2\sum_{i,j}(\sum_{\alpha} H^{\alpha} h_{ij}^{\alpha})^2,$$

$$(iii)\frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta}(\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta})^2 + 2\sum_{i,j,\alpha,\beta}(\sum_{p}(h_{ip}^{\alpha} h_{jp}^{\beta} - h_{ip}^{\beta} h_{jp}^{\alpha}))^2.$$

By Lemma 2.1, an easy calculation shows that the traceless second fundamental form satisfies the following

$$(2.1) \qquad \frac{\partial}{\partial t}|\mathring{A}|^2 = \Delta |\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + 2\sum_{\alpha,\beta}(\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta})^2 + 2\sum_{i,j,\alpha,\beta}(\sum_{p}(h_{ip}^{\alpha} h_{jp}^{\beta} - h_{ip}^{\beta} h_{jp}^{\alpha}))^2 - \frac{2}{n}\sum_{i,j}(\sum_{\alpha} H^{\alpha} h_{ij}^{\alpha})^2,$$

where $|\nabla \mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla \vec{H}|^2$.

We can choose a local orthonormal frame field $\{\upsilon_{\alpha} : n+1 \le \alpha \le n+k\}$ such that υ_{n+1} parallels $\vec{H}$. With this choice of frame the second fundamental form takes the form

$$\mathring{h}^{n+1} = h^{n+1} - \frac{|\vec{H}|}{n} g; \mathring{h}^{\alpha} = h^{\alpha}, \alpha > n+1,$$

and

$$tr h^{n+1} = |\vec{H}|; tr h^{\alpha} = 0, \alpha > n+1.$$

At a point we may choose a basis for the tangent space such that h^{n+1} is diagonal. We denote the diagonal entries of h^{n+1} and $\mathring{h}^{n+1}$ by λ_i and $\mathring{\lambda}_i$ respectively. Additionally, we denote the squared norm of the $\alpha(\neq n+1)$ -directions of the second fundamental form by $|\mathring{h}_-|^2$, that is, $|\mathring{h}|^2 = |\mathring{h}^{n+1}|^2 + |\mathring{h}_-|^2$. By direct calculations we then have the following (see [1] for details)

$$\sum_{\alpha,\beta}(\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta})^2 = |\mathring{h}^{n+1}|^4 + \frac{2}{n}|\mathring{h}^{n+1}|^2 |\vec{H}|^2 + \frac{1}{n^2}|\vec{H}|^4 + 2\sum_{\alpha>n+1}(\sum_{i,j} \mathring{h}_{ij}^{n+1} \mathring{h}_{ij}^{\alpha})^2 + \sum_{\alpha,\beta>n+1}(\sum_{i,j} \mathring{h}_{ij}^{\alpha} \mathring{h}_{ij}^{\beta})^2,$$

$$\sum_{i,j,\alpha,\beta}(\sum_{p}(h_{ip}^{\alpha} h_{jp}^{\beta} - h_{ip}^{\beta} h_{jp}^{\alpha}))^2 = 2\sum_{\alpha>n+1,i,j}(\sum_{p}(h_{ip}^{n+1} \mathring{h}_{jp}^{\alpha} - \mathring{h}_{ip}^{\alpha} h_{jp}^{n+1}))^2 + \sum_{\alpha,\beta>n+1,i,j}(\sum_{p}(\mathring{h}_{ip}^{\alpha} \mathring{h}_{jp}^{\beta} - \mathring{h}_{ip}^{\beta} \mathring{h}_{jp}^{\alpha}))^2,$$

$$\sum_{i,j}(\sum_{\alpha} H^{\alpha} h_{ij}^{\alpha})^2 = |\mathring{h}^{n+1}|^2 |\vec{H}|^2 + \frac{1}{n}|\vec{H}|^4,$$

and

$$\sum_{\alpha>n+1}(\sum_{i,j} \mathring{h}_{ij}^{n+1} \mathring{h}_{ij}^{\alpha})^2 + \sum_{\alpha>n+1,i,j}(\sum_{p}(h_{ip}^{n+1} \mathring{h}_{jp}^{\alpha} - \mathring{h}_{ip}^{\alpha} h_{jp}^{n+1}))^2 \le 2|\mathring{h}^{n+1}|^2 |\mathring{h}_-|^2 .$$

On the other hand, the following inequality from [10] is well-known

$$\sum_{\alpha,\beta>n+1}(\sum_{i,j}\overset{0}{h}{}^{\alpha}_{ij}\overset{0}{h}{}^{\beta}_{ij})^2+\sum_{\alpha,\beta>n+1,i,j}(\sum_{p}(\overset{0}{h}{}^{\alpha}_{ip}\overset{0}{h}{}^{\beta}_{jp}-\overset{0}{h}{}^{\beta}_{ip}\overset{0}{h}{}^{\alpha}_{jp}))^2\le\frac{3}{2}|\overset{0}{h}_{-}|^4.$$

Inserting the first three formulae into Lemma 2.1 (iii) and (2.1) respectively, and then using the last two estimates and $|\vec{H}|^2\le n|A|^2$, we obtain the following evolutions

Corollary 2.2. The second fundamental form and traceless second fundamental form satisfy the following inequalities

(i) $\frac{\partial}{\partial t}|A|^2\le\Delta|A|^2-2|\nabla A|^2+19|A|^4$,

(ii) $\frac{\partial}{\partial t}|\overset{0}{A}|^2\le\Delta|\overset{0}{A}|^2-2|\nabla\overset{0}{A}|^2+15|A|^2|\overset{0}{A}|^2$.

For any $m\ge1$, the following evolution equation of higher order derivatives of the second fundamental form is standard ([5, 1])

(2.2) $$\frac{\partial}{\partial t}|\nabla^m A|^2=\Delta|\nabla^m A|^2-2|\nabla^{m+1}A|^2+\sum_{i+j+k=m}\nabla^i A*\nabla^j A*\nabla^k A*\nabla^m A,$$

Where $S*T$ denotes any linear combination of tensors formed by contraction on S and T by the metric g.

Next, we recall the following interpolation inequalities for tensors proved by Hamilton in [7].

Lemma 2.3. Let M be an n-dimensional compact Riemannian manifold and Ω be any tensor on M.

(i)Suppose $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with $r\ge1$, then

$$(\int_M|\nabla\Omega|^{2r}d\mu)^{\frac{1}{r}}\le(2r-2+n)(\int_M|\nabla^2\Omega|^p d\mu)^{\frac{1}{p}}(\int_M|\Omega|^q d\mu)^{\frac{1}{q}}.$$

(ii) If $1\le i\le n-1$ and $j\ge0$ there exists a constant $C=C(n,j)$ which is independent of the metric and connection on M such that

$$\int_M|\nabla^i\Omega|^{\frac{2j}{i}}d\mu\le C\max_M|\Omega|^{2(\frac{j}{i}-1)}\int_M|\nabla^j\Omega|^2 d\mu.$$

Applying a kind of Sobolev inequality in [13], we have the following version of Michael-Simon's inequality (which can be similarly proved as that in [11]).

Lemma 2.4. Let M be a closed n-dimensional submanifold, smoothly immersed in $\mathbb{R}^{n+k}$. Let $\upsilon>0$ be any Lipschitz function on M. Then we have

(i)For any $n>2$,

$$(\int_M\upsilon^{\frac{2n}{n-2}}d\mu)^{\frac{n-2}{n}}\le C(n)(\int_M|\nabla\upsilon|^2d\mu+\int_M|\vec{H}|^2\upsilon^2d\mu).$$

(ii)For $n=2$,

$$\int_M\upsilon^2d\mu\le C(n)(\int_M|\nabla\upsilon|^2d\mu+\int_M|\vec{H}|^2\upsilon^2d\mu).$$

The following diameter estimate is from [15].

Lemma 2.5. Let M be a compact n-dimensional submanifold without boundary, which is smoothly immersed in $\mathbb{R}^{n+k}$. Then the intrinsic diameter and the mean curvature vector $\vec{H}$ of M are related by

$$diam(M)\le C(n)\int_M|\vec{H}|^{n-1}d\mu.$$

Finally we state the following version of a maximum principle in [3].

Theorem 2.6. Suppose $\mu:M\times[0,T]\to\mathbb{R}$ satisfies

$$\frac{\partial}{\partial t}\mu\le a^{ij}(t)\nabla_i\nabla_j\mu+\langle B(t),\nabla\mu\rangle+F(\mu),$$

where the coefficient matrix $a^{ij}(t)>0$ for all $t\in[0,T]$, $B(t)$ is a time-dependent vector field and F is a Lipschitz function. If $\mu\le c$ at $t=0$ for some $c>0$, then $\mu(x,t)\le U(t)$ for all $(x,t)\in M_t$, $t\ge0$, where $U(t)$ is the solution to the following initial value problem:

$$\frac{\partial}{\partial t}U(t)=F(\mu) \text{ with } U(0)=c.$$

3.PROOF OF MAIN RESULTS

It is well-known that the mean curvature flow (1.1) only exists for a finite time interval $[0,T)$, where T is the maximal existing time. Without loss of generality we assume that the origin is always in the region enclosed by the evolving submanifolds for all times $0\le t<T$.

In fact, in this section we show that the flow (1.1) indeed becomes strictly convex at some time in the sense of Andrews-Baker [1]. For this purpose, we shall start with the initial submanifold with small L^2-norm of the traceless second fundamental form, to show that $|\overset{0}{A}(\cdot,t)|$ stays uniformly small for some short time interval $[0,T_1]$. Then we can iterate this step starting at time T_1. After finitely many iterations, we see that the flow becomes strictly convex immediately, otherwise we reach time T showing that $|A|$ can not blow up at time T, which is a contradiction.

Once the flow (1.1) becomes strictly convex at some time, the following normalized flow

(3.1) $$\frac{\partial\tilde{F}}{\partial\tilde{t}}=|\tilde{\vec{H}}|+\frac{1}{n}\tilde{h}\tilde{F}$$

will exist for all time and converge uniformly to a sphere exponentially (cf. [1]). Where $\tilde{F}(\cdot,t)=\varphi(t)F(\cdot,t)$, $\varphi(t)$ is chosen a positive constant such that the total area of the rescaled submanifold $\tilde{M}_t$ is equal to the total area of M_0, $\tilde{t}(t)=\int_0^t\varphi^2(t)d\tau$, and $h=\frac{\int_{M_t}|\vec{H}|^2\,d\mu}{\int_{M_t}d\mu}$.

Lemma 3.1. Let $M_t^n\subset\mathbb{R}^{n+k}$, $n\ge 2$ be a smooth compact solution to the mean curvature flow (1.1) for $t\in[0,T)$ with $T<\infty$. Assume that

$$\max\{\max_{M_0}|A|,\int_{M_0}|\nabla^m A|^2\,d\mu\}\le\Lambda_0$$

for some $\Lambda_0\gg 1$ and all $m\in[1,\hat{m}]$ for some fixed $\hat{m}\gg 1$. Then there exist $\varepsilon=\varepsilon(n,|M_0|,\Lambda_0)>0$, $T_1=T_1(\Lambda_0)\in(0,1)$, $c_1=c_1(n,|M_0|,\Lambda_0)$ and some universal constant $\kappa\in(0,1)$ such that if

$$\int_{M_0}|\overset{0}{A}|^2\,d\mu<\varepsilon,$$

then for all $t\in[0,T_1]$ we have

$$\max_{M_t}|A|\le 2\Lambda_0$$

and

$$\max_{M_t}|\overset{0}{A}|\le c_1\varepsilon^{\kappa}.$$

Proof. By Corollary 2.2 we have

$$\frac{\partial}{\partial t}|A|^2\le\Delta|A|^2+19|A|^4,$$

On M_t for all $t\in[0,T)$. Then the maximum principle Theorem 2.6 implies

$$\max_{M_t}|A|\le\frac{1}{\sqrt{-19t+\Lambda_0^{-2}}},t\in[0,T).$$

Choose $T_1\le\frac{1}{19}\cdot\frac{3}{4}\Lambda_0^{-2}\ll 1$ so that $\max_{M_t}|A|\le 2\Lambda_0$ for all $t\in[0,T_1]$. Then standard argument yields (cf. [5])

(3.2) $$\max_{M_t}|\nabla^m A|^2\le c_2(n,\Lambda_0),\forall t\in[0,T_1].$$

Now we integrate the evolution inequality for $|\overset{0}{A}|^2$ in Corollary 2.2 (ii) from 0 to T_1 and have into account Lemma 2.1 (i) to get,

$$\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2\,d\mu+\int_{M_t}|\overset{0}{A}|^2|\vec{H}|^2\,d\mu\le -2\int_{M_t}|\nabla\overset{0}{A}|^2\,d\mu+15\int_{M_t}|A|^2|\overset{0}{A}|^2\,d\mu,$$

which yields

$$\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2\,d\mu\le 60\Lambda_0{}^2\int_{M_t}|\overset{0}{A}|^2\,d\mu.$$

Using the assumption that $\int_{M_0}|\overset{0}{A}|^2\,d\mu<\varepsilon$ we obtain for all $t\in[0,T_1]$

(3.3) $$\int_{M_t}|\overset{0}{A}|^2\,d\mu\le\varepsilon e^{60\Lambda_0{}^2 t}\le 30\varepsilon.$$

Applying Hamilton's interpolation inequality in Lemma 2.3 (i) for $r=1$, $p=q=2$ and $\Omega=\overset{0}{A}$ we have

$$\int_{M_t}|\nabla\overset{0}{A}|^2\,d\mu\le n(\int_{M_t}|\nabla^2\overset{0}{A}|^2\,d\mu)^{\frac12}(\int_{M_t}|\overset{0}{A}|^2\,d\mu)^{\frac12}\le c_3(n,\Lambda_0)\varepsilon^{\frac12},$$

where we have used (3.2), (3.3), and the fact $|\nabla^2\overset{0}{A}|\le c_4(n)|\nabla^2 A|$.

Having this estimate, we can apply Lemma 2.3 (i) inductively, and then use Lemma 2.3 (ii) to obtain the L^p-estimate of $|\nabla^m\overset{0}{A}|$ exactly same as that in [11]

$$\int_{M_t}|\nabla^m\overset{0}{A}|^p\,d\mu\le c_4(n,m,p,\Lambda_0)\varepsilon^{\frac{1}{2^{m+p/2-1}}},$$

for any $t\in[0,T_1]$, $m\in[1,\hat{m}]$ and $p<\infty$.

By the standard Sobolev embedding theorem [2] we have that for some universal constant $\kappa\in(0,1)$, all $m\in[1,\hat{m}-1]$ and $t\in[0,T_1]$

(3.4) $$\max_{M_t}|\nabla^m\overset{0}{A}|\le c_5(n,m,\Lambda_0)\varepsilon^{\kappa}.$$

It specially implies

$$\max_{M_t}|\overset{0}{A}|\le c_1\varepsilon^{\kappa},t\in[0,T_1],$$

for a constant c_1 depends on n, Λ_0, $|M_0|$ and ε. This finishes the proof of the lemma.

In the following we want to show that $\int_{M_t}|\overset{0}{A}|^2\,d\mu$ stays small along the flow so that we can use iterative type of arguments.

Proposition 3.2. Let $M_t^n\subset\mathbb{R}^{n+k}$, $n\ge 2$ be a smooth compact solution to the mean curvature flow (1.1) for $t\in[0,T)$ with $T<\infty$, where T_1 is as in Lemma 3.1. Assume there exists a c_0 so that $\sup_{M_t}|\vec{H}|(\cdot,t)\le c_0$ for all $t\in[0,T)$. Then there exists a ε such that if

$$\int_{M_0}|\overset{0}{A}|^2\,d\mu<\varepsilon,$$

then for all $t\in[0,T_1]$

(3.5) $$\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2\,d\mu\le 0,$$

and for all $m\in[1,\hat{m}-1]$

(3.6) $$\frac{\partial}{\partial t}\int_{M_t}|\nabla^m A|^2\,d\mu\le 0.$$

Proof. Using the inequality that is also true for submanifolds (cf. [1])

$$|\nabla\vec{H}|^2\le\frac{n(n+2)}{2(n-1)}|\nabla\overset{0}{A}|^2,$$

and (3.4) for $m=1$ we obtain for any $t\in[0,T_1]$

$$\max_{M_t}|\nabla\vec{H}|\le\frac{n(n+2)}{2(n-1)}c_5\varepsilon^{\kappa}=c_6\varepsilon^{\kappa}.$$

Assume there exists $\eta=c_6\varepsilon^{\frac{\kappa}{2}}>0$ such that $\max_{M_{t_0}}|\vec{H}|\ge\eta$ for some $t_0\in[0,T_1]$. By Lemma 3.1, Lemma 2.5 implies that the diameter of M_{t_0} has a upper bound. Then integrating $|\nabla|\vec{H}||$ along a geodesic in M_{t_0}, and using the inequality $|\nabla|\vec{H}||\le|\nabla\vec{H}|$ (a direct result of Cauchy-Schwarz inequality), we have by the above estimate on $|\nabla\vec{H}|$ that

$$\min_{M_{t_0}}|\vec{H}|\ge\frac{\eta}{2}>0.$$

By Lemma 3.1 again, it follows that

$$|A|^2=|\overset{0}{A}|^2+\frac{1}{n}|\vec{H}|^2\le c_1{}^2\varepsilon^{2\kappa}+\frac{1}{n}|\vec{H}|^2\le(4c_1{}^2c_6{}^{-2}\varepsilon^{\kappa}+\frac{1}{n})|\vec{H}|^2,$$

which implies that there exists c such that $|A|^2\le c|\vec{H}|^2$ (one can choose a smaller ε if necessary), where

(3.7) $$c\le\frac{4}{3n},2\le n\le4;c\le\frac{1}{n-1},n\ge4.$$

Therefore the submanifold M_{t_0} of the mean curvature flow (1.1) satisfies the pinching assumption of Andrews-Baker ([1]). In this case, we know by Andrews-Baker's theorem ([1]) the flow contracts to a round point as $t\to T$ and $|\vec{H}|$ must blow up, which contradicts with the assumption $\sup_{M_t}|\vec{H}|(\cdot,t)\le c_0$ for all $t\in[0,T)$.

Thus we have $\max_{M_t}|\vec{H}|<\eta=c_6\varepsilon^{\frac{\kappa}{2}}$ for all $t\in[0,T_1]$. It follows by Lemma 3.1 and $|A|^2=|\overset{0}{A}|^2+\frac{1}{n}|\vec{H}|^2$ that for all $t\in[0,T_1]$, there is a constant $c_7>0$ such that

$$\max_{M_t}|A|\le c_7\varepsilon^{\frac{\kappa}{2}}.$$

Using $\frac{\partial}{\partial t}d\mu=-|\vec{H}|^2d\mu$, and integrating the evolution of $|\overset{0}{A}|^2$ in Corollary 2.2 over M_t yield

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2d\mu+\int_{M_t}|\overset{0}{A}|^2|\vec{H}|^2d\mu&\le-2\int_{M_t}|\nabla\overset{0}{A}|^2d\mu+15\int_{M_t}|A|^2|\overset{0}{A}|^2d\mu\\&\le-2\int_{M_t}|\nabla\overset{0}{A}|^2d\mu+c_8\varepsilon^{\kappa}\int_{M_t}|\overset{0}{A}|^2d\mu.\end{aligned}$$

For the case $n=2$, by Lemma 2.4 (ii), we have

$$\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2d\mu\le-\int_{M_t}|\overset{0}{A}|^2|\vec{H}|^2d\mu-2\int_{M_t}|\nabla\overset{0}{A}|^2d\mu+c_9\varepsilon^{\kappa}(\int_{M_t}|\overset{0}{A}|^2|\vec{H}|^2d\mu+\int_{M_t}|\nabla|\overset{0}{A}||^2d\mu).$$

This proves (3.5) by Kato's inequality $|\nabla|\overset{0}{A}||^2\le|\nabla\overset{0}{A}|^2$ and by choosing ε small.

For the case $n>2$, we obtain by Holder's inequality for $t\in[0,T_1]$

$$\frac{\partial}{\partial t}\int_{M_t}|\overset{0}{A}|^2d\mu+\int_{M_t}|\overset{0}{A}|^2|\vec{H}|^2d\mu\le-2\int_{M_t}|\nabla\overset{0}{A}|^2d\mu+c_{10}\varepsilon^{\kappa}(\int_{M_t}|\overset{0}{A}|^{\frac{2n}{n-2}}d\mu)^{\frac{n-2}{n}},$$

this together with Lemma 2.4 (i) yields (3.5) by choosing ε sufficiently small.

For the second inequality, integrating (2.2) by parts we obtain the following estimate

$$\frac{\partial}{\partial t}\int_{M_t}|\nabla^mA|^2d\mu+\int_{M_t}|\nabla^mA|^2|\vec{H}|^2d\mu\le-2\int_{M_t}|\nabla^{m+1}A|^2d\mu+C(n,m)\max_{M_t}|A|^2\int_{M_t}|\nabla^mA|^2d\mu.$$

Using Kato's type inequality $|\nabla|\nabla^mA||^2\le|\nabla^{m+1}A|^2$, by similarly discussion as above we can get (3.6). This finishes the proof of Proposition 3.2.

Proof of Theorem 1.1.

Assume that there exists $\Lambda_0 > 0$ sufficiently large so that

$$\max\{\max_{M_0} |A|, \int_{M_0} |\nabla^m A|^2 \, d\mu\} \le \Lambda_0$$

Then there exists $T_1 = T_1(\Lambda_0) \in (0,1)$ and $\varepsilon > 0$ sufficiently small such that if

$$\int_{M_0} |\overset{0}{A}|^2 \, d\mu < \varepsilon,$$

then only two things can happen:

(i) There exists a time $t_0 \in [0,T_1]$ such that $\max_{M_{t_0}} |\vec{H}| \ge c_6 \varepsilon^{\frac{\kappa}{2}}$ (see the proof of Proposition 3.2). Then the flow will stay strictly convex for all $t \in [t_0, T)$ in the sense of Andrews-Baker [1], i.e. $|A|^2 \le c |\vec{H}|^2$, where c is the constant in (3.7). We have done in this case.

(ii) For all $t \in [0,T_1]$, $\max_{M_{t_0}} |\vec{H}| < c_6 \varepsilon^{\frac{\kappa}{2}}$. Then by discussion as in Proposition 3.3 $\max_{M_t} |A| \le c_7 \varepsilon^{\frac{\kappa}{2}} \le \Lambda_0$. By Proposition 3.2 again

$$\frac{\partial}{\partial t} \int_{M_t} |\overset{0}{A}|^2 \, d\mu \le 0 \quad \text{and} \quad \frac{\partial}{\partial t} \int_{M_t} |\nabla^m A|^2 \, d\mu \le 0.$$

These mean that at time $t = T_1$ we still have

$$\max\{\max_{M_{T_1}} |A|, \int_{M_{T_1}} |\nabla^m A|^2 \, d\mu\} \le \Lambda_0 \quad \text{and} \quad \int_{M_{T_1}} |\overset{0}{A}|^2 \, d\mu < \varepsilon.$$

Then we can iterate the arguments by using Proposition 3.2 and Lemma 3.1 for the time interval of size $T_1 = T_1(\Lambda_0)$. We see that after finitely many iterations, there must have a time t_0 such that M_{t_0} is strictly convex in the sense of Andrews-Baker([1]), otherwise we reach time T since T_1 is a uniform constant. This contradicts to the fact that T is the maximal existing time. This completes the proof of Theorem 1.1.

The proof of Corollary 1.2 follows directly as in above by using Proposition 3.2.

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