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Norms Of Hankel-Hessenberg and Toeplitz-Hessenberg Matrices Involving Pell and Pell-Lucas Numbers

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ABSTRACT

We derive some sum formulas for the squares of Pell and Pell-Lucas numbers. We construct Hankel-Hessenberg and Toeplitz-Hessenberg matrices whose entries in the first column are $HH_P = (a_{ij})$, $a_{ij} = P_{i-j}$; $HH_Q = (a_{ij})$, $a_{ij} = Q_{i-j}$ and $TH_P = (a_{ij})$, $a_{ij} = P_{i-j+1}$; $TH_Q = (a_{ij})$, $a_{ij} = Q_{i-j+1}$, respectively where P_n and Q_n denote the usual Pell and Pell-Lucas numbers. Then, we found upper and lower bounds for spectral norm of these matrices.

Keywords: Euclidean norm, Spectral norm, Toeplitz matrix, Hankel matrix, Hessenberg matrix, Pell numbers, Pell-Lucas numbers.

Mathematics Subject Classification: 15A60, 15A36, 11B99.

1. Introduction

Special matrices is a widely studied subject in matrix analysis. Especially special matrices whose entries are well known number sequences have become a very interesting research subject in recent years and many authors have obtained some good results in this area. For example, the norms of Toeplitz, Hankel and Circulant matrices involving Fibonacci, Lucas, Pell and Pell-Lucas numbers were investigated in [1, 2, 5, 6, 7]. In this study, We derive some sum formulas for the squares of Pell and Pell-Lucas numbers. We construct Hankel-Hessenberg and Toeplitz-Hessenberg matrices involving Pell and Pell-Lucas numbers.

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2} \quad for \quad n \geq 2$$

and

$$Q_0 = 2, Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2} \quad for \quad n \geq 2.$$

If start from n =0, then the Pell and Pell-Lucas sequence are given by

n	0	1	2	3	4	5	6
P_n	0	1	2	5	12	29	70
Q_n	2	2	6	14	34	11	82

The following sum formulas the Pell and Pell-Lucas numbers are well known [8, 9]:

$$\sum_{k=1}^{n-1} P_k^2 = \frac{P_n P_{n-1}}{2}$$

$$\sum_{k=1}^{n-1} Q_k^2 = \frac{Q_{2n-1} + 2(-1)^n - 4}{2}$$

$$\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}$$

$$\sum_{k=1}^{n-1} Q_{2k+1} = \frac{Q_{2n} - 6}{2}$$

$$Q_n^2 - 8P_n^2 = 4(-1)^n$$

A matrix HH is a Hankel-Hessenberg matrix if it is of the form

$$HH = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & & & a_n \\ \vdots & & & \\ a_{n-1} & a_n & & \end{bmatrix}$$

where $a_n \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

A matrix TH is a Toeplitz-Hessenberg matrix if it is of the form

$$TH = \begin{bmatrix} a_1 & a_0 & & \\ a_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & a_0 \\ a_n & \cdots & a_2 & a_1 \end{bmatrix}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$ [3].

The Euclidean norm of the matrix A is defined as

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} |\lambda_i|}$$

where the numbers λ_i are the eigenvalues of matrix A^*A. The matrix A^* is the conjugate transpose of the matrix A.

The following inequality holds,

$$\frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2 \le \|A\|_E.$$

For the matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ the Hadamard Product of these matrices is defined as

$$B \circ C = (a_{ij} b_{ij})_{m\times n}.$$

Define the maximum column lenght norm c_1 ,and the maximum row lenght norm r_1 of any matrix A by

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}$$

and

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}$$

respectively. Let A , B and C be $m \times n$ matrices. If $A = B \circ C$ then

$$\|A\|_2 \le r_1(B)c_1(C)[4].$$

2. Main Result

Lemma 1 If P_n and Q_n are nth Pell and Pell-Lucas numbers, we have

$$\sum_{k=1}^{n} kP_k^2 = \frac{(4n+2)P_{n+1}P_n - P_{2n+1} + 1}{8}$$

and

$$\sum_{k=1}^{n} kQ_k^2 = \{ \begin{array}{ll} \frac{2nQ_{2n+1} - Q_{2n} - 4n - 2}{4}, & if\ \ n\ is\ odd \\ \frac{2nQ_{2n+1} - Q_{2n} + 4n + 2}{4}, & otherwise \end{array}$$

Proof. Let $A_n = \sum_{k=1}^{n} P_k^2 = \frac{P_{n+1}P_n}{2}$, then

$$\sum_{k=1}^{n} kP_k^2 = P_1^2 + 2P_2^2 + 3P_3^2 + \ldots + nP_n^2$$

$$= \sum_{k=1}^{n} P_k^2 +$$

$$= A_n + (A_n - A_1) + (A_n - A_2) + \ldots + (A_n - A_{n-1})$$

$$= nA_n - \sum_{i=1}^{n-1} A_i = n\left(\frac{P_{n+1}P_n}{2}\right) - \sum_{i=1}^{n-1} \frac{P_{i+1}P_i}{2}$$

$$= n\left(\frac{P_{n+1}P_n}{2}\right) - \frac{1}{2}\left(\frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}\right)$$

$$= \frac{(4n+2)P_{n+1}P_n - P_{2n+1} + 1}{8}$$

So, the proof is completed. Similarly,

$$\sum_{k=1}^{n} kQ_k^2 = \begin{cases} \dfrac{2nQ_{2n+1} - Q_{2n} - 4n - 2}{4}, & \text{if } n \text{ is odd} \\ \dfrac{2nQ_{2n+1} - Q_{2n} + 4n + 2}{4}, & \text{otherwise} \end{cases}$$

Corollary 2 P_n *and* Q_n *are* nth *Pell and Pell-Lucas numbers, we have formulas for* $\sum_{k=1}^{n} kP_k^2$ *and* $\sum_{k=1}^{n} kQ_k^2$.

We can derive a formula for $\sum_{k=1}^{n}(n+1-k)P_k^2$ and $\sum_{k=1}^{n}(n+1-k)Q_k^2$.

$$\sum_{k=1}^{n}(n+1-k)P_k^2 = nP_1^2 + (n-1)P_2^2 + (n-2)P_3^2 + \ldots + 1P_n^2$$

$$= (n+1)\sum_{k=1}^{n} P_k^2 - \sum_{k=1}^{n} kP_k^2$$

$$= (n+1)\left(\frac{P_{n+1}P_n}{2}\right) - \frac{(4n+2)P_{n+1}P_n - P_{2n+1} + 1}{8}$$

$$= \frac{P_{2n+1} + 2P_{n+1}P_n - 1}{8}$$

Using the same technique, we can be show that

$$\sum_{k=1}^{n}(n+1-k)Q_k^2 = \begin{cases} \dfrac{Q_{2n+2} - 8n - 10}{4}, & n \text{ odd} \\ \dfrac{Q_{2n+2} - 16n - 14}{4}, & n \text{ even} \end{cases}$$

Theorem 3 Let A be a Hankel-Hessenberg matrix satisfying $a_{ij} = P_{i-j}$, then

$$\sqrt{\frac{8nP_n^2 + (4n-2)P_nP_{n-1} - 8P_n^2 - P_{2n-1} + 1}{8n}} \le \|A\|_2 \le \sqrt{n\left(\frac{P_{n+1}P_n}{2}\right)}$$

where $\|.\|_2$ is the spectral norm and P_n denotes the n th Pell number.

Proof. The matrix A is of the form

A=[cccc P_0   P_1   P_n-1

P_1   P_n

P_n-1   P_n   ]

Then we have,

$$\|A\|_E^2 = \sum_{k=0}^{n-1}(k+1)P_k^2 + (n-1)P_n^2$$

$$= \frac{8nP_n^2 + (4n-2)P_nP_{n-1} - 8P_n^2 - P_{2n-1} + 1}{8}$$

hence,

$$\|A\|_2 \geq \sqrt{\frac{8nP_n^2 + (4n-2)P_nP_{n-1} - 8P_n^2 - P_{2n-1} + 1}{8n}}$$

On the other hand, let the matrices B and C as

B =[cccc 1 1 1
1 1
1 1] *and* C=[cccc P_0 P_1 P_n-1
P_1 P_n
P_n-1 P_n]
such that $A = B \circ C$. Then
r_1(B)=max
i_j|b_ij|^2=_j=1^n|b_nj|^2=n *and*
c_1(C)=max
j_i|c_ij|^2=_i=1^n|c_in|^2=_i=1^nP_i^2=P_n+1P_n2
We have

$$\|A\|_2 \leq \sqrt{n\left(\frac{P_{n+1}P_n}{2}\right)}$$

Thus, the proof is completed.

Theorem 4 *Let* A *be a Hankel-Hessenberg matrix satisfying* $a_{ij} = Q_{i-j}$, *then*

{ where $\|.\|_2$ is the spectral norm and Q_n denotes the n th Pell-Lucas number.

Proof. The matrix A is of the form
A=[cccc Q_0 Q_1 Q_n-1
Q_1 Q_n
Q_n-1 Q_n]
Then we have,

$$\|A\|_E^2 = \sum_{k=0}^{n-1}(k+1)Q_k^2 + (n-1)Q_n^2$$

$$= \{$$

hence,

$$\|A\|_2 \geq \{$$

On the other hand, let the matrices B and C as

B =[cccc 1   1   1

1   1

1   1   ] *and* C=[cccc Q_0   Q_1   Q_n-1

Q_1   Q_n

Q_n-1   Q_n   ]

such that $A = B \circ C$. Then

r_1(B)=max

i_j|b_ij|^2=_j=1^n|b_nj|^2=n *and*

c_1(C)=max

j_i|c_ij|^2=_i=1^n|c_in|^2=_i=1^nQ_i^2={lcrQ_2n+1-62, if n is odd

Q_2n+1-22, otherwise

We have

$$\|A\|_2 \le \{ \begin{array}{ll} \sqrt{n\left(\dfrac{Q_{2n+1}-6}{2}\right)}, & \text{if } n \text{ is odd} \\ \sqrt{n\left(\dfrac{Q_{2n+1}-2}{2}\right)}, & \text{otherwise} \end{array}$$

Thus, the proof is completed.

Theorem 5 *Let A be a Toeplitz-Hessenberg matrix satisfying $a_{ij} = P_{i-j+1}$, then*

$$\sqrt{\frac{P_{2n+3} - 2(2n+3)P_{n+2}P_{n+1} + 8n^2 + 24n - 1}{8n}} \le \|A\|_2 \le \sqrt{n\left(\frac{P_{n+2}P_{n+1} - 2}{2}\right)}$$

where $\|\cdot\|_2$ is the spectral norm and P_n denotes the n th Pell number.

Proof. The matrix A is of the form

A=[cccc P_2   P_1

P_3

P_1

P_n+1   P_3   P_2]

Then we have,

$$\|A\|_E^2 = \sum_{k=2}^{n+1}(n+2-k)P_k^2 + (n-1)P_1^2$$

$$= \frac{P_{2n+3} - 2(2n+3)P_{n+2}P_{n+1} + 8n^2 + 24n - 1}{8}$$

hence,

$$\|A\|_2 \ge \sqrt{\frac{P_{2n+3} - 2(2n+3)P_{n+2}P_{n+1} + 8n^2 + 24n - 1}{8n}}$$

On the other hand, let the matrices B and C as

B=[cccc 1   1

1   1

1

1   1 ...   1] *and* C=[cccc P_2   P_1

P_3   P_2

P_1

P_n+1   P_n ...   P_2]

such that $A = B \circ C$. Then

r_1(B)=max

i_j|b_ij|^2=_j=1^n|b_nj|^2=n *and*

c_1(C)=max

j_i|c_ij|^2=_i=1^n|c_in|^2=_i=2^n+1P_i^2=P_n+2P_n+1-22

We have

$$\|A\|_2 \le \sqrt{n\left(\frac{P_{n+2}P_{n+1}-2}{2}\right)}$$

Thus, the proof is completed.

Theorem 6 Let A be a Toeplitz-Hessenberg matrix satisfying $a_{ij} = Q_{i-j+1}$, then

$$\begin{cases} \sqrt{\frac{Q_{2n+2}+8n-26}{4n}} \le \|A\|_2 \le \sqrt{n\left(\frac{Q_{2n+1}-6}{2}\right)}, & \text{if } n \text{ is odd} \\ \sqrt{\frac{Q_{2n+2}-30}{4n}} \le \|A\|_2 \le \sqrt{n\left(\frac{Q_{2n+1}-2}{2}\right)}, & \text{otherwise} \end{cases}$$

where $\|.\|_2$ is the spectral norm and Q_n denotes the n th Pell-Lucas number.

Proof. The matrix A is of the form

A=[cccc Q_1   Q_0

Q_2

Q_0

Q_n   Q_2   Q_1]

Then we have,

$$\|A\|_E^2 = \sum_{k=1}^{n}(n+1-k)Q_k^2 + (n-1)Q_0^2$$

$$= \begin{cases} \frac{Q_{2n+2}+8n-26}{4}, & \text{if } n \text{ is odd} \\ \frac{Q_{2n+2}-30}{4}, & \text{otherwise} \end{cases}$$

hence,

$$\|A\|_2 \geq \{ \begin{array}{ll} \sqrt{\dfrac{Q_{2n+2}+8n-26}{4n}}, & if\ \ n\ is\ odd \\ \sqrt{\dfrac{Q_{2n+2}-30}{4n}}, & otherwise \end{array}$$

On the other hand, let the matrices B and C as

B=[cccc 1   1

1   1

1

1   1 ...   1] *and*   C=[cccc Q_1   Q_0

Q_2   Q_1

Q_0

Q_n   Q_n-1 ...   Q_1]

such that $A = B \circ C$. Then

r_1(B)=max

i_j|b_ij|^2=_j=1^n|b_nj|^2=n *and*

c_1(C)=max

j_i|c_ij|^2=_i=1^n|c_in|^2=_i=1^nQ_i^2={lcrQ_2n+1-62, if n is odd

Q_2n+1-22, otherwise

We have

$$\|A\|_2 \leq \{ \begin{array}{ll} \sqrt{n\left(\dfrac{Q_{2n+1}-6}{2}\right)}, & if\ \ n\ is\ odd \\ \sqrt{n\left(\dfrac{Q_{2n+1}-2}{2}\right)}, & otherwise \end{array}$$

Thus, the proof is completed.

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