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The First Triangular Representation of The Symmetric Groups over a field K of characteristic pdivides (n-2)

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Abstract

In this paper we will study the new type of triangular representations of the symmetric groups which is called the first triangular representations of the symmetric groups over a field K of characteristic pdivides(n-2).

Keywords: symmetric group; group algebra KS_n,KS_n –module;Specht module ;exact sequence .

Academic Discipline And Sub-Disciplines

Philosophy of Mathematics ,Algebra,Grouptheory,Group representation.

SUBJECT CLASSIFICATION: 20C30

TYPE (METHOD/APPROACH)

Pure Mathematics.

1. INTRODUCTION

When Prof. W. Specht was a student under the supervision of Prof. I. Schur, he began investigating representation theory of the symmetric group .During that time it was well known that standard Young tableaux of a given partition λ of a positive integer n form a basis of an ordinary irreducible representation space of $\mathbf{S_n}$. The problem that W.Specht was facing in his investigating in that time is that the symmetric group acts in a natural way on tableaux, but the result of the application of a permutation to a standard tableau can be a nonstandard tableau, and it is by no means clear how a nonstandard tableau can be written as a linear combination of standard ones. For this reason W. Specht introduced in 1935 polynomials corresponding to the tableaux (known nowadays as Specht polynomials), and it is obvious how a given polynomial can be written as a linear combination of other polynomials (see [1]).

In 1971 Peel introduced the r^{th} Hook representations which deals with the partitions $\lambda = (n-r, 1^r)$; r≥ 1.[4]

In 2016 we introduce in our paper[5] new representations of the symmetric groups we call them the triangular representations of the symmetric groups and we study the first of them which we call it the first triangular representation of the symmetric groups when p divides (n-1).and follow it by the paper[6] which it is study the first triangular representation of the symmetric groups over a field **K** of characteristic p=0.

Throughout this paper let $\mathbf{K}$ be a field of characteristic p which is divide (n-2), and $x_1, x_2, \dots, x_n$ be linearly independent commuting variables over $\mathbf{K}$.

2. PRIEMINERIES

Definition 2.1:

Let $\mathbf{S_n}$ be the set of all permutations τ on the set $\{x_1, x_2, \dots, x_n\}$ and $\mathbf{K}[x_1, x_2, \dots, x_n]$ be the ring of polynomials in $x_1, x_2, \dots, x_n$ with coefficients in $\mathbf{K}$. Then each permutation $\tau \in \mathbf{S_n}$ can be regarded as a bijective function from $\mathbf{K}[x_1, x_2, \dots, x_n]$ onto $\mathbf{K}[x_1, x_2, \dots, x_n]$ defined by$\big(f(x_1, x_2, \dots, x_n)\big) = f\big(\tau(x_1), \tau(x_2), \dots, \tau(x_n)\big) \forall\, f(x_1, x_2, \dots, x_n) \in \mathbf{K}[x_1, x_2, \dots, x_n]$. Then $\mathbf{KS_n}$ forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group $\mathbf{S_n}$[3].

Definition 2.2:

Let n be a positive integer then the sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$is called a partition of n if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ and$\lambda_1 + \lambda_2 + \dots + \lambda_l = n$. Then the set $D_\lambda = \{(i,j) | i = 1,2, \dots, l; 1 \leq j \leq \lambda_i\}$ is called $\lambda-$diagram .And any bijective function $t : D_\lambda \to \{x_1, x_2, \dots, x_n\}$ is called a λ–tableau. A λ–tableau may be thought as an array consisting of l rows and λ_1 columns of distinct variables $t((i,j))$ where the variables occur in the first λ_i positions of the i^{th} row and each variable $t((i,j))$ occurs in the i^{th} row and the j^{th} column ((i, j)-position)of the array.$t((i,j))$will be denoted by $t(i,j)$for each $(i,j) \in D_\lambda$.

The set of all λ-tableaux will be denoted by T_λ. i.e $T_\lambda = \{t|t \text{ is a } \lambda - \text{tableau}\}$.

Then the function $g: T_\lambda \to K[x_1, x_2, \dots, x_n]$ which is defined by $g(t) = \prod_{i=1}^{l} \prod_{j=1}^{\lambda_i} (t(i,j))^{i-1}$, $\forall\, t \in T_\lambda$ .is called the row position monomial function of T_λ,and for eachλ–tableau t, $g(t)$ is called the row position monomial of t .So $M(\lambda)$ is the cyclic KS_n –module generated by $g(t)$ over KS_n.[2]

3. THE FIRST TRIANGULAR REPRESENTATIOM OF S_n

In the beginning we define some denotations which we need them in this paper.

1) Let$\sigma_1(n)= \sum_{i=1}^{n} x_i$.

2) Let $\sigma_2(n)= \sum_{1\le i \langle j \le n} x_i x_j$.

3) Let$C_l(n) = x_l^2 \left(\sigma_2(n) - \sum_{\substack{j=1 \\ j\neq l}}^{n} x_l x_j\right); l = 1,2,\dots,n$.

We denote $\overline{N}$ to be the KS_nmodule generated by $C_1(n)$over KS_n. The set $B = \{C_i(n)| i = 1,2,\dots,n\}$ is a K-basis for $\overline{N} = KS_n C_1(n)$and $\dim_K \overline{N} = n$.

4) Let$u_{ij}(n) = C_i(n) - C_j(n)$; $i,j = 1,2,\dots,n$.

we denote $\overline{N}_0$ the KS_nsubmodule of $\overline{N}$ generated by $u_{12}(n)$.

5) Let $\sigma_3(n)= \sum_{1\le i \langle j \le n} \sum_{\substack{k=1 \\ k\neq i,j}}^{n} x_i x_j x_K^2$.

Then $\sum_{l=1}^{n} C_l(n) = \sigma_3(n)$and $\dim_K(K\sigma_1(n)) = \dim_K(K\sigma_2(n)) = \dim_K(K\sigma_3(n))$ =1.

$K\sigma_1(n), K\sigma_2(n)andK\sigma_3(n)$ are all KS_n-modules, since $\tau\sigma_k(n) = \sigma_k(n)\ \forall\, k = 1,2,3$.

Definition 3.1:

The KS_n-module $M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right)$defined by

$$M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right) = KS_n x_1 x_2 \dots x_{r+1} x_{r+2}^2 \dots x_{2r+1}^2 x_{2r+2}^3 \dots x_n^{r+1}$$

is called the r^{th} triangular representation module of S_n over K ,where $n \ge \frac{(r+3)(r+2)}{2}$.

Remark 3.1.1:

The first triangular representation module of S_n over K is theKS_n –module $M(n-3,2,1)$, the second triangular representation module of S_n over K is theKS_n –module$M(n-6,3,2,1)$,the third triangular representation module of S_n over K is theKS_n –module$M(n-10,4,3,2,1)$,and so on.

Lemma 3.2:

The set $B(n-3,2,1) = \{ x_i x_j x_l^2 | 1 \le i < j \le n, 1 \le l \le n, l \neq i,j\}$ is a K-basis of $M(n-3,2,1)$,and$\dim_K M(n-3,2,1) = \binom{n}{2}(n-2)$; $n \ge 6$.

Theorem 3.3:

The set $B_0(n-3,2,1) = \{ x_i x_j x_l^2 - x_1 x_2 x_3^2 | 1 \le i < j \le n, 1 \le l \le n, l \neq i,j,(i,j,l) \neq (1,2,3)\}$ is a K-basis of $M_0(n-3,2,1)$, and $\dim_K M_0(n-3,2,1) = \binom{n}{2}(n-2) - 1$; $n \ge 6$.(see [5])

Theorem 3.4.: The set $B = \{C_i(n)| i = 1,2,\dots,n\}$ is a K- basis for $\overline{N}(n) = KS_n C_1(n)$.

Proof: Let $\tau_i = (x_1 x_i) \in S_n$; $i = 2,3,\dots,n$. Then we get $\tau_i(C_1(n)) = C_i(n)$; $i = 2, 3, \dots, n$

which implies that $C_i(n) \in \overline{N}(n) = KS_n C_1(n)$ for all $i = 1,2,3,\dots,n$. Thus

$B = \{C_i(n)| i = 1,2,\dots,n\} \subseteq \overline{N}(n)$. Now if $\omega \in \overline{N}(n) \Rightarrow \omega \in KS_n C_1(n)$.

$\Rightarrow \omega = \sum_{j=1}^{(n-1)!} \sum_{i=1}^{n} k_{ji}\, \sigma_{ji}\, C_1(n)$where$\sigma_{ji} \in S_n$ and $\sigma_{ji}(x_1) = x_i$ (i.e. $\sigma_{ji}(C_1(n)) = C_i(n) \forall\, i = 1,2,\dots,n$).

$\Rightarrow \omega = \sum_{i=1}^{n} \sum_{j=1}^{(n-1)!} k_{ji}\, \sigma_{ji}\, C_1(n) = \sum_{i=1}^{n} \left(\sum_{j=1}^{(n-1)!} k_{ji} \right) C_i(n) = \sum_{i=1}^{n} d_i C_i(n)$where$d_i = \sum_{j=1}^{(n-1)!} k_{ji}$ .Hence B generates $\overline{N}(n)$ over K .If $\sum_{i=1}^{n} k_i C_i(n) = 0$ .i.e.

$k_1 C_1(n) + k_2 C_2(n) + \dots + k_n C_n(n) = 0$ .which implies that

$k_1 = k_2 = \dots = k_n = 0$since$C_l(n) = \sum_{\substack{1 \le i \langle j \le n \\ l \ne i,j}} x_i x_j x_l^2$ .Thus B is linearly independent .Therefore B is a basis of $\overline{N}(n)$ and $dim_K \overline{N}(n) = n$.

Theorem 3.5.:$\overline{N} = KS_n C_1(n)$ and $M(n-1,1)$ are isomorphic over KS_n(see[5])

Theorem 3.6.:$\overline{N}_0 = KS_n u_{12}(n)$ and $M_0(n-1,1)$ are isomorphic over KS_n.(see[5])

Corollary 3.6.1: The KS_n-module $\overline{N}_0 = KS_n u_{12}(n)$ is irreducible over KS_n.(see [5])

Proposition 3.6.2:$\overline{N} = \overline{N}_0 \oplus K\sigma_3(n)$.(see[5])

Proposition 3.6.3:$\overline{N}$ has the following two composition series

$0 \subset \overline{N}_0 \subset \overline{N}and0 \subset K\sigma_3(n) \subset \overline{N}$. (see [5])

Definitions 3.7.:

1. the KS_n–homomorphism$d : M(n-3,2,1) \to M(n-2,2)$ is defined in terms of the partial operators by

$$d\left(x_i x_j x_l^2\right) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\left(x_i x_j x_l^2\right),$$

2. the KS_n–homomorphism $\bar{d}$which is the restriction of d to$M_0(n-3,2,1)$. i.e.

$$\bar{d}\colon M_0(n-3,2,1). \to M_0(n-3,2).$$

3. the KS_n–homomorphism $f : M(n-3,2,1) \to K$ which is defined by

$$f\Big(\sum_{1 \le i \langle j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{i,j,l}\, x_i x_j x_l^2 \Big) = \sum_{1 \le i \langle j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{i,j,l}.$$

Theorem 3.8.: The following sequence ofKS_n − modules is exact

$0 \to Ker\ d \xrightarrow{i} M(n-3,2,1) \xrightarrow{d} M(n-2,2) \to 0 \quad \dots\dots\dots (1)$(see [5])

Corollary 3.8.1: The dimension of ker$d$ over K of the KS_n − homomorphism

$d\colon M(n-3.2.1) \to M(n-2,2)is\frac{n(n-1)(n-3)}{2}$.(see [5])

Corollary 3.8.2: The following sequence ofKS_n − modules is exact

$0 \to Kerd \xrightarrow{i} M_0(n-3,2,1) \xrightarrow{\bar{d}} M_O(n-2,2) \to 0 \quad \dots\dots\dots (2)$(see [5])

Lemma 3.9.:$dim_K\ S(n-3,2,1) = \frac{n(n-2)(n-4)}{3}$.

Proposition 3.10.: $S(n-3,2,1)$ is a proper submodule of $kerd$.(see [5])

Theorem 3.11.:The following sequence over the field K is exact.

$O \to M_0(n-3,2,1) \xrightarrow{i} M(n-3,2,1) \xrightarrow{f} K \to O \qquad \dots (3)$(see [5])

Corollary 3.11.1: the exact sequence (3) is split iff p does not divide $\frac{n(n-1)(n-2)}{2}$

Proof: Assume that p does not divide$\frac{n(n-1)(n-2)}{2}$. Define $g\colon K \to M(n-3,2,1)$ by

$g(1) = \frac{2\sigma_3(n)}{n(n-1)(n-2)}$ where $\sigma_3(n) = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2$

Then g is a KS_n -homomorphism since $\tau\,\sigma_3(n) = \sigma_3(n)$ for any $\tau \in S_n$ and $\tau(1) = 1$, thus we get

$g(\tau 1) = g(1) = \frac{2\sigma_3(n)}{n(n-1)(n-2)} = \tau\left(\frac{2\sigma_3(n)}{n(n-1)(n-2)}\right) = \tau g(1).$

Moreover we have $fg(1) = f\big(g(1)\big) = f(\frac{2\sigma_3(n)}{n(n-1)(n-2)}) = \frac{2}{n(n-1)(n-2)} f\big(\sigma_3(n)\big)$

$= \frac{2}{n(n-1)(n-2)} \cdot \frac{n(n-1)(n-2)}{2} = 1$ .Hence $fg = I$ .Therefore the sequence (3) is split.

Now assume the sequence (3) is split .Hence there is a KS_n –homomorphism

$g: K \to M(n-3,2,1)\ s.t.\ fg = I$. Then g has the form

$$g(1) = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl} x_i x_j x_l^2.$$

Then for any $\tau = (x_r x_s) \in S_n; 1 \leq r < s \leq n$ we have $g(1) = g(\tau 1)$.

Thus we get $0 = g(1) - g(\tau 1)$i. e .

$$0 = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl} x_i x_j x_l^2 - \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl}\, \tau\, x_i x_j x_l^2 = \sum_{1\leq i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl} (x_i x_j x_l^2 - \tau(x_i x_j x_l^2))$$

$$= \sum_{\substack{j=r+1\\ j\neq s}}^{n} \sum_{\substack{l=1\\ l\neq r,s,j}}^{n} (k_{rjl} - k_{sjl}) x_r x_j x_l^2 + \sum_{j=s+1}^{n} \sum_{\substack{l=1\\ l\neq r,s,j}}^{n} (k_{sjl} - k_{rjl}) x_s x_j x_l^2 + \sum_{i=1}^{r-1} \sum_{\substack{l=1\\ l\neq r,s,i}}^{n} (k_{irl} - k_{isl}) x_i x_r$$

$$+ \sum_{\substack{i=1\\ i\neq r}}^{s-1} \sum_{\substack{l=1\\ l\neq r,s,i}}^{n} (k_{isl} - k_{irl}) x_i x_s x_l^2 + \sum_{\substack{i=1\\ i\neq r,s}}^{n-1} \sum_{\substack{j=i+1\\ j\neq r,s}}^{n} (k_{ijr} - k_{ijs}) x_i x_j x_r^2 + \sum_{\substack{i=1\\ i\neq r,s}}^{n-1} \sum_{\substack{j=i+1\\ j\neq r,s}}^{n} (k_{ijs} - k_{ijr}) x_i x_j x_s^2$$

$$+ \sum_{\substack{j=r+1\\ j\neq s}}^{n} (k_{rjs} - k_{sjr}) x_r x_j x_s^2 + \sum_{j=s+1}^{n} (k_{sjr} - k_{rjs}) x_s x_j x_r^2 + \sum_{i=1}^{r-1} (k_{irs} - k_{isr}) x_i x_r x_s^2 + \sum_{\substack{i=1\\ r\neq i}}^{s-1} (k_{isr} - k_{irs}) x_i x_s x_r^2$$

$$= \sum_{\substack{j=r+1\\ j\neq s}}^{n} \sum_{\substack{l=1\\ l\neq r,s}}^{n} (k_{rjl} - k_{sjl})(x_r x_j x_l^2 - x_s x_j x_l^2)$$

$$+ \sum_{i=1}^{r-1} \sum_{\substack{l=1\\ l\neq r,s}}^{n} (k_{irl} - k_{isl})(x_i x_r x_l^2 - x_i x_s x_l^2) + \sum_{\substack{i=1\\ i\neq r,s}}^{n-1} \sum_{\substack{j=i+1\\ j\neq r,s}}^{n} (k_{ijr} - k_{ijs})(x_i x_j x_r^2 - x_i x_j x_s^2)$$

$$+ \sum_{\substack{r\langle j\leq n\\ j\neq s}} (k_{rjs} - k_{sjr})\big(x_r x_j x_s^2 - x_s x_j x_r^2\big) + \sum_{1\leq i\langle r\langle n} (k_{irs} - k_{isr})(x_i x_r x_s^2 - x_i x_s x_r^2).$$

Which implies by equating the coefficient of the above equation that

$k_{rjl} = k_{sjl} \quad \forall\, r \leq j < n \ni j \neq s$ and$\forall\, 1 \leq l \leq n \ni l \neq r,s,j$.

$k_{irl} = k_{isl} \quad \forall\, 1 \leq i < r$ and$\forall\, 1 \leq l \leq n \ni l \neq r,s,j.$

$k_{ijr} = k_{ijs} \quad \forall\, 1 \leq i < j < n \ni i,j \neq r,s.$

$k_{irs} = k_{isr} \quad \forall\, 1 \leq i < r$.

$$k_{rjs} = k_{sjr} \quad \forall\, r < j \leq n, j \neq s.$$

Hence for each $r, s\ s.t\ 1 \leq r < s \leq n$ we get $k_{ijl} = k\ ; 1 \leq i < j \leq n$

and $1 \leq l \leq n \ni l \neq i,j.$ Thus

$$g(1) = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl} x_i x_j x_l^2 = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k\, x_i x_j x_l^2 = k \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2 = k\,\sigma_3(n).$$

$\because f\ g = I$ .Hence $1 = f\ g\ (1) = f\left(k\ \sigma_3(n)\right) = k\ f\left(\sigma_3(n)\right) = \frac{k}{2}n(n-1)(n-2)$

Thus p does not divide $\frac{n(n-1)(n-2)}{2}$.

Corollary 3.11.2: $M_0(n-3,2,1)$ is not a direct summand of $M(n-3,2,1)$ when p divides $\frac{n(n-1)(n-2)}{2}$.

Proof: Assume that $M_0(n-3,2,1)$ is a direct summand of $M(n-3,2,1)$ when p divides

$\frac{n(n-1)(n-2)}{2}$ .Hence there is a KS_n submodule of $M(n-3,2,1)$, say H , s.t. $M(n-3,2,1) =$

M_0(n-3,2,1) $\oplus H$ .Thus the exact sequence (3) is split and this is a contradicts corollary(3.11.1).

Therefore M_0(n-3,2,1) is not a direct summand of $M(n-3,2,1)$.

Proposition 3.12: If p divides $2(n-2)$, then $K\sigma_3(n) \subset ker\ d$.

Proof : By the definition of $\sigma_3(n)$ we have that

$\sigma_3(n) = \sum_{1\le i\langle j\le n} \sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_i x_j x_l^2$ .Thus $d\left(\sigma_3(n)\right) = 2(n-2) \sum_{1\le i\langle j\le n} x_i x_j = 2(n-2)\ \sigma_2(n)$ .Which implies that $K\sigma_3(n) \subset$ $ker\ d$ when p divides $2(n-2)$.

Theorem3.13: If $p \neq 2$ and p divides $(n-2)$ then we have the following series :

1) $0 \subset K\sigma_3 \subset ker\ d \subset \overline{N}_0 \oplus ker\ d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$

2) $0 \subset \overline{N}_0 \subset \overline{N} \subset \overline{N}_0 \oplus ker\ d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

3) $0 \subset K\sigma_3 \subset \overline{N} \subset \overline{N}_0 \oplus ker\ d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

Proof :

Since $p \neq 2$ and p divides $(n-2)$,then by proposition (3.12) we get that $K\sigma_3(n) \subset ker\ d$.

Moreover when p divides $(n-2)$ implies p does not divides n.Thus we get that $\overline{N}_0$ is irreducible submodule over KS_n by corollary (3.6.1).Hence $\overline{N}_0 \oplus \ker d \subset M_0(n-3,2,1)$.

Therefore we get the following series

1) $0 \subset K\sigma_3 \subset \ker d \subset \overline{N}_0 \oplus \ker d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$

2) $0 \subset \overline{N}_0 \subset \overline{N} \subset \overline{N}_0 \oplus \ker d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

3) $0 \subset K\sigma_3 \subset \overline{N} \subset \overline{N}_0 \oplus \ker d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

Theorem 3.14: The following sequence of a KS_n -submodule is exact.

$0 \to \ker d_1 \overset{i}{\to} T \overset{d_1}{\to} S(n-2,2) \to 0$ (4)

where $T = KS_n\ (x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2)$.

Proof: The same proof of theorem(3.13) in [6].

Corollary 3.14.1: The exact sequence (4) over the field K is split.

Proof : As the proof of corollary(3.13.1) in[6].

Proposition 3.15: $S(n-3,2,1)$ is a proper KS_n $-$submodule of T.

Proof: As the proof of proposition(3.13.2)in[6].

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