ISSN 2347-1921

(U,R) – STRONGLY DERIVATION PAIRS ON LIE IDEALS IN RINGS

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ABSTRACT

Let R be an associative ring , U be a nonzero Lie ideal of R. In this paper , we will present the definition of (U,R) – strongly derivation pair (d,g) , then we will get d=0 (resp. g=0 **)** under certain conditions on d and g for (U,R) – strongly derivation pair (d,g) on semiprime ring . After that we will study prime rings , semiprime rings ,and rings that have a commutator left nonzero divisor with (U,R) – strongly derivation pair (d,g) , to obtain the notation of (U,R) – derivation .

Indexing terms/Keywords

prime ring; semiprime ring; derivation; strongly derivation pair.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .10, No.8
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

INTRODUCTION

Let R be an associative ring , U be a nonzero Lie ideal . This paper consists of two section . In section one , we recall some basic definitions and other concepts ,which will be used in our paper , then we explain these concepts by examples ,remarks .In section two , we will present the definition of (U,R) –strongly derivation pair (d,g) and it will be denoted by (U,R) – S- derivation pair (d,g) and we put certain conditions on d and g for (U,R) – strongly derivation pair (d,g) on semiprime ring to obtain d=0 or g=0 . Then we will prove that (1) If R is a 2- torsion free semiprime ring, and (d,g) be a (U,R) – strongly derivation pair , then d and g are (U,R) – derivations , (2) If R is a prime ring , and (d,g) be a (U,R) – strongly derivation pair , then d and g are (U,R) – derivations, (3) If R is a ring which has a commutator left nonzero divisor and (d,g) be a (U,R) – strongly derivation pair ,then d and g are (U,R) – derivations.

1.BASIC CONCEPTS

Definition 1.1:[1] A ring R is called a prime ring if for any a,b∈R, aRb = {0},implies that either a=0 or b=0 .

Examples 1.2:[1]

1.Any integral domain is a prime ring .

2.Any matrix ring over an integral domain is a prime ring .

Definition 1.3:[1] A ring R is called a semiprime ring if for any a∈R,aRa={0},implies that a=0 .

Remark 1.4 [1] Every prime ring is a semiprime ring ,but the converse in general is not true.

Definition 1.5:[2] A ring R is said to be n-torsion free,where n≠0 is an integer if whenever na=0,with a∈R , then a=0 .

Definition 1.6:[2] A ring R is said to be a commutator right (resp. left) nonzero divisor ,if there exists elements a and b of R such that c[a,b]=0 (resp. [a,b]c=0) implies c=0 , for every a∈R .

Definition 1.7:[3] Let R be a ring . Define a lie product [.,.] on R as follows [x ,y] = xy – yx , for all x,y∈R.

Definition 1.8:[3] An additive subgroup U of a ring R is said to be a Lie ideal of R if the commutator [u,r] = ur – ru ∈U , for all u∈U , r∈R .

Definition 1.9:[3] A Lie ideal verifies that u^2∈U, for all u∈U is called square closed Lie ideal .

Definition 1.10:[4] Let R be a ring . An additive mapping d:R→ R is called a derivation if : d(xy) =d(x)y + xd(y) , for all x,y∈R .

Example 1.11:[4] Let R={$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$,a,b∈N,where N is the ring of integers } be a ring of 2x2 matrices with respect to usual addition and multiplication .

Let d: R → R , defined by d $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ ∈R .Then d is a derivation of R .

Definition 1.12:[4] Let U be a Lie ideal of a ring R. An additive mapping d: R→ R is

called a (U,R) – derivation ,if d(xy) = d(x)y + xd(y) ,for all x∈U ,y∈R .

Example 1.13:[3] Let R= {$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$: x,y,z,w ∈N where N is the ring of integers } be a ring of 2x2 matrices with respect to the usual addition and multiplication .

Let d: R → R , defined by d $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix}$, for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ ∈R and

Let U= { $\begin{pmatrix} x & y \\ z & 0 \end{pmatrix}$, x,y ,z ∈ N }

It is clear that U is a Lie ideal of R .

Then d is a (U,R) – derivation .

Definition 1.14 :[1] Let R be a ring , additive mappings d,g : R $\rightarrow$ R is called strongly derivation pair (d,g) if satisfy the following equations :

d(xy) = d(x)y + xg(y) , for all x,y $\in$R .

g(xy) = g(x)y + xd(y) , for all x,y $\in$R .

2. (U, R) – STRONGLY DERIVATION PAIRS

We first introduce the basic definition in this section

Definition 2.1: Let R be a ring , U a nonzero Lie ideal of R , additive mappings d,g:U $\rightarrow$ R is called (U,R) – S – derivation pair (d,g) if satisfy the following equations:

d(xy) = d(x)y + xg(y) , for all x$\in$U , y$\in$R .

g(xy) = g(x)y + xd(y) , for all x$\in$U , y$\in$R .

Example 2.2: Let R be a non commutative ring , U a nonzero Lie ideal of R and let a,b$\in$R , such that xa = xb = 0 , for all x$\in$U . Define d,g : U $\rightarrow$ R ,as follows :

d(x) = ax , g(x) = bx . Then (d,g) is a (U,R) – S- derivation pair .

Let x$\in$U , y$\in$R , so :

d(xy) = d(x)y + xg(y) = axy + xby = axy

On the other hand : d(xy) = axy

g(xy) = g(x)y + xd(y) = bxy + xay = bxy

On the other hand : g(xy) = bxy

Hence (d,g) is a (U,R) – S- derivation pair .

Theorem 2.3: Let R be a semiprime ring , U a nonzero Lie ideal of R with $u^2$$\in$U , for all u$\in$U. If R admits a (U,R) – S-derivation pair (d,g), such that d(x)g(y)=0 (resp. g(x)d(y)=0) ,for all x,y$\in$U, then d=0 (resp. g=0) .

Proof: we have

d(x)g(y) = 0, for all x,y $\in$U (1)

Replacing yx for y in (1) and using (1) , we have :

d(x)yd(x)=0 ,for all x,y $\in$U (2)

By semiprimeness of R , (2) gives :

d(x)=0 , for all x$\in$U (3)

Similarly , if g(x)d(y)=0 ,for all x,y $\in$U ,then g=0 .

Theorem 2.4: Let R be a semiprime ring, U a nonzero Lie ideal of R with $u^2$$\in$U, for all u$\in$U. If R admits a (U,R)-S-derivation pair (d,g) , such that d(x) =$\pm$x (resp. g(x) =$\pm$x) ,for all x$\in$U, then g=0 (resp. d=0) .

Proof: we have

d(x) = x , for all x$\in$U (1)

Replacing x by xy in (1) and using (1) , we get :

xg(y) = 0, for all x,y $\in$U (2)

Left multiplication of (2) by g(y) ,leads to :

$g(y)xg(y) = 0$, for all $x,y \in U$ (3)

By semiprimeness of R , we get :

$g(y) = 0$, for all $y \in U$ (4)

Similarly , we can show if $d(x) = -x$, for all $x \in U$, then $g = 0$

In the same way , if $g(x) = \pm$, for all $x \in U$, then $d = 0$.

Theorem 2.5 : Let R be a 2- torsion free semiprime ring , U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d,g) be a (U,R)-S-derivation pair , then d and g are (U,R)-derivations .

Proof: Suppose that (d,g) is (U,R)-S-derivation pair . Then:

$d(xyx) = d(x(yx)) = d(x)yx + xg(yx)$, for all $x \in U$, $y \in R$ (1)

That is :

$d(xyx) = d(x)yx + xg(y)x + xyd(x)$, for all $x \in U$, $y \in R$ (2)

Also:

$d(xyx) = d((xy)x) = d(xy)x + xyg(x)$, for all $x \in U$, $y \in R$ (3)

That is :

$d(xyx) = d(x)yx + xg(y)x + xyg(y)$, for all $x \in U$, $y \in R$ (4)

From (2) and (4) , we get :

$xy(d(x) – g(x)) = 0$, for all $x \in U$, $y \in R$ (5)

Replace y by $(d(x) – g(x))yx$ in (5) , we get :

$x(d(x) – g(x)) yx (d(x) – g(x)) = 0$, for all $x \in U$, $y \in R$ (6)

Since R is semiprime , we get :

$xd(x) = xg(x)$, for all $x \in U$ (7)

It follows that :

$d(x^2) = d(x)x + xd(x)$, for all $x \in U$ (8)

And $g(x^2) = g(x)x + xg(x)$, for all $x \in U$ (9)

Thus , by using [1,Theorem 1.1.14] , we obtain that d and g are (U,R)-derivations .

Theorem 2.6 : Let R be a prime ring , U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d,g) be a (U,R)-S-derivation pair , then d and g are (U,R)-derivations .

Proof:

Since (d,g) is (U,R)-S-derivation pair , we have (see how relation (5) was obtained from relation (1) in the proof of Theorem 2.5)

$xy (d(x) – g(x)) = 0$, for all $x \in U$, $y \in R$ (1)

And , by primeness of R ,we get :

$d(x) = g(x)$, for all $x \in U$ (2)

And hence d and g are (U,R)-derivations .

Theorem 2.7: Let R be a ring which has a commutator left nonzero divisor , U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d,g) be a (U,R)-S-derivation pair, then d and g are (U,R)-derivations .

Proof: we have :

$d(yx^2) = d(y)x^2 + yg(x^2)$, for all $x \in U$, $y \in R$ (1)

That is :

$d(yx^2) = d(y)x^2 + yg(x)x + yxd(x)$, for all $x \in U$, $y \in R$ (2)

On the other hand :

$d(yx^2) = d(yx)x + yxg(x)$, for all $x \in U$, $y \in R$ (3)

That is :

$d(yx^2) = d(y)x^2 + yg(x)x + yxg(x)$, for all $x \in U$,$y \in R$ (4)

From (2) and (4) , we obtain :

$y(xd(x) - xg(x)) = 0$, for all $x \in U$, $y \in R$ (5)

Replacing y by yr in (5) , to get :

$yr(xd(x) - xg(x)) = 0$, for all $x \in U$,$y,r \in R$ (6)

Again , left multiplying of (5) by r , to get :

$ry(xd(x) - xg(x)) = 0$, for all $x \in U$, $y ,r \in R$ (7)

Subtracting (7) from (6) , we get :

$[y,r] (xd(x) - xg(x)) = 0$,for all $x \in U$, $y,r \in R$ (8)

Since R has a commutator left nonzero divisor , we get :

$xd(x) = xg(x)$, for all $x \in U$ (9)

Linearizing (9) , we get :

$xd(y) + yd(x) = xg(y) + yg(x)$, for all $x \in U$, $y \in R$ (10)

That is : $x(d - g)(y) + y(d - g)(x) = 0$, for all $x \in U$, $y \in R$ (11)

Replacing y by ry in (11) , to get :

$x(d - g)(ry) + ry(d - g)(x) = 0$, for all $x \in U$, $y,r \in R$ (12)

Again , left multiplying of (11) by r , to get :

$rx(d - g)(y) + ry(d - g)(x) = 0$, for all $x \in U$, $y ,r \in R$ (13)

Subtracting (12) from (13) , we get :

$rx(d - g)(y) - x(d - g)(ry) = 0$, for all $x \in U$, $y , r \in R$ (14)

Replacing x by sx in (14) , to get :

$rsx(d - g)(y) - sx(d - g)(ry) = 0$, for all $x \in U$, $y , r , s \in R$ (15)

Also , left multiplying of (14) by s , to get :

$srx(d - g)(y) - sx(d - g)(ry) = 0$, for all $x \in U$, $y,r,s \in R$ (16)

Subtracting (16) from (15) ,we get :

$[r,s] x (d - g)(y) = 0$, for all $x \in U$,$y,r,s \in R$ (17)

Since R has a commutator left nonzero divisor , we get :

$x(d - g)(y) = 0$, for all $x \in U$, $y \in R$. (18)

That is :

$xd(y) = xg(y)$, for all $x \in U$,$y \in R$ (19)

And hence d and g are (U,R)-derivations .

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