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A NOTE ON SOLVABILITY OF FINITE GROUPS

Rola. A. Hijazi

Department of Mathematics, Faculty of Science 14466, King Abdulaziz University, Jeddah 21424, Saudi Arabia

Abstract. Let G be a finite group. A subgroup H of G is said to be c -normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the largest normal subgroup of G contained in H. In this note we prove that if every Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are c-normal (S-permutable) in G, then G is solvable. This results improve and extend classical and recent results in the literature.

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1 INTRODUCTION

All groups considered in the sequel will be finite. Most of the notation is standard and can be found in Huppert [10].

The relationship between the properties of the Sylow subgroups of a group G and its structure has been investigated by a number of authors. In particular, Gaschütz and Itô [10, p. 436, Satz 5.7] proved that a group G is solvable if all its minimal subgroups are normal (a subgroup of prime order is called a minimal subgroup). Buckley [5] proved that a group of odd order is supersolvable if all its minimal subgroups are normal. Srinivasan [14] got the supersolvability of G under the assumption that the maximal subgroups of all Sylow subgroups are S-permutable in G (a subgroup which permutes with all Sylow subgroups of a group G is called S-permutable in G; see Kegel [11]). Recall that a subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H. This concept was introduced by Wang [15] in 1996 and has been studied extensively by many authors. In fact, Wang extended the above results by proving that a group G is supersolvable when all minimal subgroups and the cyclic subgroups of order 4 are c-normal in G or the maximal subgroups of all Sylow subgroups of G are c-normal in G. In 2000, Ballester-Bolinches et al. [4] introduced the concept of c-supplementation of a finite group which is weaker than c-normality. A subgroup H of a group G is said to be c-supplement in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. By using this concept, Ballester-Bolinches et al. [4] proved that a group G is solvable if and only if every Sylow subgroup of G is c-supplemented in G. Moreover, as applications, they proved that if all minimal subgroups and the cyclic subgroups of order 4 of a group G are c-supplemented in G, then G is supersolvable. In 2008, Asaad and Ramadan [2] dropped the assumption that every cyclic subgroup of order 4 is c-supplemented in G and proved that: If every minimal subgroup of G is c-supplemented in G, then G is solvable. In 2012, Asaad [1] achieved interesting results about the structure of the group G when certain subgroups of prime power orders are c-supplemented in G. In 2014, Heliel [9] continued the above mentioned studies and obtained results improved and generalized the results of Hall [7-8], Ballester-Bolinches and Guo [3], Ballester-Bolinches et al. [4] and Asaad and Ramadan [2] as follows:

Theorem A. If each subgroup of prime odd order of a group G is c-supplemented in G, then G is solvable.

Theorem B. Let G be a group. Then G is solvable if and only if every Sylow subgroup of odd order of G is c-supplemented in G.

In connection with the above two Theorems, the following conjecture is posed at the end of Heliel [9].

Conjecture. Let G be a finite group such that every non-cyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < |D| \leq |P|$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Is G solvable?

In the same year 2014, Li et al. [12] presented a counterexample to show that the answer of this conjecture is negative and also gave a generalization of Theorems A and B.

Based on the above mentiond results, the main goal of this note is to prove the following results:

Theorem C. Suppose that each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then G is solvable.

Theorem D. Suppose that each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are c-normal in G. Then G is solvable.

Remark. The research on c-normal subgroups has formed a sreies, which is similar to the series of S-permutable subgroups. However, the two series are independent of each other.

2 Proofs

First we give an improvement of Gaschütz and Itô result that was mentioned in the introduction as follows:

Theorem 3.1. Suppose that each Sylow subgroup P of a finite group G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are normal in G. Then G is solvable.

Proof. Assume that the result is false and let G be a counterexample of minimal order. If all minimal subgroups of G are normal in G, then G is solvable by Gaschütz and Itô result [10, p. 436, satz 5.7], a contratdiction. Thus there exists a subgroup L of G of prime order, say p, such that L is not normal in G. Let P be a Sylow p-subgroup of G such that $L \le P$. Then there exists a subgroup H of P such that $L \le H < P$ with $|H| = |D|$. By the hypothesis, H is normal in G and since L is not normal in G, we have $L < H < P$. Clearly, $\Phi(H)$ is characteristic in H and since $H \lhd G$, we have $\Phi(H) \lhd G$. If $\Phi(H) \ne 1$, then $G/\Phi(H)$ satisfies the hypothesis of the theorem and so $G/\Phi(H)$ is solvable by the minimal choice of G. Hence G is solvable as the class of solvable groups is a saturated formation, a contradiction. Thus $\Phi(H) = 1$ and H is elementary abelian p-group by [6, p. 174, **Theorem 1.3].** In fact, $|H| > p$ and so H is noncyclic. We argue that $|P/H| \ne p$. If not, $|P| = p|H|$ and P is noncyclic. Then P contains a subgroup N such that $|P : N| = p$ and $N \ne H$. By hypothesis, H and N are both normal in G and so $P = HN$ is normal in G. Then, by Schur-Zassenhaus Theorem [6, p. 221,

Theorem 1.2], there exists a subgroup K of G such that $G = PK$ and $P \cap K = 1$. But K is solvable by the minimal choice of G, then G is solvable, a contradiction. Thus $|P/H| = p^n$, where $n \ge 2$. Let L_1/H be a subgroup of P/H of order p. Then $|L_1| = p|H|$ and since L_1 is noncyclic as above, we have $L_1 \lhd G$ and so $L_1/H \lhd G/H$. Hence G/H is solvable by the minimal choice of G and so G is solvable, a final contradiction completing the proof of the theorem.0.3cm

Proof of Theorem C. Assume that the result is false and let G be a counterexample of minimal order. Then, by **Theorem 3.1**, there exists a subgroup H of P with $|H| = |D|$ such that H is not normal in G. By the hypothesis, H is S-permutable in G. By [13, Lemma A], $O^p(G) \le N_G(H)$ and since H is not normal in G, we have $N_G(H) < G$. Let M be a maximal subgroup of G such that $N_G(H) \le M < G$. Then M is normal in G and $|G/M| = p$ (recall that P is a Sylow p-subgroup of G). Clearly, $P \cap M$ is a Sylow p-subgroup of M and $H \le P \cap M$. Hence if $1 < H < P \cap M$, M satisfies the hypothesis of the Theorem and so M is solvable by the minimal choice of G and consequently so G is solvable, a contradiction. Thus we may assume that $H = P \cap M$, so $|P : H| = p$, that is, $H \lhd P$. Hence $G = \langle P, O^p(G) \rangle \le N_G(H) \le M < G$, a contradicition completing the proof of the Theorem.

Proof of Theorem D.

Assume that the result is false and let G be a counterexample of minimal order. Then, by Theorem 3.1, there exists a subgroup H of G such that $|H| = |D|$ and H is not normal in G. Without loss of generality we may assume that $H < P$, where P is a Sylow p-subgroup of G for some prime p dividing the order of G. Then, by the hypothesis, H is c-normal in G, that is, there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \le H_G$. As H is not normal in G, we have $H_G < H$. Hence if $H_G \ne 1$, G/H_G satisfies the hypothesis of the theorem by [15, Lemma 2.1], and so G/H_G is solvable and since H_G is of prime power order, it follows that G is solvable, a contradiction. Thus we may assume that $H_G = 1$. Since $G/K \cong H$ and $H < P$, where P is a Sylow p-subgroup of G, it follows that there exists a subgroup M of G such that $K \le M, M \lhd G$ and $|G/M| = p$. Clearly, $P \cap M$ is a Sylow p-subgroup of M. Hence if $|D| = |H| < |P \cap M|$, M satisfies the hypothesis of the theorem by [15, Lemma 2.1], and so M is solvable by the minimal choice of G. But $|G/M| = p$, that is, G/M is solvable, then G is solvable, a contradiction. So we may assume that $|P \cap M| \le |D|$. Then by the hypothesis, $|P \cap M| = |D|$. Set $P \cap M = L$. By [15, Lemma 2.1], L is c-normal in M, that is there exists a normal subgroup N of M such that $M = LN$ and $L \cap N \le L_M$. Hence if $L_M = 1$, N is a normal p'-Hall subgroup of M. Clearly, N is a p'-Hall subgroup of G and N satisfies the hypothesis of the thoerem and so N is solvable by the minimal choice of G. Then M is solvable and so G is solvable, a contradiction. Thus we may assume that $L_M \ne 1$. Hence if

$L_M = L = P \cap M \lhd M$, $M = LN$, $N \lhd M$ and $L \cap N = 1$ by Schur-Zassenhaus Theorem [6, p. 221, Theorem 1.2]. As above, N is solvable and so G is solvable, a contradiction. Thus $1 \neq L_M < L$. Now we consider the normal closure of L_M, that is, $L_M^G = < L_M^g : g \in G >$. Since $G = MH$, we have $L_M^g = L_M^{mh} = L_M^h \leq P$ (where $m \in M$ and $h \in H$) and so $L_M^G \leq P$. Hence if $L_M^G = P$, once again Schur-Zassenhaus Theorem implies that $G = PK$, $P \cap K = 1$ and K is solvable by the minimal choice of G and so G is solvable, a contradiction. Thus we may assume that $L_M^G < P$. Hence if $\left|L_M^G\right| < |D|$, G/L_M^G is solvable by the minimal choice of G and so G is solvable, a contradiction. Now we may assume that $\left|L_M^G\right| \geq |D|$. Since $|P \cap M| = |D|$ and $|P/P \cap M| = p$ and $L_M^G < P$, we should have $\left|L_M^G\right| = |D|$. Also, $L_M^G \neq P \cap M$ (otherwise, G is solvable, a contradiction). Then $G = L_M^G M$ and $L_M^G \cap M \lhd G$ and $\left|L_M^G \cap M\right| < |D|$. Hence $G/(L_M^G \cap M)$ is solvable by the minimal choice of G and so G is solvable, a final contradiction completing the proof of the theorem.

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