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Some methods for calculating limits

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ABSTRACT

The article presents some methods of calculating limits

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Below we present some methods for calculating the limits of the numerical sequence.

We recall the following definition.

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Definition. A function $f : N \to X$ whose domain of definition is the set of natural numbers is called a sequence.

The values $f(n)$ of the function are called the terms of the sequence. It is customary to denote them by a symbol for an element of the set into which the mapping goes, endowing each symbol with the corresponding index of the argument. Thus, $x_n = f(n)$. In this connection the sequence itself is denoted $\{x_n\}$ and also written as $x_1, x_2, \dots, x_n, \dots$ — It is called a sequence in X or a sequence of elements of X. The element x_n is called the nth term of the sequence. Throughout the next few sections we shall be considering only sequences $f : N \to R$ of real numbers.

A number $a \in R$ is called the limit of the sequence $\{x_n\}$ if for every $\varepsilon > 0$ there exists an index N such that $|x_n - a| < \varepsilon$ for all n > N. We now write these formulations of the definition of a limit in the language of symbolic logic, agreeing that the expression $\lim_{n\to\infty} x_n = a$ is to mean that $n \to \infty$ a is the limit of the sequence $\{x_n\}$.

Let us consider some examples.

Example 1. Let $a \in R,\ |a| > 1$. Prove that

$$\lim_{n\to\infty} \frac{1}{a^n} = 0.$$

Solution. Let $|a| = 1 + \delta$. Then $\delta = |a| - 1 > 0$ and $\forall n \in N$ by inequality Bernoulli's we obtain $(1+\delta)^n \geq 1 + n\delta > n\delta$ therefore

$$\frac{1}{|a|^n} < \frac{1}{n\delta}.$$

Thus $$\left|\frac{1}{a^n} - 0\right| = \frac{1}{a^n} < \varepsilon \ (\varepsilon > 0)$$

Inequality holds for all $$n > \frac{1}{\varepsilon\delta}.$$

If

$$n_0 = \left[\frac{1}{\varepsilon\delta}\right] + 1$$

then, for $\forall n > n_0$

$$\left|\frac{1}{a^n} - 0\right| < \varepsilon.$$

Thus $$\lim_{n\to\infty} \frac{1}{a^n} = 0.$$

Example 2. Let, $a \in R, |a| > 1$ и $\alpha \in R$. Prove that

$$\lim_{n\to\infty} \frac{n^\alpha}{a^n} = 0.$$

Solution. Suppose that for a number k holds the inequality $k \geq \alpha + 1$. Since $|a|^{\frac{1}{k}} > 1$ therefore, assuming that $|a|^{\frac{1}{k}} = 1 + \delta$, т. е. $\delta = |a|^{\frac{1}{k}} - 1 > 0$. Thenby inequality Bernoulli's $\forall n \in N$, we obtain $|a|^{\frac{n}{k}} = (1+\delta)^n \geq 1 + n\delta > n\delta$.

Hence $\frac{n^{k-1}}{a^n} < \frac{1}{n\delta^k}$

Let $n_0 = \left[\frac{1}{\delta^k \cdot \varepsilon}\right] + 1 \quad (\varepsilon > 0)$

For $\forall\ n > n_0$ we obtain

$$\left|\frac{n^\alpha}{a^n} - 0\right| = \frac{n^\alpha}{|a|^n} \leq \frac{n^{k-1}}{|n|^n} < \varepsilon.$$

Thus $\lim_{n\to\infty} \frac{n^\alpha}{a^n} = 0$.

Example 3. Prove equality $\lim_{n\to\infty} \frac{\lg n}{n} = 0$.

Solution. Since for $\forall\ \varepsilon > 0$ and $\forall\ n \in N$ we have

$$0 \leq \frac{\lg n}{n} < \varepsilon \iff \lg n < n\varepsilon \iff n < 10^{n\varepsilon} \iff \frac{n}{(10^\varepsilon)^n} < 1.$$

Noting that $10^\varepsilon > 1$ and using the Example 2 we obtain

$$\frac{n}{(10^\varepsilon)^n} \to 0 \ npu\, n \to \infty \text{ and } \exists\ n_0 \in N,\ \forall\ n > n_0 :\ \frac{n}{(10^\varepsilon)^n} < 1.$$

And so, for $\forall\ n > n_0$ $\frac{\lg n}{n} < \varepsilon$. Hence, $\lim_{n\to\infty} \frac{\lg n}{n} = 0$.

Example 4. Take limit $\lim_{n\to\infty}\left(\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}\right)$.

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*Solution.*Assuming that $S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + ... + \frac{2n-1}{2^n}$. Then

$$S_n - \frac{1}{2}S_n = \frac{1}{2} + (\frac{3}{2^2} - \frac{1}{2^2}) + (\frac{5}{2^3} - \frac{3}{2^3}) + ... + (\frac{2n-1}{2^n} - \frac{2n-3}{2^n}) - \frac{2n-1}{2^{n+1}} =$$

$$\frac{1}{2} + (\frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}) - \frac{2n-1}{2^{n+1}}, S_n = 1 + 1 + \frac{1}{2} + ... + \frac{1}{2^n} - \frac{2n-1}{2^n} = 1 + \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} - \frac{2n-1}{2^n}.$$

Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 + \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} - \frac{2n-1}{2^n}) = \lim_{n\to\infty}(1 + 2 - \frac{1}{2^{n-2}} - \frac{2n-1}{2^n}) = \lim_{n\to\infty} 3 - \lim_{n\to\infty}\frac{1}{2^{n-2}} -$$

$$2\lim_{n\to\infty}\frac{n}{2^n} + \lim_{n\to\infty}\frac{1}{2^n} = 3.$$

Since

$$\left|\frac{n}{2^n}\right| = \frac{n}{(1+1)^n} = \frac{n}{1 + n + \frac{n(n-1)}{2} + ... + 1} < \frac{n}{\frac{n(n-1)}{2}} = \frac{2}{n-1} < \varepsilon$$

for an arbitrary $\varepsilon > 0$, if $n > 1 + \frac{2}{\varepsilon}$, i.e. $\lim_{n\to\infty}\frac{n}{2^n} = 0$.

Example 5.Take limit $\lim_{n\to\infty}\frac{1^2 + 3^2 + ... + (2n-1)^2}{2^2 + 4^2 + ... + (2n)^2}$.

*Solution.*We have $2^2 + 4^2 + ... + (2n)^2 = 4(1^2 + 2^2 + ... + n^2) = \frac{2n(n+1)(2n+1)}{3}$,

$1^2 + 2^2 + ... + (2n-1)^2 + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$. Subtracting the second equation from the first, we obtain $1^2 + 3^2 + ... + (2n-1)^2 = \frac{n(2n+1)(4n+1)}{3}$ - $\frac{2n(n+1)(2n+1)}{3}$ = $\frac{n(4n^2-1)}{3}$. Thus

$$\lim_{n\to\infty}\frac{1^2 + 3^2 + ... + (2n-1)^2}{2^2 + 4^2 + ... + (2n)^2} = \lim_{n\to\infty}\frac{n(4n^2-1)}{2n(n+1)(2n+1)} = 1.$$

Example 6. Prove that if the sequence $\{a_n\}$ converges, then the sequence of arithmetic means $\{\xi_n\}$, where $\xi_n = \frac{a_1 + a_2 + ... + a_n}{n}$ also converges and $\lim_{n\to\infty}\xi_n = \lim_{n\to\infty} a_n$.

*Solution.*We use Theorem Stolz: i.e.if

$a)\forall n \in N, y_{n+1} > y_n, b)\lim_{n\to\infty} y_n = +\infty, c)\exists\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}}$ then $\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}}$.

Putting $x_n = a_1 + a_2 + ... + a_n$ и $y_n = n$ we obtain

$$\lim_{n\to\infty}\xi_n = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} a_n.$$

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