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Consistency of least squares estimators of AR(2) Model

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Abstract

In this paper, ordinary least squares (OLS) method will be used to estimate the parameters of the auto-regressive model without constant of order two. Moreover, the convergence in probability (the consistency property) of the estimates is proved.

Keywords and phrases: Time series; AR(2) model; consistency; ordinary least squares; probability limit; normal approximation.

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INTRODUCTION

Mann and Wald (1943) showed that if a temporal stochastic process described by linear difference equations is damped without disturbances, it is stable (the expectations of the squares of all variables are uniformly bounded) with disturbances and the "maximum likelihood" estimates of the parameters involved are consistent. However, a system which is stable or explosive without disturbances is explosive with them. Rubin (1950) proved consistency in a simple example of the explosive case.

He considered the first order autoregressive model without constant as:

$$y_t = \rho y_{t-1} + \varepsilon_t , \qquad (1)$$

Where, ρ is a real number and ε_t are real stochastic variables independently distributed with mean zero and variance σ^2, and y_0 is a given real number. The results derived here hold equally well if ρ, ε_t and y_0 are complex numbers, quaternions, or Cayley numbers. The maximum likelihood estimate $\hat{\rho}$ is defined as:

$$\hat{\rho} = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_t^2} , \qquad (2)$$

In case of complex number, quaternions, or Cayley number, equation (2) will take the form:

$$\hat{\rho} = \frac{\sum_{t=1}^{n} y_t \tilde{y}_{t-1}}{\sum_{t=1}^{n} y_{t-1} \tilde{y}_{t-1}} , \qquad (3)$$

Where, $\tilde{y}$ denotes the conjugate of y and showed that $\lim \hat{\rho} = \rho$

White (1958, 1959) showed that, for $|\rho| > 1$ and ε_t NID$(0, \sigma^2)$, the distribution of $\hat{\rho}$ normalized by a function of n has a Cauchy distribution in the limit. His result can also be used to demonstrate that $(\hat{\rho} - 1)$ normalized by $\left(\sum_{t=2}^{n} y_{t-1}^2\right)^{\frac{1}{2}}$ has normal distribution in the limit although white obtained the limit moment generating functions for the case $|\rho| < 1$, $|\rho| = 1$ and $|\rho| > 1$. Anderson and Hsiao (1982) presented several estimation methods for various regression models with autoregressive of order one covariance structure using maximum likelihood method. Blundell and Bond (1998) proposed an alternative Generalized Method of Moments (GMM) estimator that imposed a restriction on the initial conditions y_{i1}. Lung and Jihai (2010), established asymptotic properties of quasi-maximum likelihood estimators for spatial autoregressive (SAR) panel data models with fixed effects and SAR disturbances. Youssef, et al (2011), suggested an OLS estimator for AR (2) with constant term and the properties of the estimated parameters of AR (2), have been studied. Also, closed form of the variance of the estimated parameters has been derived. El-Sayed, et al (2014), suggested an OLS estimator for AR (2) with and without constant in penal date. In this article, the estimators of AR (2) model without constant have been obtained and the consistency of the estimators has been proved.

OLS Estimators of AR(2) Model

Consider the second order auto-regressive process $\{y_t$, t = 3, …,n $\}$ be defined as

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t , \qquad (4)$$

Where, ε_t is a sequence of independent identically distributed random variables with mean zero and variance σ^2. The values of ρ_1, ρ_2 and the form of y_t will determine the nature of the time series. The least square estimator for (ρ_1, ρ_2), can be obtained by minimized the error sum of squares as:

$$S_2(\rho_1,\rho_2) = \sum_{t=3}^{n} \varepsilon_t^2 = \sum_{t=3}^{n} (y_t - \rho_1 y_{t-1} - \rho_2 y_{t-2})^2, \qquad (5)$$

By equating the partial derivatives with respect to $(\rho_1 \text{ and } \rho_2)$ of equation (5), the following estimators will be obtained:

$$\hat{\rho}_1 = \frac{\sum_{t=3}^{n} y_t y_{t-1} \sum_{t=3}^{n} y_{t-2}^2 - \sum_{t=3}^{n} y_t y_{t-2} \sum_{t=3}^{n} y_{t-1} y_{t-2}}{\sum_{t=3}^{n} y_{t-2}^2 \sum_{t=3}^{n} y_{t-1}^2 - (\sum_{t=3}^{n} y_{t-1} y_{t-2})^2}, \qquad (6)$$

and $\hat{\rho}_2$ will take the form:

$$\hat{\rho}_2 = \frac{\sum_{t=3}^{n} y_t y_{t-2} \sum_{t=3}^{n} y_{t-1}^2 - \sum_{t=3}^{n} y_t y_{t-1} \sum_{t=3}^{n} y_{t-1} y_{t-2}}{\sum_{t=3}^{n} y_{t-2}^2 \sum_{t=3}^{n} y_{t-1}^2 - (\sum_{t=3}^{n} y_{t-1} y_{t-2})^2}, \qquad (7)$$

Consistency $\hat{\rho}_1$ and $\hat{\rho}_2$

Based on the OLS estimators of ρ_1 and ρ_2 of equations (6) and (7) , the estimate $\hat{\rho}_1$ can be rewritten as

$$\hat{\rho}_i = \rho_i + \frac{\sum_{t=3}^{n} c_{1t}\varepsilon_t - \sum_{t=3}^{n} c_{2t}\varepsilon_t \sum_{t=3}^{n} c_{1t} y_{t-2}}{1 - [\sum_{t=3}^{n} c_{2t} y_{t-1} \cdot \sum_{t=3}^{n} c_{1t} y_{t-2}]}, \qquad (8)$$

Where,

$$c_{it} = \begin{cases} c_{1t} = \dfrac{y_{t-1}}{\sum_{t=3}^{n} y_{t-1}^2}, & i = 1 \\ c_{2t} = \dfrac{y_{t-2}}{\sum_{t=3}^{n} y_{t-2}^2}, & i = 2 \end{cases}$$

[Khalifa (2011)]

Lemma: The OLS estimates $\hat{\rho}_1$ and $\hat{\rho}_2$ are consistent as $n \to \infty$.

Proof

We shall prove the consistency by showing that n^{-1} time the numerator (denominator) converges to zero (constant) in probability respectively.

Assume that,

$$I = \sum_{t=3}^{n} c_{1t}\varepsilon_t = \frac{\sum_{t=3}^{n} y_{t-1}\varepsilon_t \Big/ n}{\sum_{t=3}^{n} y_{t-1}^{2} \Big/ n}$$

According to the assumptions of the model, it can be seen that $\mathrm{E}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}\varepsilon_t\right] = 0$

And since, $\mathrm{var}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}\varepsilon_t\right] = n^{-2}\sum_{t=3}^{n}\mathrm{var}(y_{t-1})\,\mathrm{var}(\varepsilon_t)$ (Amemiya (1985))

So, $\mathrm{var}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}\varepsilon_t\right] = \frac{(n-2)\sigma_y^2\,\sigma_\varepsilon^2}{n^2}$, (9)

Hence, $\left[n^{-1}\sum_{t=3}^{n} y_{t-1}\varepsilon_t\right] \xrightarrow{p} o$, (10)

On the other hand, $\mathrm{E}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}^{2}\right] = (1-\tfrac{2}{n})\sigma_y^2$

Hence, $\lim_{n\to\infty}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}^{2}\right] = \sigma_y^2$, (11)

It can be proved that, $\mathrm{var}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}^{2}\right] = \frac{(n-2)\sigma_y^2}{n^2}$, (12)

Therefore, $n^{-1}\sum_{t=3}^{n} y_{t-1}^{2} \xrightarrow{p} \sigma_y^2$ *or* $p\lim(n^{-1}\sum_{t=3}^{n} y_{t-1}^{2}) = \sigma_y^2$, (13)

$\Rightarrow p\lim(Part\ \mathrm{I}) = 0$

By the same procedures, it can be easily prove that $p\lim(Part\ \mathrm{II}) = 0$, where

$$Part\ \mathrm{II} = \sum_{t=3}^{n} c_{2t}\varepsilon_t \sum_{t=3}^{n} c_{1t}y_{t-2},$$

Now, assume that, $III = \sum_{t=3}^{n} c_{1t}y_{t-2} = \frac{\sum_{t=3}^{n} y_{t-1}y_{t-2} \Big/ n}{\sum_{t=3}^{n} y_{t-1}^{2} \Big/ n}$

By taking the expectation of the nominator the following result will be obtained

$$\mathrm{E}\left[n^{-1}\sum_{t=3}^{n} y_{t-1}y_{t-2}\right] = (1\text{-}\tfrac{2}{n})\sigma_{yy}$$

Hence, $\lim_{n\to\infty}\left[n^{-1}\sum_{t=3}^{n} y_{t-1} y_{t-2}\right] = \sigma_{yy}$, (14)

Since, $\operatorname{var}\left[n^{-1}\sum_{t=3}^{n} y_{t-1} y_{t-2}\right] = \frac{(n-2)V_{n(yy)}}{n^2}$, where $V_{n(yy)}$ is a finite constant

Therefore, $\lim_{n\to\infty}\left[\operatorname{var}(n^{-1}\sum_{t=3}^{n} y_{t-1} y_{t-2})\right] = 0$, (15)

Based on equations (13) and (14), we find that,

$$\frac{p\lim(n^{-1}\sum_{t=3}^{n} y_{t-1} y_{t-2})}{p\lim(n^{-1}\sum_{t=3}^{n} y_{t-1}^{2})} = \frac{\sigma_{yy}}{\sigma_y^2}, \quad (16)$$

By taking the $p\lim$ of equation (8), the consistency of $\hat{\rho}_1$ will be reached i.e.

$$\hat{\rho}_1 \xrightarrow{p} \rho_1$$

Similarly, by the same way, it can be proved that $\hat{\rho}_2$ is consistent estimator of ρ_2 .

REFERENCES

1- Amemiya,T. 1985. Advanced Econometrics, United States of America

2- Anderson,T.W. and Hsiao,C. 1982. Formulation and Estimation of Dynamic Models Using Panel Data. Journal of Econometrica, 18,47-82

3- Lung-fei Lee and Jihai Yu. 2010. Estimation of Spatial Autoregressive Panel Data Models with Fixed Effect. Journal of Econometrics, 154(2), 165-185.

4- Mann, H. B. and Wald, A. 1943. On the Statistical Treatment of the Linear Stochastic Difference Equations. Econometrica, 11, 173-220.

5- Rubin, Herman 1950. Consistency of Maximum-Likelihood Estimates in the Explosive Case, in Statistical Inference in Dynamic Economic Models, ed. T. C. Koopmans, New York: John Wiley and Sons.

6- White, J.S. 1958. Obtained the Limiting Joint Moment Generating Function for The Properly Normalized Numerator. The Annals of Mathematical Statistics, 29, 1188-1197.

7- White, J.S. 1959. Asymptotic Expansions for the Mean and Variance of Serial Correlation Coefficient. Biometrika, 48, 85-94.

Khalifa, M. A. 2011. "Some properties of the unit root estimators". Unpublished MSc Thesis, Institute of Statistical Studies and Research, Cairo University

8- El-Sayed, S. M, El-Sheikh, A and Khalifa, M, A. 2014. new estimators of AR (2) panel data model. Advances and Applications in Statistics, 41, 129-136

9- Youssef, A. H, El-Sheikh, A and Khalifa, M, A. 2011. Inference of the Second Order Autoregressive Model with Unit Roots. International Mathematical Forum, 6(52), 2595 - 2604