ISSN 2347-1921

THE FUNDAMENTAL SOLUTIONS OF DIFFERENTIAL OPERATOR WITH BLOCK - TRIANGULAR OPERATOR COEFFICIENTS

A. Kholkin

Department of higher and applied mathematics, faculty of information technologies,

Pryazovkyi State Technical University, Mariupol, Ukraine

a.kholkin@gmail.com

ABSTRACT

For the non-selfadjoint Sturm-Liouville equation with a block-triangular operator potential that increases at infinity, both increasing and decreasing at infinity operator solutions are found. The asymptotics of these solutions at infinity is defined.

Keywords: differential operators; block-triangular operator coefficients

Subject Classification: 34K11, 47A10

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol.10, No.6

www.cirjam.com , editorjam@gmail.com

INTRODUCTION

In the study of the connection between spectral and oscillation properties of non-selfadjoint differential operators with block-triangular operator coefficients there is a question on about sufficient terms, at that the discrete spectrum of non-selfadjoint operator is real and coincides with the union of spectra of the self-adjoint operators answering diagonal coefficients. For an operator with a triangular matrix potential decaying at infinity which first moment is bounded, due to the inverse scattering problem, the spectral structure was established in [1], [2]. For at operator with block-triangular matrix potential that increases at infinity these questions are considered in paper [3]. Two matrix solutions were built for this purpose, one of that is decreasing at infinity, and the second growing.

In this paper we construct the fundamental system of solutions of differential equation with block-triangular operator potential that increases at infinity.

BASIC CONCEPTS AND SUPPOSITIONS

Denote by $H_k,\ k=\overline{1,r}$ finite-dimensional or infinite-dimensional separable Hilbert space. We will put $\mathbf{H}=H_1\oplus H_2\oplus\ldots\oplus H_r$. Element $\overline{h}\in\mathbf{H}$ will be written in the form $\overline{h}=\mathrm{col}\left(\overline{h}_1,\overline{h}_2,\ldots,\overline{h}_r\right)$, where $\overline{h}_k\in H_k$, $k=\overline{1,r}$, I_k, I - identity operators in H_k and $\mathbf{H}$ accordingly.

Consider the equation with block-triangular operator potential

$$l\left[\overline{y}\right]=-\overline{y}''+V(x)\overline{y}=\lambda\overline{y},\quad 0\le x<\infty, \tag{1}$$

where

$$V(x)=v(x)\cdot I+U(x),\ U(x)=\begin{pmatrix} U_{11}(x) & U_{12}(x) & \ldots & U_{1r}(x)\\ 0 & U_{22}(x) & \ldots & U_{2r}(x)\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & U_{rr}(x)\end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function such that $0<v(x)\to\infty$ monotonically, as $x\to\infty$, and it has monotone absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation, e. g. $\left|U(x)\right|\cdot v^{-1}(x)\to 0$ as $x\to\infty$ or $\left|U\right|v^{-1}\in L^{\infty}\left(\mathbb{R}_+\right)$. The diagonal blocks $U_{kk}(x),\ k=\overline{1,r}$ are assumed bounded self-adjoint operators in H_k.

Let $v(x)=x^{2\alpha}$, $0<\alpha\le 1$ and coefficients of equation (1) satisfy the condition

$$\int_a^{\infty}\left|U(t)\right|\cdot t^{-\alpha}dt<\infty,\ \ a>0. \tag{3}$$

Rewrite equation (1) in the form

$$-\overline{y}''+\left(x^{2\alpha}-\lambda+q(x,\lambda)\right)\overline{y}=\left(q(x,\lambda)\cdot I-U(x)\right)\overline{y}, \tag{4}$$

where $q(x,\lambda)$ determined by a formula (cf. with the monograph of E. Ch. Titchmarsh [4])

$$q(x,\lambda)=\frac{5\alpha^2}{4}\left(\frac{x^{2\alpha-1}}{x^{2\alpha}-\lambda}\right)^2-\frac{\alpha(2\alpha-1)x^{2\alpha-2}}{2\left(x^{2\alpha}-\lambda\right)}. \tag{5}$$

Denote

$$\gamma_0(x,\lambda)=\frac{1}{\sqrt[4]{4\left(x^{2\alpha}-\lambda\right)}}\cdot\exp\left(-\int_a^x\sqrt{u^{2\alpha}-\lambda}du\right)$$

$$\gamma_\infty(x,\lambda)=\frac{1}{\sqrt[4]{4\left(x^{2\alpha}-\lambda\right)}}\cdot\exp\left(\int_a^x\sqrt{u^{2\alpha}-\lambda}du\right)$$

There solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z''+\left(x^{2\alpha}-\lambda+q(x,\lambda)\right)z=0\,, \tag{6}$$

in such a way that for all $x\in[0,\infty)$ one has

$$W(\gamma_0,\gamma_\infty):=\gamma_0(x,\lambda)\cdot\gamma_\infty'(x,\lambda)-\gamma_0'(x,\lambda)\cdot\gamma_\infty(x,\lambda)=1.$$

In [3], there was established the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ as $x\to\infty$.

THEOREM

Under condition (3) equation (1) has a unique decreasing at infinity operator solution $\Phi(x,\lambda)\in B(\mathbf{H})$, satisfying the conditions

$$\lim_{x\to\infty}\frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)}=I\,,\qquad \text{and}\qquad \lim_{x\to\infty}\frac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)}=I\,. \tag{7}$$

Also, there exists increasing at infinity operator solution $\Psi(x,\lambda)\in B(\mathbf{H})$, satisfying the conditions

$$\lim_{x\to\infty}\frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)}=I\,,\qquad \text{and}\qquad \lim_{x\to\infty}\frac{\Psi'(x,\lambda)}{\gamma_\infty'(x,\lambda)}=I\,. \tag{8}$$

PROOF OF THEOREM

1. Equation (4) equivalently to integral equation

$$\Phi(x,\lambda)=\gamma_0(x,\lambda)I+\int_x^\infty K(x,t,\lambda)\cdot\Phi(t,\lambda)dt\,, \tag{9}$$

where

$$K(x,t,\lambda)=C(x,t,\lambda)\cdot\left[q(t,\lambda)I-U(t)\right] \tag{10}$$

$$C(x,t,\lambda)=\gamma_\infty(x,\lambda)\cdot\gamma_0(t,\lambda)-\gamma_\infty(t,\lambda)\cdot\gamma_0(x,\lambda)\,, \tag{11}$$

with $C(x,t,\lambda)$ being the Cauchy function that in each variable satisfies equation (6) and initial conditions

$$C(x,t,\lambda)\big|_{x=t}=0,\quad C_x'(x,t,\lambda)\big|_{x=t}=1,\ C_t'(x,t,\lambda)\big|_{x=t}=-1\,.$$

Set $\chi(x,\lambda)=\dfrac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)}$ to rewrite equation (9) in form

$$\chi(x,\lambda)=I+\int_x^\infty R(x,t,\lambda)\chi(t,\lambda)dt\,, \tag{12}$$

where $R(x,t,\lambda)=K(x,t,\lambda)\cdot\dfrac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}$. Thus

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right|=\left|\gamma_0^2(t,\lambda)\cdot\frac{\gamma_\infty(x,\lambda)}{\gamma_0(x,\lambda)}-\gamma_0(t,\lambda)\cdot\gamma_\infty(t,\lambda)\right|=$$

$$=\left|\frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot\exp\left(-2\int_a^t\sqrt{u^{2\alpha}-\lambda}du\right)\cdot\exp\left(2\int_a^x\sqrt{u^{2\alpha}-\lambda}du\right)-\frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\right|=$$

$$=\frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot\left|\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du-1\right)\right|.$$

As $x\le t$, one has $\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du\right)\le 1$, and that is why

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right|\le\frac{1}{\sqrt{t^{2\alpha}-\lambda}}. \tag{13}$$

Hence

$$|R(x,t,\lambda)|=\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\cdot\left[q(t,\lambda)\cdot I-U(t)\right]\right|\le\frac{1}{\sqrt{t^{2\alpha}-\lambda}}\left(|q(t,\lambda)|+|U(t)|\right).$$

By virtue of (3) and (5),

$$\frac{1}{\sqrt{t^{2\alpha}-\lambda}}\left(|q(t,\lambda)|+|U(t)|\right)\in L(a,\infty), \tag{14}$$

and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)|\le\text{const}$. By (12), one has that $\lim_{x\to\infty}\chi(x,\lambda)=I$, where the first part of formula (7) follows from.

Differentiable (9) to get $\dfrac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)}=I+\int_x^\infty S(x,t,\lambda)\chi(t,\lambda)dt$, where

$$S(x,t,\lambda)=K_x'(x,t,\lambda)\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}=C_x'(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}\cdot\left[q(t,\lambda)\cdot I-U(t)\right].$$

We have similarly (13), that $\left|C_x'(x,t,\lambda)\cdot\dfrac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right|\le\dfrac{1}{\sqrt{t^{2\alpha}-\lambda}}$, and therefore

$$|S(x,t,\lambda)|\le\frac{1}{\sqrt{t^{2\alpha}-\lambda}}\left(|q(t,\lambda)|+|U(t)|\right)\in L(a,\infty),$$

where the second part of formula (7) follows from.

2. Denote by

$$\hat{\Psi}(x,\lambda)=\begin{pmatrix}\Psi_{11}(x,\lambda) & \Psi_{12}(x,\lambda) & \dots & \Psi_{1r}(x,\lambda)\\ 0 & \Psi_{22}(x,\lambda) & \dots & \Psi_{2r}(x,\lambda)\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \Psi_{rr}(x,\lambda)\end{pmatrix}$$

the operator solution of equation (1) that increases at infinity. The diagonal blocks $\Psi_{kk}(x)\in B(H_k,H_k)$ are the bounded operators. Now equation (4) is equivalent to the integral equation

$$\hat{\Psi}(x,\lambda)=\gamma_\infty(x,\lambda)\cdot I-\int_0^x K(x,t,\lambda)\cdot\hat{\Psi}(t,\lambda)dt, \tag{15}$$

where, just as in (9), the kernel $K(x,t,\lambda)$ is given by (10). Now set $\chi(x,\lambda)=\dfrac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}$ to rewrite equation (15) in form

$$\chi(x,\lambda)=I-\int_0^x R(x,t,\lambda)\cdot\chi(t,\lambda)dt, \tag{16}$$

where $R(x,t,\lambda)=C(x,t,\lambda)\cdot\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\cdot\left[q(t,\lambda)\cdot I-U(t)\right]$. Proved similarly (13), that at $t\le x$ $\left|C(x,t,\lambda)\cdot\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\right|\le\frac{1}{\sqrt{t^{2\alpha}-\lambda}}$, and therefore by virtue of (3) and (5) $\left|R(x,t,\lambda)\right|\in L(a,\infty)$, consequently integral equation has a unique solution $\chi(x,\lambda)$ and $\left|\chi(x,\lambda)\right|\le\text{const}$. Pass in (16) to a limit as $x\to\infty$ to get $\lim_{x\to\infty}\chi(x,\lambda)=I+\tilde{C}(\lambda)$, where $\tilde{C}(\lambda)$ is f block-triangular operator in $\mathbf{H}$, that is

$$\lim_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}=I+\tilde{C}(\lambda). \tag{17}$$

Now consider another operator solution that increases at infinity

$$\tilde{\Psi}(x,\lambda)=\begin{pmatrix}\tilde{\Psi}_{11}(x,\lambda) & \tilde{\Psi}_{12}(x,\lambda) & \dots & \tilde{\Psi}_{1r}(x,\lambda)\\ 0 & \tilde{\Psi}_{22}(x,\lambda) & \dots & \tilde{\Psi}_{2r}(x,\lambda)\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \tilde{\Psi}_{rr}(x,\lambda)\end{pmatrix},$$

where

$$\tilde{\Psi}_{kk}(x,\lambda)=\Phi_{kk}(x,\lambda)\int_a^x\Phi_{kk}^{-1}(t,\lambda)\left(\Phi_{kk}^*(t,\lambda)\right)^{-1}dt,\ \ k=\overline{1,r},\ (a\ge 0),$$

$\Phi_{kk}(x,\lambda)$ are the diagonal blocks of operator solution $\Phi(x,\lambda)$ as in Section1.

In view of (7) and definitions of $\gamma_0(x,\lambda)$, $\gamma_\infty(x,\lambda)$, one has

$$\tilde{\Psi}_{kk}(x,\lambda)=\gamma_0(x,\lambda)\left(I_k+o_x(I_k)\right)\int_0^x\frac{I_k+o_t(I_k)}{\gamma_0^2(t,\lambda)}dt=\gamma_0(x,\lambda)\int_0^x\frac{dt}{\gamma_0^2(t,\lambda)}\left(I_k+o(I_k)\right)=$$

$$=\gamma_\infty(x,\lambda)\cdot\left(I_k+o(I_k)\right) \tag{18}$$

Hence

$$\lim_{x\to\infty}\frac{\tilde{\Psi}_{kk}(x,\lambda)}{\gamma_\infty(x,\lambda)}=I_k,\quad k=\overline{1,r}\ . \tag{19}$$

Since $\hat{\Psi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$ are the operator solutions of equation (10) that increase at infinity,

$$\hat{\Psi}(x,\lambda)=\tilde{\Psi}(x,\lambda)+\Phi(x,\lambda)\cdot C_0(\lambda), \tag{20}$$

where $C_0(\lambda)$ is some block-triangular operator. Thus

$$\lim_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}=\lim\frac{\tilde{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}.$$

Hence, by virtue of (19),

$$\lim_{x\to\infty}\frac{\Psi_{kk}(x,\lambda)}{\gamma_\infty(x,\lambda)}=I_k,\ k=\overline{1,m}$$

and in (17) one has

$$\tilde{C}(\lambda)=\begin{pmatrix} 0 & C_{12}(\lambda) & \dots & C_{1r}(\lambda) \\ 0 & 0 & \dots & C_{2r}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The solution $\Psi(x,\lambda)$ given by $\Psi(x,\lambda)=\hat{\Psi}(x,\lambda)\left(I+\tilde{C}(\lambda)\right)^{-1}$ is subject to first from condition (8).

Use (7) to differentiate (20), then find the asymptotes of $\Psi'(x,\lambda)$ as $x\to\infty$ similarly to (18) to obtain the second part of formula (8). Theorem is proved**.**

CONCLUSION

Corollary: If $\alpha=1$, *i.e. the coefficient* $v(x)=x^2$, *then, under the condition (3), theorem holds true for functions* $\gamma_0(x,\lambda)=x^{\frac{\lambda-1}{2}}\cdot\exp\left(-\frac{x^2}{2}\right)$, $\gamma_\infty(x,\lambda)=x^{-\frac{\lambda+1}{2}}\cdot\exp\left(\frac{x^2}{2}\right)$. *If* $\alpha=\frac{1}{2}$, *i.e. the coefficient* $v(x)=x$, *and the condition (3) holds, then* $\gamma_0(x,\lambda)=x^{-\frac{1}{4}}\cdot\exp\left(-\frac{2}{3}x^{\frac{3}{2}}+\lambda x^{\frac{1}{2}}\right)$, $\gamma_\infty(x,\lambda)=x^{-\frac{1}{4}}\cdot\exp\left(\frac{2}{3}x^{\frac{3}{2}}-\lambda x^{\frac{1}{2}}\right)$.

Remark: It is known that scalar equation

$$-\varphi''+x^2\cdot\varphi=\lambda\varphi \tag{21}$$

for $\lambda=2n+1$ has the solution $\varphi_n(x)=H_n(x)\cdot exp\left(-\frac{x^2}{2}\right)$, where $H_n(x)$ is the Chebyshev – Hermitre polynomial, that at $x\to\infty$ has next asymptotics $H_n(x)=(2x)^n\left(1+o(1)\right)$. Hence the solution $\varphi_n(x)$ of the equation (21) for $\lambda=2n+1$ will have the following asymptotics at infinity: $\varphi_n(x)=(2x)^n\cdot exp\left(-\frac{x^2}{2}\right)\cdot\left(1+o(1)\right)$.

In the case of $U(x)=0, v(x)=x^2$ in (2), the equation (1) is splitting into infinity system scalar equations of the form (21). The operator solution $\Phi(x,\lambda)$ will be diagonal in this case. Denote by $\varphi(x,\lambda)$ the diagonal elements of the operator $\Phi(x,\lambda)$. Then, by Corollary, the solution $\varphi(x,\lambda)$ will have the following asymptotics at infinity: $\varphi(x,\lambda)=(x)^{\frac{\lambda-1}{2}}\cdot\exp\left(-\frac{x^2}{2}\right)\left(1+o(1)\right)$. In particular, for $\lambda=2n+1$, this yields the solution proportional to $\varphi_n(x)$.

REFERENCE

[1] Rofe-Beketov F. S. and Zubkova E. I. (2011) Inverse scattering problem on the axis for the triangular 2×2 matrix potential a system with or without a virtual level. Azerbaijan Journal of Mathematic, **1** (2). 3-69.

[2] Bondarenko E.I. and Rofe-Beketov F. S. (2003) Inverse scattering problem on semiaxis for a system with a triangular matrix potential. Math. Physics, Analysis and Geometry, 10 (3), 412-424 (Russian).

[3] Kholkin A. M. and Rofe-Beketov F. S. (2014) On spectrum of differential operator with block-triangular matrix coefficients. Journal Mathematical Physics, Analysis, Geometry, **10** (1), 44-63.

[4] Titchmarsh E. Ch. (1958) Eigenfunction expansions associated with second – order differential equations, Vol. 2, Clarendon Press, Oxford.