ISSN 2347-1921

AN EXISTENCE THEOREM FOR QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT

A class of set-valued quasi-variational inequalities is studied in Banach spaces. The concept of QVI was earlier introduced by A. Bensoussan and J. L. Lions [4]. In this paper we give a generalization of the existence theorem du to Kano et al [11] by proving the existence of a fixed point of the variational selection.

Keywords

Quasi-variational inequalities; variational inequalities; pseudo-monotone; fixed point.

SUBJECT CLASSIFICATION

49J20, 90C51, 90C30.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol.10, No.6

www.cirjam.com , editorjam@gmail.com

INTRODUCTION

Let B is a reflexive Banach space and B^* is its topological dual. We assume that B has been renormed so that B and B^* are locally uniformly convex. We denote the duality pairing between B and B^* by $\langle \cdot \rangle$, whereas $\langle \cdot \rangle$ stands for the norm in B as well as the associated norm in B^*. Let $C \subset B$ be nonempty, closed, and convex set, and let $K : C \rightrightarrows C$ be a set-valued map such that for every $v \in C$, the set $K(v)$ is a nonempty, closed, and convex subset of C. Given a nonlinear operator F from B into B^* an element $f \in B^*$, the set-valued quasi-variational inequality (QVI) is formulated as a problem to find $u \in C$ such that $u \in K(u)$, and there exists $w \in F(u)$ satisfying the variational inequality

$$\langle w - f, v - u \rangle \geq \varphi(u) - \varphi(v), \forall v \in K(u) \tag{1}$$

The above QVI includes many important problems of interest as particular cases. For example, if F is single valued, then (1) recovers the folowing QVI: find $u \in C$ such that $u \in K(u)$, and

$$\langle F(u) - f, v - u \rangle \geq \varphi(u) - \varphi(v), \forall v \in K(u) \tag{2}$$

The above problem was introduced by Bensoussan and Lions [4] in connection with a problem of impulse control. A general treatment was made by Mosco [14]. If additionally $K(x) = C$ for every $x \in C$, then (1) recovers the following variational inequality: find $x \in C$ such that

$$\langle F(u) - f, v - u \rangle \geq \varphi(u) - \varphi(v), \forall v \in C. \tag{3}$$

Notice that if for every $x \in C$, $K(x)$ is a closed and convex cone with its apex at the origin and $f = 0$, then equation (1) collapses to the generalized complementarity problem:

Find $x \in C$ such that

$$x \in K(x), w \in F(x) \cap K^*(x), \langle w, x \rangle = 0 \tag{4}$$

where $K^*(x)$ denotes the positive polar of $K(x)$.

If additionally $K(x) \equiv C$, then (4) recovers the classical complementarity problem (see [7]).

QVIs turned out to be a powerful modeling tool capable of describing complex equilibrium situations that can appear in such different fields as generalized Nash games (see [3, 8, 10], mechanics (see [2, 5, 9], economics (see [10, 15]. We refer the reader to the monographs Mosco [14] and Baiochi and Capelo [2] for a more comprehensive analysis of QVIs.

The objective of this paper is to generalize the result in [12] to the case where $F : B \rightrightarrows B^*$ is the set-valued pseudo-monotone operator, $F(x) = \hat{F}(x, x)$, generated by a semi-monotone operator $\hat{F} : B \times B \rightrightarrows B^*$.

In such a case, our quasi-variational inequality is of the form of the equation (1):

Find $u \in K(u)$ such that for some $w \in F(u)$,

$$\langle w - f, v - u \rangle \geq \varphi(u) - \varphi(v), \quad \forall v \in K(u) \tag{5}$$

The technique that will be used to prove the existence of a solution of this QVI is to find fixed points of the associated variational selection (see [4, 1,14]).

The content of this paper will be organized as follows. Section 2 recalls the basic definitions and results for their later use in this work. The main result is given in section 3, it deals with an existence theorem for quasi-variational inequalities.

PRELIMINARIES

Throughout this paper, B is a reflexive Banach space and B^* is its topological dual. By J we denote the associated normalized duality map. We assume that B has been renormed so that B and B^* are locally uniformly convex. We denote the duality pairing between B and B^* by $\langle \cdot \rangle$, whereas $\|.\|$ stands for the norm in B as well as the associated norm in B^*. Let $C \subset B$ be nonempty, closed, and convex, and let $K : C \rightrightarrows C$ be a set-valued map such that for every $v \in C$, the set $K(v)$ is a nonempty, closed, and convex subset of C.

Let $\hat{F} : B \times B \rightrightarrows B^*$ be a given set-valued map, let $F : B \rightrightarrows B^*$ such that $F(x) = \hat{F}(x, x)$, let $\varphi : B \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be a given functional, and let $f \in B^*$. The domain and the graph of F are given by $D(F) := \{x \in B \mid F(x) \neq \varnothing\}$ and $G(F) := \{(x, y) \in B \times B^* : x \in D(F), y \in F(x)\}$, respectively. The strong convergence and the weak convergence in B as well as in B^* are specified by $\to$ and $\rightharpoonup$, respectively.

In this work, we study the following quasi-variational inequality:

Find $u \in K(u)$ such that for some $w \in F(u)$, we have

$$\langle w - f, v - u \rangle \geq \varphi(u) - \varphi(v), \quad \forall v \in K(u) \tag{6}$$

Definition 1: An operator $\hat{F} : B \times B \rightrightarrows B^*$ is called semimonotone, if $D(\hat{F}) = B \times B$ and the following conditions (SM1) and (SM2) are satisfied:

- (SM1) For any fixed $v \in B$ the mapping $u \to \hat{F}(v,u)$ is maximal monotone form $D(\hat{F}(v,)) = B$ into B^*.
- (SM2) Let u be any element of B and $\{v_n\}$ be any sequence in B such that $v_n \rightharpoonup v$ weakly in B.

Then, for every $u^* \in \hat{F}(v,u)$, there exists a sequence $\{u_n^*\}$ in B^* such that $u_n^* \in \hat{F}(v_n,u)$ and $u_n^* \to u^*$ in B^* as $n \to \infty$.

Theorem 1 :[13]

Let Z be a reflexive Banach space and let $C \subset Z$ be nonempty, convex, and closed. Assume that $\Psi : C \rightrightarrows C$ is a set-valued map such that for every $u \in C$, the set $\Psi(u)$ is nonempty, closed, and convex, and the graph of Ψ is sequentially weakly closed. Suppose that the set $\Psi(C)$ is bounded. Then the map Ψ has at least one fixed point in C.

Definition 2: Let $F : B \rightrightarrows B^*$ be a set-valued map. The map F is said to be:

- monotone, if $\langle u - v, x - y \rangle \geq 0$ for all $(x,u),(y,v) \in G(F)$,
- strictly monotone, if $\langle u - v, x - y \rangle > 0$ for all $(x,u),(y,v) \in G(F)$ with $x \neq y$,
- m-relaxed monotone, if $\langle u - v, x - y \rangle \geq m \| x - y \|$ for all $(x,u),(y,v) \in G(F)$, where $\mathrm{m} > 0$,
- maximal monotone, if the graph of the monotone map F is not included in the graph of any other monotone map with the same domain;
- coercive, if $\langle u, x \rangle \geq m(\| x \|) \| x \|$ for all $(x,u) \in G(F)$, where $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r \to \infty} m(r) = \infty$.

Definition 3: Let $F : B \rightrightarrows B^*$ be a set-valued map.

- The map F is called pseudo-monotone, if for any sequence $(x_n, w_n) \in G(F)$ such that $x_n \rightharpoonup x$ and $\limsup \langle w_n, x_n - x \rangle \leq 0$, it holds that for each $y \in B$, there exists $w(y) \in F(x)$ satisfying $\liminf \langle w_n, x_n - y \rangle \geq \langle w(y), x - y \rangle$
- The map F is called generalized pseudo-monotone,if for any $(x_n, w_n) \in G(F)$ with $x_n \rightharpoonup x$ and $w_n \rightharpoonup w$ such that $\limsup \langle w_n, x_n \rangle \leq \langle w, x \rangle$, we have $w \in F(x)$ and $\langle w_n, x_n \rangle \to \langle w, x \rangle$.
- The map F is said to possess S_+ property if for any sequence $(x_n, w_n) \in G(F)$ with $x_n \rightharpoonup x \in D(F)$ and $\limsup \langle w_n, x_n - x \rangle \to 0$, we have $x_n \to x$.

Definition 4: The map F is called M-continuous relative to φ, if the following conditions hold:

- (M1) For any sequence $x_n \subset C$ with $x_n \rightharpoonup x$, and for each $y \in K(x)$, there exists $\{y_n\}$ such that $y_n \in K(x_n), y_n \to y$ and $\varphi(y_n) \to \varphi(y)$.
- (M2) For $y_n \in K(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in K(x)$, which means that $\mathrm{G}(\mathrm{K})$ is sequentially weakly closed.

Lemme 1: Let Z be a reflexive Banach space with Z^* as its dual. Let $A : Z \rightrightarrows Z^*$ be a monotone map with $\bar{x} \in int(D(A))$. Then there exists a constant $r = r(\bar{x}) > 0$ such that for every $(x,w) \in G(A)$ and corresponding $c := \sup\{\| w' \| \| x' - \bar{x} \| \leq r\}$, and $w' \in A(x') < \infty$, we have

$$\langle w, x - \bar{x} \rangle \geq r \| w \| - (\| x - \bar{x} \| + r)c$$

Lemme 2: Let Z be a Banach space with Z^* as its dual and let $\{x_n\} \subset Z$. Suppose that there exists a sequence $\{s_n\} \subset \mathbb{R}_+$ with $s_n \downarrow 0$ such that for every $h \in Z^*$, there exists a constant C_h such that $\langle h, x_n \rangle \leq s_n \| x_n \| + C_h$, for every n. Then the sequence $\{\mathrm{x}_n\}$ is bounded.

MAIN RESULT

The main result of this paper is the existence Theorem for quasi-variational inequalities cited as follows:

Theorem 2: Assume that the following conditions hold:

$(A_{\hat{F}})$: $\hat{F}$ is a bounded semi-monotone operator.

(A_φ) : $\varphi : B \to \overline{\mathbb{R}}$ is a proper, convex, and lower-semicontinuous functional.

$(A_C) : C \subset int(D(\partial\varphi))$

(A_K): K is M-continuous relative to φ

$(A_{coer}) : \forall s \in B^*, \exists x_s \in \cap_{v\in C} K(v), \varphi(x_s) < \infty$ such that $\forall y \in \cup_{v\in C} K(v)$ with $\| y \|$ sufficiently large and $\forall w \in \cup_{v\in C} \hat{F}(v, y)$, we have:

$$\langle w - s, y - x_s \rangle + \varphi(y) \geq -\sigma(\| y \|) \| y \| \tag{7}$$

Then the set of solutions of the quasi-variational inequality (6) is nonempty and bounded.

Proof. We will divide the proof into several parts. Our objective is to show that the solution map $S : C \rightrightarrows C$ satisfies the assumptions imposed on the map Ψ in Theorem 1. However, instead of assuming that the underlying set C is bounded, we show below that $S(C)$ is bounded. We have to show that $G(S)$ is sequentially weakly closed. The proof is done in five steps.

Step I. For every $v \in C$, the set $S(v)$ is nonempty. Let $v \in C$ be arbitrary. We will show that there exists $x \in K(v)$ such that for some $w \in \hat{F}(v, x)$, we have

$$\langle w - f, z - x \rangle \geq \varphi(x) - \varphi(z), \forall z \in K(v) \tag{8}$$

Define a set-valued map $T : B \rightrightarrows B^*$ by $T(x) = \hat{F}(v, x) + N_{K(v)}(x) + \partial\varphi(x)$ where $N_{K(v)}$ is the normal map of $K(v)$. It is known that $N_{K(v)}$ is maximal monotone. Since

$$D(N_{K(v)}) \cap int(D(F) \cap int(D(\partial\varphi)) = K(v) \cap int(D(\partial\varphi)) \subset C \cap int(D(\partial\varphi)) \neq \emptyset$$

we notice that T is a maximal monotone map with $D(T) = K(v)$. Hence, we have $R(T + \varepsilon J) = B^*, \forall \varepsilon > 0$ and then for every $n \in N$, there exists $x_n \in D(T)$ such that $f \in T(x_n) + \varepsilon_n J(x_n)$, , where $\{\varepsilon_n\} \subset \mathbb{R}_+$ is such that $\varepsilon_n \downarrow 0$. Therefore, for some $w_n \in \hat{F}(v, x_n)$, $v_n \in N_{K(v)}(x_n)$, $u_n \in \partial\varphi(x_n)$ we have $f = w_n + v_n + u_n + \varepsilon_n J(x_n)$, which, due to the definitions of $N_{K(v)}(.)$ and $\partial\varphi(.)$, implies that

$$\langle w_n + \varepsilon_n J(x_n) - f, y - x_n \rangle + \geq \varphi(x_n - \varphi(y) \quad \text{for every } y \in K(v) \tag{9}$$

We claim that $\{x_n\}$ is bounded. Indeed, if this is not the case, then there exists a subsequence $\{x_n\}$ such that $\| x_n \| \to \infty$ as $n \to \infty$. In view of the above inequality, for every $y \in K(v)$, we have

$$\begin{aligned} \langle w_n - f, x_n - y \rangle + \varphi(x_n) &\leq \langle \varepsilon_n J(x_n), y - x_n \rangle + \varphi(y) \\ &\leq -\varepsilon_n \| x_n \| (\| x_n \| - \| y \|) + \varphi(y) \end{aligned}$$

where the second inequality follows from the properties of the duality map. Let $s \in B^*$ be arbitrary and take x_S provided by A_{coer} . By substituting $y = x_s$ in the above inequality, and using A_{coer} we obtain

$$\begin{aligned} -\sigma(\| x_n \|) \| x_n \| &\leq \langle w_n - s, x_n - x_s \rangle + \varphi(x_n) \\ &\leq -\langle s - f, x_n - x_s \rangle - \varepsilon_n \| x_n \| (\| x_n \| - \| x_s \|) + \varphi(x_s) \\ &\leq -\langle s - f, x_n - x_s \rangle + \varphi(x_s) \end{aligned}$$

because $\varepsilon_n \| x_n \| (\| x_n \| - \| x_s \|)$ is positive for $\| x_n \|$ sufficiently large. Therefore,

$$\langle s - f, x_n \rangle \leq \sigma(\| x_n \|) \| x_n \| + \langle s - f, x_s \rangle + \varphi(x_s)$$

hence Lemma 2 with $h := s - f, s_n := \sigma(\| x_n \|)$ and $C_s := \langle s - f, x_s \rangle + \varphi(x_s)$ ensures that $\{x_n\}$ is bounded. Due to the reflexivity of B, we extract a subsequence $\{x_n\}$ converging weakly to some x. The Minty formulation (see (10) below) of (9) reads $\langle w_z + \varepsilon_n J(z) - f, z - x_n \geq \varphi(x_n) - \varphi(z)$, for every $z \in K(\nu)$ and $w_z \in \hat{F}(\nu, z)$ and by invoking the Minty formulation once again, we obtain (8).

Step II. The Minty formulation holds. If $x \in K(\nu)$ satisfies (8), then it is a solution of the following Minty variational inequality and vice versa: for every $z \in K(\nu)$ and for every $u \in \hat{F}(\nu, z)$ we have

$$\langle u - f, z - x \rangle \geq \varphi(x) - \varphi(z) \tag{10}$$

The proof of the statement can be found in F. Giannessi and A. Khan [7].

Step III. The set $S(C)$ is bounded. This follows from the condition (7) in a similar way as in part (I).

Step IV. For every $\nu \in C$, $S(\nu)$ is closed and convex set. This is a direct consequence of (10) (see Giannessi and A. Khan [6]).

Step V. The graph of the variational selection S is sequentially weakly closed. Let $\{(\nu_n, y_n)\} \subset G(S)$ be such that $y_n \rightharpoonup y$ and $\nu_n \rightharpoonup \nu$. We will show that $(\nu, y) \in G(S)$. The set C being convex and closed is also weakly closed, and consequently $\nu \in C$. From the containment $\{(\nu_n, y_n)\} \in G(S)$, we infer that $y_n \in K(\nu_n)$ and that there exists $w_n \in \hat{F}(\nu_n, y_n)$ such that $\langle w_n - f, z - y_n \rangle \geq \varphi(y_n) - \varphi(z)$, for every $z \in K(\nu_n)$.

We have (w_n) is bounded because $\hat{F}$ is bounded. Moreover let $z \in K(\nu)$ and $w \in \hat{F}(\nu, z)$ by (SM2) there exists $\hat{w}_n \in \hat{F}(\nu_n, z)$ such that $\hat{w}_n \to w$.

we have

$$\begin{aligned} \langle \hat{w}_n, y_n - z \rangle &\leq \langle \hat{w}_n, y_n - z \rangle + \langle w_n - f, z_n - y_n \rangle + \varphi(z_n) - \varphi(y_n) \\ &= \langle \hat{w}_n, y_n - z \rangle + \langle w_n, z_n - z \rangle + \langle w_n, z - y_n \rangle + \langle f, y_n - z_n \rangle + \varphi(z_n) - \varphi(y_n) \\ &= \langle w_n - \hat{w}_n, z - y_n \rangle + \langle w_n, z_n - z \rangle + \langle f, y_n - z_n \rangle + \varphi(z_n) - \varphi(y_n) \end{aligned}$$

But since $w_n \in \hat{F}(\nu_n, y_n)$, $\hat{w}_n \in \hat{F}(\nu_n, z)$ and by the monotonicity of $\hat{F}(\nu, .)$ for all $\nu \in B$ we have $\langle w_n - \hat{w}_n, y_n - z \rangle \geq 0$ and then $\langle \hat{w}_n, y_n - z \rangle \leq \langle w_n, z_n - z \rangle + \langle f, y_n - z_n \rangle + \varphi(z_n) - \varphi(y_n)$.

By taking the limit, we have

$$\langle w, y - z \rangle \leq \langle f, y - z \rangle + \varphi(z) - \varphi(y)$$

Which means that: $\forall z \in K(\nu), \forall w \in \hat{F}(\nu, z)$ we have $\langle w - f, z - y \rangle \geq \varphi(z) - \varphi(y)$ and by using the Minty formulation, we deduce that $(y, \nu) \in G(S)$.

ACKNOWLEDGMENTS

The authors would like to thank the Professor A. El Hami inside the LOFIMS Laboratory of INSA Rouen for his scientific invitation and his collaboration during the preparation of this manuscript.

The first author would like to thank the Professor A. Khan for the fruitfull scientific exchanges.

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