ISSN 2347-1921

Controlled 2-Frames in 2-Hilbert Spaces

G. Upender Reddy

Department of Mathematics, Mahatma Gandhi University, Nalgonda, T.S

upendermathsmgu@gmail.com

ABSTRACT

Controlled frames in Hilbert spaces and 2-frames in 2-Hilbert spaces are studied, some results on them are presented. The controlled 2-frames in 2-Hilbert spaces is introduced. Some results on controlled 2-frames are established.

Keywords

controlled frames; frames; 2-frames; 2-inner product spaces.

AMS classifications

42C15, 42C40, 42C50.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol.10, No.6

www.cirjam.com , editorjam@gmail.com

1. INTRODUCTION

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaefer in 1952 to study some deep problems in nonharmonic Fourier series. Peter G. Casazza [2] presented a tutorial on frame theory and he suggested the major directions of research in frame theory.

The concept of linear 2-normed spaces has been investigated by S.Gahler in 1965[5] and has been developed extensively in different subjects by many authors. The concept of 2-frames for 2-inner product spaces was introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi [1] and described some fundamental properties of them. Y.J.Cho, S.S.Dragomir, A.White and S.S.Kim[4] are presented some inequalities in 2-inner product spaces. Some results on 2-inner product spaces are described by H.Mazaherl and R.Kazemi[6]. Properties of bounded 2-linear operators from a 2-normed set into a 2-normed space are studied by Zofia Lewandowska[7]. Peter Balazs, Jean-Pierre Antoine and Anna Grybos[3] are developed controlled frames and they will shown that the controlled frames are equivalent to standared frames.

In this paper controlled frames in Hilbert spaces and 2-frames in 2-Hilbert spaces are studied, some results on them are presented. The controlled 2-frames in 2-Hilbert spaces is introduced. Some results on controlled 2-frames are established.

2.Preliminaries

The following definitions from [2] are useful in our discussion.

Definition2.1. A sequence $\{x_i\}_{i=1}^{\infty}$ of vectors in a Hilbert space X is called a frame if there exist constants 0 < A ≤ B <∞ such that

$$A\,\|x\|^2 \le \sum_{i=1}^{\infty}\left|\langle x, x_i\rangle\right|^2 \le B\,\|x\|^2 \quad \text{for all } x \in X.$$

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively.

Definition2.2. A synthesis operator T : $l_2 \to X$ is defined as $Te_i = x_i$ where $\{e_i\}$ is an orthonormal basis for l_2.

Definition2.3. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for X and $\{e_i\}$ be an orthonormal basis for l_2. Then, the analysis operator

T^* : X → l_2 is the adjoint of synthesis operator T and is defined as $T^*x = \sum_{i=1}^{\infty}\langle x, x_i\rangle e_i$ for all x ∈ X.

Definition2.4. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for the Hilbert space X. A frame operator $S = TT^*$: X→ X is defined as

$$Sx = \sum_{i=1}^{\infty}\langle x, x_i\rangle x_i \quad \text{for all } x \in X.$$

Here we give the basic definitions of 2-normed spaces and 2-inner product spaces**.**

Definition2.5. X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on XxX satisfying the following conditions.

a) $\|x, y\| \ge 0$ and $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

b) $\|x, y\| = \|y, x\| \quad forall \;\; x, y \in X$

c) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$ for any real number α and $forall \;\; x, y \in X$

d) $\|x + y, z\| \le \|x, z\| + \|y, z\| \quad forall \;\; x, y, z \in X$

Then $\|\cdot,\cdot\|$ is called 2-norm on X and $(X, \|\cdot,\cdot\|)$ called a linear 2-normed space.

Definition2.6. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $x, y \in X$, then x is said to be orthogonal to y if and only if there exists $b \in X$ such that for all scalar α, $\|x, b\| \ne 0$ and $\|x, b\| \le \|x + \alpha y, b\|$, in this case we write $x \perp^b y$.

Definition2.7. Let X be a linear space of dimension greater than 1 over the field K(= R or C). Suppose that $(.,./.)$ is K-valued function on XxXxX which satisfies the following conditions.

a) $(x,x/z)\geq 0$ and $(x,x/z)=0$ if and only if x and z are linearly dependent.

b) $(x,x/z)=(z,z/x)$

c) $(y,x/z)=\overline{(x,y/z)}$

d) $(\alpha x,y/z)=\alpha(x,y/z)\ for all\ \alpha\in K$

e) $(x_1+x_2,y/z)=(x_1,y/z)+(x_2,y/z)$

Then $(.,./.)$ is called a 2-inner product on X and $(X,(.,./.))$ is called a 2- inner product space(or 2-pre Hilbert space).

If $(X,\langle\ \rangle)$ is an inner product space, then the standard 2-inner product space $(.,./.)$ is defined on X by

$$(x,y/z)=\begin{vmatrix}\langle x,y\rangle & \langle x,z\rangle\\ \langle z,y\rangle & \langle z,z\rangle\end{vmatrix}=\langle x,y\rangle\langle z,z\rangle-\langle x,z\rangle\langle z,y\rangle\ \ for all x,y,z\in X .$$

Let $(X,(.,./.))$ be a 2-inner product space, we can define a 2-norm on XxX by

$\|x,y\|=(x,x/y)^{\frac{1}{2}}$, for all $x,y\in X$.

Let $(X,(.,./.))$ be a 2-inner product space, $b\in X$ and $x,y\in X\setminus\langle b\rangle$. Then $x\perp^{b} y\Leftrightarrow(x,y/b)=0$.

Using the above properties, we can prove the Cauchy-Schwartz inequality $(x,y/b)^2\leq\|x,b\|^2\|y,b\|^2$

A 2-inner product space X is called a 2-Hilbert space if it is complete.

Definition 2.8. Let X be a 2-inner product space, a sequence $\{a_n\}_{n=1}^{\infty}$ of X is said to be convergent if there exists an element $a\in X$ such that $Lim_{n\to\infty}\|a_n-a,x\|=0$ for all $x\in X$.

3. Controlled frames and 2-Frames

The following definitions from [1] are useful in consequent sections.The set of all bounded linear operators with a bounded inverse from X to X is denoted by $GL(X)$.

Definition 3.1. Let $C\in GL(X)$. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{x_i\}_{i=1}^{\infty}$ such that there exist constants $0<A_{cl}\leq B_{cl}<\infty$, satisfying

$$A_{cl}\|x\|^2\leq\sum_{i=1}^{\infty}\langle x_i,x\rangle\langle x,Cx_i\rangle\leq B_{cl}\|x\|^2,\ \ for all\ x\in X .$$

Definition3.2. Let $\{x_i\}_{i=1}^{\infty}$ be a C-controlled frame for the Hilbert space X. A controlled frame operator $S_{cl}:X\to X$ is defined as

$$S_{cl}x=\sum_{i=1}^{\infty}\langle x,x_i\rangle Cx_i ,\quad \text{for all } x\in X.$$

Note that the operator S_{cl} is positive, therefore self adjoint.

Proposition3.3. Let $\{x_i\}_{i=1}^{\infty}$ be a C-controlled frame in X. Then $\{x_i\}_{i=1}^{\infty}$ is a frame. Further $CS=SC^{*}$ and so

$$\sum_{i=1}^{\infty}\langle x,x_i\rangle Cx_i=\sum_{i-1}^{\infty}\langle x,Cx_i\rangle x_i .$$

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a C-controlled frame in X. By using the definition and proposition (2.4) of [3] we have $S_{cl} \in GL(X)$.

Consider $C^{-1}S_{cl}x = C^{-1}\sum_{i=1}^{\infty}\langle x, x_i\rangle Cx_i = \sum_{i=1}^{\infty}\langle x, x_i\rangle x_i = Sx$

Which shows that $C^{-1}S_{cl} = S$, therefore $S \in GL(X)$, thus $\{x_i\}_{i=1}^{\infty}$ is a frame in X.

Now $CS = S_{cl} = S_{cl}^{*} = (CS)^{*} = SC^{*}$. (1) □

Proposition3.4. Let $C \in GL(X)$ be a self- adjoint. The vectors $\{x_i\}_{i=1}^{\infty}$ is a frame controlled by C if and only if it is a frame for X, C is positive and commutes with the frame operator S.

Proof. Suppose $\{x_i\}_{i=1}^{\infty}$ is a C-controlled frame in X. From the proposition (3.3) we have $\{x_i\}_{i=1}^{\infty}$ is a frame in X. Since C is self adjoint, by using equation (1) we have $CS = SC$. We have S_{cl}, S^{-1} are positive operators hence $C = S_{cl}S^{-1} = S^{-1}S_{cl}$ is also positive.

Conversely suppose $\{x_i\}_{i=1}^{\infty}$ is a frame in X by definition we have

$$A\|x\|^2 \le \sum_{i=1}^{\infty}\langle x_i, x\rangle\langle x, x_i\rangle \le B\|x\|^2, \quad forall\ x \in X$$

Multiplying above equation by C , we get

$$CA\|x\|^2 \le \sum_{i=1}^{\infty}\langle x_i, x\rangle\langle Cx, x_i\rangle \le CB\|x\|^2, \quad forall\ x \in X$$

By using C is self adjoint, the above equation becomes

$$A_1\|x\|^2 \le \sum_{i=1}^{\infty}\langle x_i, x\rangle\langle x, Cx_i\rangle \le B_1\|x\|^2, \quad forall\ x \in X$$, where $CA = A_1$ and $CB = B_1$

Hence, $\{x_i\}_{i=1}^{\infty}$ is a C-controlled frame in X. □

The definition of 2-frame from [1] as follows.

Definition 3.5 Let $(X, (.,./.))$ be a 2-Hilbert space and $\xi \in X$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in X is called a 2-frame associated to ξ if there exist 0 < A ≤ B <∞ such that

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty}\left|(x, x_i/\xi)\right|^2 \le B\|x,\xi\|^2 \ forall\ x \in X.$$

The above inequality is called the 2-frame inequality. The numbers A and B are called the lower and upper 2-frame bounds respectively.

The following proposition [1] shows that in the standard 2- inner product spaces every frame is a 2-frame.

Proposition 3.6. Let $(X, \langle\ \rangle)$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for H. Then, it is a 2-frame with the standard 2- inner product space on X.

Proof: Suppose that $\{x_i\}_{i=1}^{\infty}$ is a frame for X with frame bounds A and B.

Then $\sum_{i=1}^{\infty}\left|(x, x_i/\xi)\right|^2 = \sum_{i=1}^{\infty}\left|\langle x, x_i\rangle\langle\xi,\xi\rangle - \langle x,\xi\rangle\langle\xi, x_i\rangle\right|^2$

$$= \sum_{i=1}^{\infty} \left| \langle x, x_i \rangle - \langle x, \xi \rangle \langle \xi, x_i \rangle \right|^2 = \sum_{i=1}^{\infty} \left| \langle x - \langle x, \xi \rangle \xi, x_i \rangle \right|^2$$

$$\leq B \left\| x - \langle x, \xi \rangle \xi \right\|^2$$

$$= \text{B} \left(\|x\|^2 - \left| \langle x, \xi \rangle \right|^2 \right) = \text{B} \left(x, x / \xi \right) = \text{B} \|x, \xi\|^2$$

Similarly we can prove that A $\|x, \xi\|^2 \leq \sum_{i=1}^{\infty} \left| (x, x_i / \xi) \right|^2$ Hence $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for 2-Hilbert space. □

Suppose $(X, (.,./.))$ is a 2-Hilbert space and L_ξ the subspace generated with a fixed element ξ in X. Let M_ξ be denote the algebraic complement of L_ξ in X. So we have $L_\xi \oplus M_\xi = X$ .We define the inner product $\langle .,. \rangle_\xi$ on X as follows $\langle x, z \rangle_\xi = \langle x, z / \xi \rangle$.

A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in X is a 2-frame associated to ξ with frame bounds A and B, then the definition of 2-frame can be written as $A\|x\|_\xi^2 \leq \sum_{i=1}^{\infty} \left| (x, x_i)_\xi \right|^2 \leq B\|x\|_\xi^2$, $forall\ x \in X$.

Definition 3.7. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in X. Then, the 2-Synthesis operator $T_\xi : l^2 \to X_\xi$ is defined by $T_\xi \{c_i\} = \sum_{i=1}^{\infty} c_i\, x_i$.

Definition 3.8. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in X. Then, the 2-Analysis operator $T_\xi^* : X_\xi \to l^2$ is defined by $T_\xi^*(x) = \{(x, x_i / \xi)\}_{i=1}^{\infty}$.

Definition 3.9. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ with frame bounds A and B in a 2-Hilbert space X. A 2-frame operator $S_\xi : X_\xi \to X_\xi$ is defined by $S_\xi\, x = \sum_{i=1}^{\infty} (x, x_i / \xi) x_i$.

Theorem 3.10. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a sequence in 2-Hilbert space X, with $x = \sum_{i=1}^{\infty} (x, x_i / \xi) x_i$ holds for all $x \in X$ if and only if $\{x_i\}_{i=1}^{\infty}$ is a 2-normalized tight frame for X.

Proof: Since $\{x_i\}_{i=1}^{\infty}$ is a 2-normalized tight frame for X, for all $x \in X$

$$\Leftrightarrow \|x, \xi\|^2 = \sum_{i=1}^{\infty} \left| (x, x_i / \xi) \right|^2 \Leftrightarrow \|x, \xi\|^2 = \sum_{i=1}^{\infty} (x, x_i / \xi)(x_i, x / \xi)$$

$$\Leftrightarrow (x, x / \xi) = \left(\sum_{i=1}^{\infty} (x, x_i / \xi) x_i, x / \xi \right)$$

$$\Leftrightarrow x = \sum_{i=1}^{\infty} (x, x_i / \xi) x_i \text{ for all } x \in X .$$ □

Theorem 3.11. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for Hilbert space X, and T is co-isometry. Then $\{Tx_i\}_{i=1}^{\infty}$ is a 2-frame for X.

Proof: Since $\{x_i\}_{i=1}^{\infty}$ is a 2- frame for X, by Definition 3.1, we have

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, x_i/\xi)|^2 \le B\|x,\xi\|^2, (x \in X) \tag{2}$$

Since $T^*: X \to X$ is an operator, for all $x \in H$, we have $T^* x \in X$

Therefore, the above equation (2) is true for $T^* x \in X$

$$A\|T^* x,\xi\|^2 \le \sum_{i=1}^{\infty} |(T^* x, x_i/\xi)|^2 \le B\|T^* x,\xi\|^2$$

$$A\|T^* x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, Tx_i/\xi)|^2 \le B\|T^* x,\xi\|^2, \ forall\ x \in X$$

By using the fact that T is co-isometry, we have

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, Tx_i/\xi)|^2 \le B\|x,\xi\|^2, \ forall\ x \in X$$

Which shows that $\{Tx_i\}_{i=1}^{\infty}$ is a 2- frame for X. □

4. Controlled 2-frames

Throughout this section X is 2-Hilbert space.

Definition 4.1. Let X be a 2-Hilbert space and $C \in GL(X)$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in X is called a controlled 2-frame associated to $\xi \in X$ or C-controlled

2-frame for X if there exist constants $0 < A_{cl} \le B_{cl} < \infty$ such that

$$A_{cl}\|x,\xi\|^2 \le \sum_{i=1}^{\infty} (x_i/\xi, x)(x, Cx_i/\xi) \le B_{cl}\|x,\xi\|^2 \ forall\ x \in X.$$

Proposition4.2. Let $C \in GL(X)$ be a self- adjoint and $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for 2-Hilbert space X if and only if it is a controlled 2-frame for X.

Proof. Suppose $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for 2-Hilbert space X

$$\Leftrightarrow A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, x_i/\xi)|^2 \le B\|x,\xi\|^2$$

$$\Leftrightarrow A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} (x, x_i/\xi)(x_i/\xi, x) \le B\|x,\xi\|^2$$

$$\Leftrightarrow CA\|x,\xi\|^2 \le \sum_{i=1}^{\infty} (Cx, x_i/\xi)(x_i/\xi, x) \le CB\|x,\xi\|^2$$

$$\Leftrightarrow CA\|x,\xi\|^2 \le \sum_{i=1}^{\infty} (x_i/\xi, x)(x, C^* x_i/\xi) \le CB\|x,\xi\|^2$$

$$\Leftrightarrow A_1\left\|x,\xi\right\|^2 \le \sum_{i=1}^{\infty}\left(x_i/\xi,x\right)\left(x,Cx_i/\xi\right) \le B_1\left\|x,\xi\right\|^2$$

$\Leftrightarrow \{x_i\}_{i=1}^{\infty}$ is a controlled 2-frame for X. □

Acknowledgement:

The research of the author is partially supported by the UGC (India) [Letter No. F.20-4(1)/2012(BSR)].

References:

[1]. Ali Akbar Arefijammaal and Ghadir Sadeghi, " Frames in 2-inner Product Spaces", International Journal of Mathematical Sciences and Informatics, Vol.8,No.2(2013),pp.123-130.

[2]. P.G. Casazza, "The Art of Frame theory", Taiwanese Journal of Mathematics Vol. 4, No.2(2000), pp 129-201

[3]. Peter Balazs, Jean-Pierre Antoine and Anna Grybos "Weighted and controlled frames", arXiv:math/0611729v2 [math.F.A] ,12 Feb 2009.

[4]. Y.J.Cho, S.S.Dragomir, A.White and S.S.Kim, "Some Inequalities in 2-inner Product Spaces",Reprint.

[5]. S.Gahler,"Lineare 2-normierte Raume,Math, Nacher,28(1965),pp1-43.

[6]. H.Mazaherl and R.Kazemi," Some Results on 2-inner Product Spaces", Novi Sad J.Math, Vol.37,No.2,2007,pp.35-40.

[7].Zofia Lewandowska," Bounded 2-Linear operators on 2-Normed sets", Glasnik Matematicki, Vol.39(59),2004,pp.303-314.

Author' biography with Photo

Dr.G.Upender Reddy, is an Assistant Professor and Incharge Head, Departmant of Mathematics, Mahatma Gandhi University, Nalgonda since May 2012. He is also working as Assistant Controller of Examinations. He has completed his education from Osmania University i.e., M.Sc., 1998, M.Phil 2006 and Ph.D 2009. He has 9 publications to his credit of which 4 at International and 5 at National Journals. He presented 8 papers in International and National Conferences. One M.Phill awarded under his supervision.Presently he is guiding two Ph.D scholors. His research areas of interest are Frame theory, Functional Analysis and Operator theory. He has been awarded Start-up project worth of Rs 6 lakhs by UGC in 2012.