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Extension of Eulerian Graphs and Digraphs

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Abstract

In this paper the concept of extensibility number has been studied. The Eulerian graphs(digraphs) which have extensibility number 1, 2 or 3 have been characterized.

Keywords: Extension of graphs; Eulerian graphs; Regular graphs.

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1 Introduction

Attar [2009] introduced the concept of extension graphs (digraphs). In fact he characterized the extensibility number for some graphs(digraphs). Akram and Ahmed [2013] introduced a new definition for extension graphs(digraphs). Further, they defined the extensible class of graphs(digraphs), and they characterized regular graphs(digraphs) which have extensibility number 1; 2 or 3. In this work, we characterized the Eulerian graphs(digraphs), which have extensibility number 1, 2 or 3.

If e is an edge of a graph G having end vertices v, w then e is said to join thevertices v and w and these vertices are said to be adjacent. In this case we also saythat e is incident to v and w, and that W is a neighbor of v. An independent set ofvertices in G is a set of vertices of G no two of which are adjacent.

Let v be a vertex of the graph G, if V joined to itself by an edge, such an edge is called loop. The degree of v denoted by $d(v)$ is the number of edges of G incident with v, counting each looptwice. If two (or more) edges of G have the same end vertices then these edges arecalled parallel. A graph is called simple if it has no loops and parallel edges.

A trail in G is called Euler trail if it includes every edge of G. A tour of G is aclosed walk of G which includes every edge of G at least once. An Euler tour of G is a tour which includes each edge exactly once. A graph G is called Eulerian or Eulerif it has an Euler tour.

The following theorem characterize the Eulerian graphs.

Theorem 1 [3]: A connected graph G is Eulerian if and only if the degree of every vertex is even.

A digraph D is said to be weakly connected (or connected) if its underlying graph is connected. A digraph D is called simple if, for any pair of vertices u and v of D, there is at most one arc from u to v and there is no arc from itself.

Let v be a vertex in the digraph D. The indegree $id(v)$ of v is the number of arcs of D that have v as its head, i. e., the number of arcs that go to v. Similarly the outdegree $od(v)$ of v is the number of arcs of D that have v as its tail, i. e., that go out of v. Let D be a connected digraph. Then a directed Euler trail in D is a directed open trail of D containing all the arcs of D (once and only once). A directed Euler tour of D is a directed closed trail of Dcontaining all the arcs of D (once and only once). A digraph D containing a directed Euler tour is called an Euler digraph.

The following theorem characterize the Eulerian digraph.

Theorem 2 [4]: Let D be a weakly connected digraph with at least one arc. Then D is Eulerian if and only if $od(v) = id(v)$ for every vertex v of D.

All the graphs (digraph) through this paper are simple. For the undefined concepts and terminology we refer the reader to [3, 4, 5].

2 Extensibility of Graphs

In this section, we introduced the concepts, extension of graph, extensible classof graphs and the extensibility number of graphs.

Definition 1 [3]

Let G_1 and G_2be two graphs with no vertex in common. We define the join of G_1 and G_2 denoted by $G_1 + G_2$to be the graph with vertex set and edges set given as follows:

$$V(G1 + G2) = V(G1) \cup V(G2),$$

$$E(G1 + G2) = E(G1) \cup E(G2) \cup J$$

Where $J = \{x1x2 : x1 \in V(G1),\ x2 \in V(G2)\}$

Thus J consists of edges which join every vertex of G_1 to every vertex of G_2.

Attar[2] defined the extension of graphs as follows:

Definition 2 [2]

Let G be a nontrivial graph. The extension of G is a graph denoted by $G + S$ obtained from G by adding a nonempty set of independent vertices S such that every vertex in S is adjacent to every vertex in G exactly once.

Akram and Ahmed [2013] defined the extension of graphs as follows:

Definition 3 [1]

Let G be a nontrivial graph. The extension of G is a simple graph denoted by $G * S$ obtained by adding a nonempty set S of independent vertices to G such that every vertex in S is adjacent to at least one vertex in G. In such a way S is called extension set of G. In particular if S consists of a single element v, then v is called extension vertex of G. The graph $G * S$ have vertex set and edge set as follows:

$$\mathrm{V}(\mathrm{G} * \mathrm{S}) = \mathrm{V}(\mathrm{G}) \cup \mathrm{S},$$

$$\mathrm{E}(\mathrm{G} * \mathrm{S}) = \mathrm{E}(\mathrm{G}) \cup \mathrm{J}$$

where J is a set consists of edges join every vertex of S to at least one vertex of G.

Here, Akram and Ahmed [2013] define the extensible class of graphs.

Definition 4 [1]

Let τ be the class of graphs with certain property. Then τ is called extensible class of graphs, if for every graph $G \in \tau$ there exists an extension vertex v such that $G * v \in \tau$.

Proposition 1 [1]

The class of Eulerian graphs is not extensible class.

Now, we introduce the definition of extensibility number.

Definition 5 [1]

Let τ be the class of graphs with certain property, and $G \in \tau$ be a non trivial. The extensibility number of G with respect to τ is the smallest positive integer m, ifexists such that there exists an extension set S of G with cardinality m in which the new graph $G * S \in \tau$. We write $m = ext_{\tau}(G)$. If such a number dose not exist for Gthen we say that the corresponding extensibility number is ∞.

One can see immediate, the class of graphs is extensible class if and only if theextensibility number of every graph $G \in \tau$ is one.

3 Extension of Eulerian Graphs

In this section, we characterized the Euleriangraphs which have extensibility number equal to 1, 2 or 3.

Remark 1

Let τ be the class of Eulerian graphs,$G \in \tau$, then $ext_{\tau}(G) \geq 2$.

Proof:

The proof is obvious from Definition 5.

Theorem 3

Let τ be the class of connected Eulerian graphs, $G \in \tau$ with n vertices,$n > 2$. Then $ext_{\tau}(G) = 2$, if and only if there exist two independent vertices u, v different from the vertices of G, such that each of u and v is adjacent to even number of vertices in G exactly once, and $N(u) = N(v)$.

Proof:

Letτ the class of connected Eulerian graphs, and $G \in \tau$ with n vertices. Suppose that $ext_{\tau}(G) = 2$. Then by Definition5, there exists an extension set of two vertices u and v, such that $G * \{u, v\} \in \tau$, and u, v are independent vertices. As $G * \{u, v\}$ is Eulerian. Then by Theorem 1, every vertex of $G * \{u, v\}$ has even degree. Thus each of u, v has even degree, and by Definition 3, $G * \{u, v\}$ is simple, then each of u, v is adjacent to even number of vertices in G exactly once. Suppose that x is a vertex in G such that x is adjacent by exactly one vertex from u, v. Then x has odd degree in $G * \{u, v\}$which is a contradiction to our assumption. Thus x is adjacent by both u and v. Hence u, and v has the same neighbors in G. That is $N(u) = N(v)$.

Conversely, Suppose that there exist two vertices u, v different from the vertices of G such that each of u and v is adjacent to even number of vertices in G exactly once and $N(u) = N(v)$. Since u, v are independent vertices, and each of them is adjacent to even number in G, then $\{u, v\}$ is extension set of vertices to G. Since the degree of every vertex in G is even and $N(u) = N(v)$, then each vertex in $N(u)$ and $N(v)$ has even degree. Hence every vertex in $G * \{u, v\}$has even degree. By Theorem 1, $G * \{u, v\} \in \tau$. Hence $ext_\tau(G) \geq 2$. By Remark 1, above, $ext_\tau(G) \neq 1$. Hence $ext_\tau(G) = 2$.

Theorem 4

Let τ be the class of connected Eulerian graphs, $G \in \tau$ with n verties $n \geq 3$ then $ext_\tau(G) = 3$ if and only if there exist three independent vertices u, v, w different from the vertices of G such that each of them is adjacent to even number of vertices in G exactly once and each vertex in each of $N(u), N(v), N(w)$ is a neighbor to exactly two vertices from the vertices u, v, w and no two vertices from u, v, w have the same neighbors in G.

Proof:

Let τ be the class of connected Eulerian graphs, $G \in \tau$ with n verties $n \geq 3$. Suppose that $ext_\tau(G) = 3$. By Definition 5, there exists a set of three vertices u, v, w different fromthe vertices of G such that $G * \{u, v, w\} \in \tau$ and u, v, w are independent vertices. As $G * \{u, v, w\}$ is Eulerian, by Theorem1, every vertex of $G * \{u, v, w\}$ has even degree. Thus each of u, v, w has even degree in $G * \{u, v, w\}$. By Definition 3, $G * \{u, v, w\}$ is simple, then each of u; v; w is adjacent to even number of vertices in G exactly once. Suppose that x is a vertex in G such that x is adjacent by exactly one vertex from $\{u, v, w\}$. In this case we get $d(x)$ is odd in $G * \{u, v, w\}$ which is a contradiction. We get a similar contradiction if x in G is a common neighbor for $\{u, v, w\}$. Hence each vertex in each of $N(u)$, $N(v)$, $N(w)$ is adjacent to exactly two vertices fromu, v, w. If two vertices from u, v, w have the same neighbors. Then by Theorem 3, $ext_\tau(G) = 2$a contradiction to our assumption that $ext_\tau(G) = 3$.

Conversely, Suppose that there exist three independent vertices u, v, w different from the vertices of G such that each of them is adjacent to even number of vertices in G exactly once and each vertex in each of $N(u)$; $N(v)$; $N(w)$ is a neighbor to exactly two vertices from the vertices u, v, w and no two vertices from u, v, w have the same neighbors. Since u, v, w are independent vertices, and each of them is adjacent to even number of vertices in G. Then $\{u, v, w\}$ is extension set of vertices of G. From our assumption every vertex in each of $N(u)$; $N(v)$ and $N(w)$ has even degree. Thus every vertex in $G * \{u, v, w\}$ has even degree. By Theorem 1, $G * \{u, v, w\}$is Eulerian. As such $\{u, v, w\}$ is extensionset of G with respect to Eulerian. Hence$ext_\tau(G) \leq 3$. By Remark 1, $ext_\tau(G) \neq 1$. Suppose that $ext_\tau(G) = 2$ then by Theorem 1, there exist two independent vertices u, v different from the vertices of G and $N(u) = N(v)$, a controduction to ourassumption that no two of u, v, w have the same neighbors in G. Hence $ext_\tau(G) = 3$.

4 Extension of Digraphs

In this section, we introduced the concepts, extension of digraph, extensible class of digraphs and the extensibility number of digraphs.

Attar [2] defined the extension of digraphs as follows

Definition 6 [2]

Let D be a non trivial digraph. The extension of D is a simple digraph denoted by $D + S$ obtained from D by adding a nonempty set of independent vertices S such that every vertex in S is adjacent or adjacent by but not both every vertex in D.

Akram and Ahmed[2013] defined the extension of digraphs as follows:

Definition 7 [1]

Let D be a non trivial digraph. The extension of D is a simple digraph denoted by $D * S$ obtained from D by adding a nonempty set of independent vertices S different from the vertices of D such that every vertex in S is adjacent or adjacent by butnot both at least one vertex in D. In such away S is called extension set of D. Inparticular, If S consists of single element v, then v is called extension vertex of D.The graph $D * S$ have vertex set and edge set as follows:

$$V(D * S) = V(D) \cup S$$

$$E(D * S) = E(D) \cup J$$

where J is a set consists of edges join every vertex of S to at least one vertex of D

Here, Akram and Ahmed [2013] defined the extensible class of digraphs.

Definition 8 [1]

Let φ be the class of digraphs with certain property. Then φ is called extensible class if for every digraph $D \in \varphi$, there exists an extension vertex v such that $D * v \in \varphi$.

Proposition 2 [1]

The class of Eulerian digraphs is not extensible class.

Now, we introduce the definition of extensibility number.

Definition 9 [1]

Let φ be the class of digraphs with certain property and $D \in \varphi$ be a non trivial. The extensibility number of D with respect to φ is the smallest positive integer m, if exists such that there exists an extension set of vertices of D with cardinality m in which the new digraph $D * S \in \varphi$. We write $m = ext_{\varphi}(D)$. If such a number dosenot exist for D, then we say the correspoding extensibility number is ∞.

5 Extension of Eulerian Digraphs

In this section, we characterized Eulerian digraphs which have extensibility number equal to 1; 2 or 3.

Remark 2

Let τ be the class of Eulerian digraphs, $D \in \tau$, then $ext_{\tau}(D) \geq 2$.

Proof:

The proof is obvious from Definition 9.

Theorem 5

Let τ be the class of Eulerian digraphs, $D \in \tau$ with n vertices, $n > 2$, then $ext_{\tau}(G) = 2$ if and only if there exist two independent vertices u, v different from the vertices of D such that each of u, v is adjacent to set of vertices S in D, where $|S| \leq [n/2]$, $N^+(u) \cap N^+(v) = \emptyset$ and each vertex in $N^+(u)$ is adjacent to v and each vertex in $N^+(v)$ is adjacent to u.

Proof:

Let D be an Eulerian digraph with order $n > 2$. Suppose that $ext_{\tau}(D) = 2$. Then by Definition 9, there exist two vertices u, v different from the vertices of D such that $D * \{u, v\} \in \tau$ and u, v are independent vertices. Then by Theorem $2, id(x) = od(x), \forall\, x \; in \; D * \{u, v\}$.

Suppose that $N^+(u) \cap N^+(v) = y$, then y is a common neighbor to u and v, then $od(y) \neq id(y) in\, D * \{u, v\}$a contradiction. It is easy to see that if$|S| > [n/2]$, is impossible. Suppose that h is avertex in $N^+(u)$ such that h is not adjacent to v. Then $id(h) \neq od(h) \; in \; D * \{u, v\}$ a contradiction to $D * \{u, v\} \in \tau$. we get a similar contradiction for the vertices in $N^+(v)$ which is not adjacent to u.

Conversely, Suppose that there exist two independent vertices u, v different from the vertices of D, such that each of u, v is adjacent to set of vertices S in D, where$|S| \leq [n/2]$, $N + (u) \cap N + (v) = \emptyset$ and each vertex in $N^+(u)$ is adjacent to v and each vertex in $N^+(v)$ is adjacent to u. Since u, v are independent vertices and each of them is adjacent to S vertices in D. Then by definition 7, $\{u, v\}$is an extension set of vertices of D. From our assumption $id(x) = od(x) \forall\, x \; in \; D * \{u, v\}$. Then by Theorem $2, D * \{u, v\}$ is Eulerian digraph. Hence $ext_{\tau}(D) \leq 2$. By Remark 2,above $ext_{\tau}(D) > \neq 1$. Hence $ext_{\tau}(D) = 2$.

Theorem 6

Let τ be the class of Eulerian digraphs, $D \in \tau$ with n vertices, $n \geq 3$, then $ext_{\tau}(D) = 3$ if and only if there exist three independent vertices u, v, w different fromthe vertices of D and the following holds:

1. Each of u, v, w is adjacent to at least one vertex in D exactly once and $N^+(u), N^+(v), N^+(w)$, are mutually disjoint.
2. u is adjacent by exactly $|N^+(u)|$ vertices from $N^+(v) \cup N^+(w)$, v is adjacent by exactly$|N^+(v)|$

vertices from $N^+(u) \cup N^+(w)$, w is adjacent by exactly $|N^+(w)|$ vertices from $N^+(u) \cup N^+(v)$ and no two vertices from u, v, w such that each of them adjacent by exactly the neighbors of the other.

3. $|N+(u)| \leq |N+(v|) + |N+(w)|$,for u, v, w.

Proof:

Let D be an Eulerian digraph with $n \geq 3$, vertices. Suppose that $ext_\tau(D) = 3$. Then by Definition 9, there exist three vertices u, v, w different from the vertices of D such that $D * \{u, v, w\} \in \tau$ and u, v, w are independent vertices. Then by Theorem 2, $id(y) = od(y) \forall y \in D * \{u, v, w\}$. If u is adjacent by a set of vertices h from the vertices $N^+(v) \cup N^+(w)$, with $h \neq |N^+(u)|$ then $od(u) \neq id(u)$ in $D * \{u, v, w\}$ which is a contradiction. Similarly for the vertices v and w. Suppose that x is a vertex in D which is a common neighbor for two or three vertices from u, v, w in this case $id(x) \neq od(x)$ in $D * \{u, v, w\}$ which is a contradiction to $D * \{u, v, w\}$ is Eularian. Suppose that. $|N^+(u)| > |N^+(v|) + |N^+(w)|$ then it is easy to see that $D * \{u, v, w\}$ is not Eulerian. If $n < 3$ it is easy to see this case is impossible. Suppose that there exist two vertices from u, v, w each of them is adjacent by exactly the neighbors of the other. Then by Theorem 5, $ext_\tau(D) = 2$ a contradiction.

Conversely, Suppose that there exist three independent vertices u, v, w different from the vertices of D and the conditions 1, 2 and 3 in our theorem are holds. Since u, v, w are independent vertices and each of them is adjacent to at least one vertex in D. Then $\{u, v, w\}$ is extension set of D. From assumption we can see that in degree equals to the out degree for every vertex of $D * \{u, v, w\}$. Then $id(y) = od(y) \forall y \in D * \{u, v, w\}$. Then by Theorem 2, $D * \{u, v, w\}$ is Eulerian digraph. Thus $ext_\tau(D) \leq 3$. By Remark 2,$ext_\tau(D) \neq 1$ if $ext_\tau(D) = 2$ if then by Theorem 5, there exist two independent vertices u, v different from the vertices of D such that each of u, v is adjacent to S vertices in D where $|S| \leq [n/2], N^+(u) \cap N^+(v) = \emptyset$ and each vertex in $N^+(u)$ is adjacent to v, and each vertex in $N^+(v)$ is adjacent to u a contradiction to our assumption that $N^+(u), N^+(v)$ and $N^+(w)$ are mutually disjoint. Hence $ext_\tau(D) = 3$.

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