ON SOME CONCEPTS RESPECT TO WEAK STRUCTURES

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ABSTRACT

In this paper, we use the concept of a weak structure to introduce some new concepts such as sub weak structure, separation, connectedness and sub connected space. Furthermore, investigate some theorems with their proofs and state examples.

Indexing terms/Keywords

Sub weak structures; separation; connectedness; weak structures.

Academic Discipline And Sub-Disciplines

Topology

1. INTRODUCTION

In 2011, ´A. Cs´asz´ar[2] defined concept of weak structureon a nonempty set and he shown that these structures can replaced in many situations minimal structures [4] or generalized topologies [1]. More recently, E. Ekici [3] in 2012, further introduced and studied properties of sets on a weak structure. Furthermore, M. Navaneethakrishnan and S. Thamaraiselvi [5] extended the study of weak structure and $m-structure$. Further, [6] in 2013, they defined and studied properties of some subsets.

2. PRELIMINARIES

Throughout this paper, let X be a nonempty set, if $A \subseteq X$ then A^c is denoted the complement of a subset A , and the set $P(X)$ is denoted the power set of X. A^0_{μ} it called the interior of A, and $\overline{A}_{\mu}$ denoted the closure of A.

The pair (X, μ) is denoted X with a weak structure μ, simply called a space (X, μ).

Definition 2.1

[2] Let X be a nonempty set and P its power set. $\omega \subset P$ is a weak structure (briefly, WS) on X if and only if $\phi \in \omega$.

Definition 2.2

[6] Let μ be a weak structure on X , and $A \subseteq X$. The elements of μ are called $\mu-open$ sets and their complements called $\mu-closed$ sets, for a WS μ on X the the intersection of all $\mu-closed$ containing A is denoted by $c_{\mu}(A)$ and it called the closure of A, and the union of all $\mu-open$ sets contained in a subset A is denoted by

3. SOME NEW CONCEPTS ON A WEAK STRUCTURE

In this section we study our aim of this paper which discusses sub weak structure, separation, connectedness and sub connected space. Further, state examples and investigate some theorems with their proofs.

Definition 3.1

We say that a set A is $\mu - clopen$ if A is both $\mu - open$ and $\mu - closed$ set.

3.1 Sub weak structure

Definition 3.1.1

Let (X, μ) be a WS space and $X^* \subseteq X$ and let μ_{X^*} be the collection of all subsets of X^* that are of the form $X^* \cap U$ for a $U \in \mu$ then μ_{X^*} is a WS on the set X^*. The weak structure μ_{X^*} on X^* is referred to as the sub weak structure space on X^*.

Theorem 3.1.1

μ_{X^*} is a sub weak structure space on X^* .

Proof

Let (X, μ) be a weak structure and X^* be a subset of X , a subset U of X^* is open respect to the (X^*, μ_{X^*}) because

$$\mu_{X^*} = \{U : U = X^* \cap G \ : G \in \mu, X^* \subseteq X \}$$

Since $\phi \in \mu$, and $X^* \subseteq X$ then

$X^* \cap \phi = \phi$ then $\phi \in \mu_{X^*}$

Hence μ_{X^*} is a weak structure space.

Theorem 3.1.2

Let (X^*, μ_{X^*}) be a sub weak structure respect to (X, μ) and $A \subseteq X^* \subseteq X$ if $A \in \mu$ then $A \in \mu_{X^*}$.

Proof

Since $\mu_{X^*} = \{U : U = X^* \cap G : G \in \mu\}$

Then $A = A \cap X^* \in \mu$

Since $A \subseteq X^*$, $A \in \mu$.

Example 3.2.1

Let $X = \{a, b, c, d\}$, $\mu = \{\phi, \{a\}, \{b, c\}, \{a, c\}\}$, $A = \{a\}$

Then $\mu_A = \{\phi, \{a\}\}$.

Theorem 3.1.3

If $(X^{**}, \mu_{X^{**}})$ is a sub weak structure respect to (X^*, μ_{X^*}) and (X^*, μ_{X^*}) is a sub weak structure with respect to (X, μ) then $(X^{**}, \mu_{X^{**}})$ is a sub weak structure from (X, μ).

Proof

$\mu_{X^{**}} = \{U^* : U^* = X^{**} \cap U : U \in \mu_{X^*}\}$

Since $(X^{**}, \mu_{X^{**}})$ is a sub weak structure respect to (X^*, μ_{X^*}).

$\mu_{X^{**}} = \{U^* : U^* = X^{**} \cap (X^* \cap G) : G \in \mu\}$

Since (X^*, μ_{X^*}) is a sub weak structure respect to (X, μ).

$\mu_{X^{**}} = \{U^* : U^* = (X^{**} \cap X^*) \cap G : G \in \mu\}$

Since the intersection is associated.

While $X^{**} \subseteq X^*$ then $\mu_{X^{**}} = \{U^* : U^* = X^{**} \cap G : G \in \mu\}$.

Hence $(X^{**}, \mu_{X^{**}})$ is a sub weak structure respect to (X, μ).

Example 3.2.2

Let $X = \{a, b, c, d\}$, $\mu = \{\phi, \{a\}, \{b, c\}, \{a, c\}\}$, $A = \{a, b\}$, $B = \{b, c, d\}$

Then $\mu_A = \{\phi, \{a\}, \{b\}\}$, $\mu_B = \{\phi, \{b, c\}, \{c\}\}$.

3.2. Separation with respect to WS space:

Definition 3.2.1

Let (X, μ) be a weak structure and E be a nonempty subset of X we say that A, B are mutually separated sets the set E and we denoted that by $E = A / B$ if and only if

1- $A, B \neq \phi$

2- $E = A \cup B$

3- $A \cap B = \phi$

4- $A \cap d(B) = \phi$ & $d(A) \cap B = \phi$

Or we can instead the third and fourth conditions by the separation condition

$(A \cap \bar{B}_{\mu}) \cup (\bar{A}_{\mu} \cap B) = \phi$.

Corollary 3.2.1

Let (X, μ) be a weak structure then $X = A / B$ if and only if A, B both are nonempty $\mu-clopen$ sets.

3.3. Connectedness

In this part we go to define the connectedness in a weak structure space.

Definition 3.3.1

A weak structure space X is said to be connected if and only if X cannot be written as a union of two non empty mutually separated sets.

Example 3.3.1

Let $X = \{a, b, c\}$, $\mu = \{\phi, \{a\}, \{b, c\}\}$. $A = \{a\}$, $B = \{b, c\}$

Hence closed sets are $X, \{b, c\}, \{a\}$

Then $\bar{A}_{\mu} = \{a\} = A$, $\bar{B}_{\mu} = \{b, c\} = B$, $A \cap B = \phi$, $A \cup B = X$

Then $A \cap \bar{B}_{\mu} = \phi$, $\bar{A}_{\mu} \cap B = \phi$

This implies that $(A \cap \bar{B}_{\mu}) \cup (\bar{A}_{\mu} \cap B) = \phi$

Therefore, X is not connected.

Example 3.3.2

$X = \{a, b, c\}$, $\mu = \{\phi, \{a, b\}, \{b, c\}\}$. $A = \{a\}$, $B = \{b, c\}$ then closed sets are $X, \{c\}, \{a\}$

- $A \cap B = \phi$, $A \cup B = X$

- $\bar{A}_{\mu} = X \cap \{a\} = \{a\} = A$, $\bar{B}_{\mu} = X$

- $(\bar{A}_{\mu} \cap B) \cup (A \cap \bar{B}_{\mu}) = A \neq \phi$

The space (X, μ) has no separation on μ then (X, μ) is connected.

Definition 3.3.1

We say that the subset A of X is said to be connected if it connected in (A, μ_A) where (A, μ_A) is the sub weak structure respect to (X, μ).

Remark 3.3.1

A subset A of X is disconnected if and only if there exist two open sets U, V in X with $A \cap U \neq \phi$, $A \cap V \neq \phi$, $A \cap U \cap V \neq \phi$, $A \subseteq U \cup V$.

Theorem 3.3.1

If A is a connected set and H, K are mutually separated sets with $A \subseteq H \cup K$ then either $A \subseteq H$ or $A \subseteq K$.

Theorem **3.3.2**

If A is connected and $B \subseteq X$ with $A \subseteq B \subseteq \bar{A}_{\mu}$ then B is connected.

Proof

Suppose B is disconnected then there exist two open U, V in X with $B \cap U \neq \phi$, $B \cap V \neq \phi$, $B \cap U \cap V \neq \phi$, $B \subseteq U \cup V$

Now since A is connected and $A \subseteq U \cup V$ so either $A \subseteq V$ or $A \subseteq U$

Suppose $A \subseteq U$

If $b \in B$ and $b \notin A$ and with $b \in V$ then V is an open set containing b and with $A \cap V = \phi$ so $b \notin \bar{A}_\mu$. A contraction since $B \subset \bar{A}_\mu$.

So $b \in U$ for any $b \in B$ and hence $B \subset U$ and therefore $B \cap V = \phi$ which is a contraction so B is a connected set.

Corollary 3.3.1

If A is connected then $\bar{A}_\mu$ is connected.

Theorem **3.3.3**

If every two points in E contained in a connected set in E then E is connected.

Proof

Suppose E is disconnected then there exist two mutually separated sets with $E = A / B$.

Since $A \neq \phi$, $B \neq \phi$.

Let $a \in A$, $b \in B$.

Then there exist a connected set C contains a, b with $C \subset E$.

From theorem 1, then $A \subseteq C$ or $B \subseteq C$ which is a contraction, because if $A \subseteq C$ then $a, b \in A$ which impossible since $A \cap B = \phi$, also if $B \subseteq C$ then $a, b \in B$ that is impossible since $A \cap B = \phi$ therefore E is connected.

Theorem 3.3.4

(X, μ) is connected if and only if for any nonempty subsets A, B of X and $A, B \neq X$ with A, B are not $\mu - clopen$ sets.

Proof

$\Rightarrow$ Suppose (X, μ) is connected, if A is $\mu - clopen$ set then $B = A^c$ is $\mu - clopen$ set.

Then A, B both are $\mu - clopen$ subsets, from corollary 1 then we get $X = A / B$ which a contraction since X is connected.

$\Leftarrow$ Suppose X is disconnected then there exist two mutually separated sets A, B such that $X = A / B$, by (corollary 3.2.1), A, B both are nonempty $\mu - clopen$ sets, which is a contraction.

Theorem 3.3.5

If (X^*, μ_{X^*}) is a sub weak structure with respect to (X, μ) then the subset $E \subseteq X$ is connected in (X, μ) if and only if is connected in (X^*, μ_{X^*}).

Theorem 3.3.6

If $\{A_j\}_{j \in \Gamma}$ is a collection of connected sets with $\underset{j \in \Gamma}{\cap} A_j \neq \phi$ then $\underset{j \in \Gamma}{\cup} A_j$ is connected.

Proof

Suppose $\underset{j \in \Gamma}{\cup} A_j$ is disconnected then there exist two mutually separated sets H, K with $(\underset{j \in \Gamma}{\cup} A_j) \cap H \neq \phi$, $(\underset{j \in \Gamma}{\cup} A_j) \cap K \neq \phi$, $(\underset{j \in \Gamma}{\cup} A_j) \cap H \cap K = \phi$ and $\underset{j \in \Gamma}{\cup} A_j \subset H \cup K$

Since $\underset{j\in\Gamma}{\cap}A_j\neq\phi$ let $x\in\underset{j\in\Gamma}{\cap}A_j$

Then $x\in K$ or $x\in H$ suppose $x\in H$. Since A_j is connected for all $j\in\Gamma$ and $A_j\subset H\cup K$ then by the theorem (3.3.1) either $A_j\subset H$ or $A_j\subset K$.

But since $x\in A_j$ for all $j\in\Gamma$ and $x\in H$ so $A_j\subset H$ for all $j\in\Gamma$ and so $\underset{j\in\Gamma}{\cup}A_j\subset H$, that is $(\underset{j\in\Gamma}{\cup}A_j)\cap K=\phi$ which is a contraction so $\underset{j\in\Gamma}{\cup}A_j$ is connected.

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