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Some class of generalized entire sequence of Modal Interval numbers

[1]T.Balasubramanian [2]S.Zion Chella Ruth
[1]Department of Mathematics, Kamaraj college, Tuticorin, Tamilnadu, India
satbalu@yahoo.com
[2]Department of Mathematics, Dr.G.U.Pope college of Engineering, Sawyerpuram, Tuticorin,Tamilnadu,India.
ruthalwin@gmail.com

ABSTRACT

The history of modal intervals goes back to the very first publications on the topic of interval calculus. The modal interval analysis is used in Computer graphics and Computer Aided Design (CAD), namely the computation of narrow bounds on Bezier and B-Spline curves. Since modal intervals are used in different fields, we have constructed a new sequence space $\lambda_0^p(gI)$ of modal intervals. Also , we have given some new definitions and theorems about the sequence space $\lambda_0^p(gI)$ of modal interval numbers.

Keywords

Banach space; Modal interval number; Modal fundamental sequence

SUBJECT CLASSIFICATION

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1 INTRODUCTION

Interval arithmetic was first suggested by Dwyer [1] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [8] in 1959 and Moore and Yang [9] 1962. Furthermore, Moore and others[10] have developed applications to differential equations.

Chiao in 2002 [4] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmax [11] in 2010 studied bounded and convergent sequence space of interval numbers and showed that these spaces are complete metric space. Recently, Zararsız and Sengönül[13] introduced null, bounded and convergent sequence space of modal interval numbers.

Let us denote the set of all real valued closed interval by I, the set of positive integers by N and the set of all real numbers by $\Re$. Any element of I is called interval number and it is denoted by $\hat{x}$. That is $\hat{x}=\{x\in\Re:\underline{x}\leq x\leq\overline{x}\}$. An interval number $\hat{x}$ is a closed subset of real numbers. Let $\underline{x}$ *and* $\overline{x}$ be be respectively first and last points of the interval number $\hat{x}$. Therefore, when $\underline{x}>\overline{x}$, $\hat{x}$ is not an interval number. But in modal analysis $[\overline{x},\underline{x}]$ is a valid interval. A modal interval number $\widetilde{x}=\{[\underline{x},\overline{x}]:\underline{x},\overline{x}\in\Re\}$ is defined by a pair of real numbers $\overline{x},\underline{x}$. Let us denote the set of all modals by gI. Let us suppose that $\widetilde{x},\widetilde{y}\in gI$. Then the algebraic operations between $\widetilde{x}$ and $\widetilde{y}$ are defined in the Kaucher arithmetic, [6]. For a modal $\widetilde{x}=[\underline{x},\overline{x}]$ $dual$ operator is defined as $dual\,\widetilde{x}=[\overline{x},\underline{x}]$. Thus, if $\widetilde{x}\in gI$, then $\widetilde{x}-dual\widetilde{x}=[0,0]=\widetilde{0}$, $dual\widetilde{x}\in gI$. Let us suppose that $\widetilde{x}\in gI$, then $\widetilde{x}$ is called symmetric modal if $\underline{x}=-\overline{x}$ or vice-versa.

The set of all modals gI is metric space defined as

$$d(\widetilde{x}_1,\widetilde{x}_2)=\max\{|\underline{x}_1-\underline{x}_2|,|\overline{x}_1-\overline{x}_2|\} \tag{1.1}$$

If $\widetilde{x},\widetilde{y}\in gI$ and $\underline{x}\leq\overline{x},\underline{y}\leq\overline{y}$ then the set gI is reduced ordinary set of interval numbers which is complete metric space with the metric d defined in (1.1) [6]. If we take $\widetilde{x}_1=[a,a]$ and $\widetilde{x}_2=[b,b]$, we obtain the usual metric of $\Re$ with $d(\widetilde{x}_1,\widetilde{x}_2)=|a-b|$, where $a,b\in\Re$

Let f be a function from N to gI which is defined by $k\rightarrow f(k)=\widetilde{x},\widetilde{x}=(\widetilde{x}_k)$. Then $(\widetilde{x}_k)$ is called sequence of modals. We will denote the set of all sequences of modals by $\omega(gI)$. For two sequences of modals $(\widetilde{x}_k)$ and $(\widetilde{y}_k)$, the addition, scalar product and multiplication are defined as follows $(\widetilde{x}_k+\widetilde{y}_k)=[\underline{x}_k+\underline{y}_k,\overline{x}_k+\overline{y}_k]$, $(\alpha\widetilde{x}_k)=[\alpha\underline{x}_k,\alpha\overline{x}_k],\ \alpha\in\Re$, $(\widetilde{x}_k\widetilde{y}_k)=[\min(\underline{x}_k\underline{y}_k,\underline{x}_k\overline{y}_k,\overline{x}_k\underline{y}_k,\overline{x}_k\overline{y}_k),\max(\underline{x}_k\underline{y}_k,\underline{x}_k\overline{y}_k,\overline{x}_k\underline{y}_k,\overline{x}_k\overline{y}_k)]$ respectively.

The set $\omega(gI)$ is a vector space since the vector space rules are clearly provided. The zero element of $\omega(gI)$ is the sequence $\widetilde{\theta}=(\widetilde{\theta}_k)=([0,0])$ all terms of which are zero interval. If $(\widetilde{x}_k)\in\omega(gI)$ then inverse of $(\widetilde{x}_k)$, according to addition, is $dual(\widetilde{x}_k)$.

Proposition 1.1. If $(\widetilde{x}_k)$, $(\widetilde{y}_k)$, $(\widetilde{r}_k)$ are sequences of symmetric modal, then the following equality holds:

$$(\widetilde{x}_k)\{(\widetilde{y}_k)-(\widetilde{r}_k)\}=(\widetilde{x}_k)(\widetilde{y}_k)-(\widetilde{x}_k)(\widetilde{r}_k) \tag{1.2}$$

Definition 1.2. A sequence $\widetilde{x}=(\widetilde{x}_k)\in\omega(gI)$ of modals is said to be convergent to the modal $\widetilde{x}_0$ if for each $\varepsilon>0$ there exists a positive integer n_0 such that $d(\widetilde{x}_k,\widetilde{x}_0)<\varepsilon$ for all $k\geq n_0$ and we denote it by writing $\lim_k\widetilde{x}_k=\widetilde{x}_0$. Thus, $\lim_{k\to\infty}\widetilde{x}_k=\widetilde{x}_0\Leftrightarrow\lim_{k\to\infty}\underline{x}_k=\underline{x}_0\ and\ \lim_{k\to\infty}\overline{x}_k=\overline{x}_0$.

Definition 1.3. A sequence of modals, $\widetilde{x}=(\widetilde{x}_k)\in\omega(gI)$, is said to be modal fundamental sequence if for every $\varepsilon>0$ there exists $k_0\in\mathrm{N}$ such that $d(\widetilde{x}_k,\widetilde{x}_n)<\varepsilon$ whenever $n,k>k_0$.

Let $p = (p_k)$ be any sequences of strictly positive real numbers. The class of sequences of modal interval numbers is defined by $c_0^p(gI)$. For each fixed k, we define

$$c_0^p(gI) = \{\tilde{x} = (\tilde{x}_k) \in \omega(gI : [d(\tilde{x}_k, \tilde{0})]^{p_k} < \varepsilon\}$$

If $p = (p_k) \in l_\infty$,then $c_0^p(gI)$ becomes a locally convex FK space, under paranorm

$$g(x) = \sup_k [d(\tilde{x}_k, \tilde{0})]^{p_k/M} \text{ where } M = \max\{1, \sup p_k\}$$

When all the terms of $\{p_k\}$ are constants and all equal to $p > 0$, we have $c_0^p(gI) = c_0(gI)$ the space of all null sequences space of modal interval numbers. $c_0(gI) = \{\tilde{x} = \omega(gI) : \lim_k \tilde{x}_k = \theta\}$.

The class of entire sequence space of modal interval numbers is defined by

$\chi(gI) = \{\tilde{x} = (\tilde{x}_k) \in \omega(gI) : [d(k!\tilde{x}_k, \tilde{0})]^{1/k} < \varepsilon\}$. Let $\tilde{\lambda} = \{\tilde{\lambda}_1, \tilde{\lambda}_2, ...\}$ be a fixed sequence of modal interval numbers such that $\tilde{\lambda}_k \neq \theta$ for all k.

Consider the subspace $\lambda_0^p(gI)$ of all those sequences $\tilde{x} = (\tilde{x}_k)$ in $c_0^p(gI)$ such that $\tilde{\lambda}\tilde{x} \in c_0^p(gI)$ Here $\tilde{\lambda}\tilde{x}$ is the sequence $\{\tilde{\lambda}_k \tilde{x}_k\}$. If $p_k = p > 0$ for all k, then $\lambda_0^p(gI) = \lambda_0(gI)$ where $\lambda_0(gI) = \{\tilde{x} \in c_0(gI) : \tilde{\lambda}\tilde{x} \in c_0(gI)\}$

If $p_k = \frac{1}{k}$ and $\lambda_k = [k!, k!]$ for all k, then $\lambda_0^p(gI) = \chi(gI)$

Throughout this paper, $p = (p_k)$ a bounded sequence of strictly positive real numbers and $M = \max\{1, \sup p_k\}$. Now $\lambda_0^p(gI)$ is endowed with two topologies. One is the metric topology $\tau(gI)$ given by the metric $\tilde{d}$,

where $\tilde{d}(\tilde{x}, \tilde{y}) = \sup_k [d(\tilde{x}_k, \tilde{y}_k)]^{p_k/M}, \tilde{x}_k, \tilde{y}_k \in \lambda_0^p(gI)\}$ (1.3)

The metric $\tilde{d}$ is induced by the paranorm (1.3) .

The other is the topology $\tau_{\tilde{\lambda}}(gI)$ whose metric $\tilde{d}_{\tilde{\lambda}}$ is given by

$$\tilde{d}_{\tilde{\lambda}}(\tilde{x}, \tilde{y}) = \sup_k \{d(\tilde{\lambda}_k, \tilde{0})[d(\tilde{x}_k, \tilde{y}_k)]^{p_k/M}, \tilde{x}_k, \tilde{y}_k \in \lambda_0^p(gI)\} \quad (1.4)$$

2 MAIN RESULTS

Theorem 2.1. $\lambda_0^p(gI) = c_0^p(gI)$ if and only if $\tilde{\lambda} \in l_\infty^p(gI)$

Proof. Suppose that $\tilde{\lambda} \in l_\infty^p(gI)$

Always $\lambda_0^p(gI) \subset c_0^p(gI)$ (2.1)

Since $\tilde{\lambda} \in l_\infty^p(gI)$, we have $\tilde{\lambda}\tilde{x} \in c_0^p(gI)$ for every $\tilde{x} \in c_0^p(gI)$

Consequently, $\tilde{x} \in \lambda_0^p(gI)$ and so

$$c_0^p(gI) \subset \lambda_0^p(gI) \quad (2.2)$$

From equations (2.1) and (2.2), $\lambda_0^p(gI) = c_0^p(gI)$

On the other hand, Suppose that $\lambda_0^p(gI) = c_0^p(gI)$

If $\tilde{\lambda} \notin l_\infty^p(gI)$, then for each positive integer r there is a k(r) such that $d(\tilde{\lambda}_{k(r)}, 0) > r^{1/p_{k(r)}}$

Define $\tilde{x}$ by

$$\tilde{x}_k = \begin{cases} r^{-1/p_{k(r)}}[1,1] & k = k(r), r = 1,2,... \\ [0,0] & otherwise \end{cases}$$

then $\tilde{x} \in c_0^p(gI)$ and $d(\tilde{\lambda}_k \tilde{x}_k, \tilde{0}) > r^{1/p_{k(r)}} \times \frac{1}{r^{1/p_{k(r)}}} = 1$

This shows that $\tilde{\lambda}\tilde{x} \notin c_0^p(gI)$. This contradiction shows that $\tilde{\lambda} \in l_\infty^p(gI)$

Corollary . $\lambda_0(gI) = c_0(gI)$ if and only if $\tilde{\lambda} \in l_\infty(gI)$

Theorem 2.2. In order that $\lambda_0^p(gI) \subset \mu_0^p(gI)$ it is necessary and sufficient that

$$\min\{[d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0})]^{p_k}, d(\tilde{\mu}_k, \tilde{0})^{p_k}\} \tag{2.3}$$

is bounded

Proof. Let A denote the set of those positive integers k for which $[d(\tilde{\lambda}_k, \tilde{0})]^{p_k} > 1$

Let B denote the set of those positive integers k for which $[d(\tilde{\mu}_k, \tilde{0})]^{p_k} \le 1$

$k \in A$ implies $\min\{[d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0})]^{p_k}, d(\tilde{\mu}_k, \tilde{0})^{p_k}\} = [d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0})]^{p_k}$

$k \in B$ implies $\min\{[d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0})]^{p_k}, d(\tilde{\mu}_k, \tilde{0})^{p_k}\} = d(\tilde{\mu}_k, \tilde{0})^{p_k}$

Hence (2.3) is equivalent to the assertion that $[d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0})]^{p_k}$ is bounded for $k \in A$

$d(\tilde{\mu}_k, \tilde{0})^{p_k}$ is bounded for $k \in B$

Suppose that this holds and that $\tilde{x} \in \lambda_0^p(gI)$

If $k \in A$, write $\tilde{x}_k \tilde{\mu}_k \le (\tilde{x}_k \tilde{\lambda}_k) \tilde{\mu}_k / \tilde{\lambda}_k$

If $k \in B$, write $\tilde{x}_k \tilde{\mu}_k \le (\tilde{x}_k) \tilde{\mu}_k$

In either case $[d(\tilde{x}_k \tilde{\mu}_k, 0)]^{p_k}$ is arbitrary small for sufficiently large k. Hence $\tilde{x} \in \mu_0^p(gI)$

Thus $\lambda_0^p(gI) \subset \mu_0^p(gI)$

On the other hand if (2.3) is false ,we can find an increasing sequence of positive integers {k(r)} such that

$$[d(\tilde{\mu}_{k(r)} / \tilde{\lambda}_{k(r)}, 0)]^{p_{k(r)}} \ge r \tag{2.4}$$

and $d(\tilde{\mu}_{k(r)}, \tilde{0})^{p_{k(r)}}\} \ge r$ for r=1,2,... (2.5)

If $d(\tilde{\lambda}_{k(r)}, \tilde{0})^{p_{k(r)}}\} \ge 1$ choose $\tilde{x}_{k(r)} = \begin{cases} r^{-1/p_{k(r)}}[1,1] / \tilde{\lambda}_{k(r)} \text{ if } & k = k(r), r = 1,2,... \\ [0,0] & otherwise \end{cases}$

Then (2.4) gives $[d(\tilde{\mu}_{k(r)} \tilde{x}_{k(r)}, \tilde{0})]^{p_{k(r)}} \ge 1$

If $d(\tilde{\lambda}_{k(r)}, \tilde{0})^{p_{k(r)}}\} \le 1$ choose $\tilde{x}_{k(r)} = r^{-1/p_{k(r)}}[1,1]$. Then (2.5) gives $[d(\tilde{\mu}_{k(r)} \tilde{x}_{k(r)}, \tilde{0})]^{p_{k(r)}} \ge 1$

Thus in either case $\tilde{x} \in \lambda_0^p(gI)$ but $\tilde{x} \notin \mu_0^p(gI)$

This contradicts our present hypothesis that $\lambda_0^p(gI) \subset \mu_0^p(gI)$

This proves the theorem.

Corollary . $\lambda_0(gI) \subset \mu_0(gI)$ if and only if $\min\{d(\tilde{\mu}_k / \tilde{\lambda}_k, \tilde{0}), d(\tilde{\mu}_k, \tilde{0})\}$ is bounded

Theorem 2.3. The sequence space of modals $(\lambda_0^p(gI), \tau_{\tilde{\lambda}}(gI))$ is a complete metric space if and only if

$$\lim_{k\to\infty} \inf\{d(\tilde{\lambda}_k, \tilde{0})^{p_k/M}\} > 0 \qquad (2.6)$$

Proof. Suppose (2.6) holds. Let $(\tilde{x}^n)$ be a modal fundamental sequence in $(\lambda_0^p(gI), \tilde{d}_{\tilde{\lambda}})$. Then for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\tilde{d}_{\tilde{\lambda}}(\tilde{x}_k{}^n, \tilde{x}_k{}^m) < \varepsilon \quad \text{for all } n,m \geq n_0$$

then $\{d(\tilde{\lambda}_k, 0)[d(\tilde{x}_k{}^n, \tilde{x}_k{}^m)]\}^{p_k/M} < \varepsilon$ for all $n,m \geq n_0$ (2.7)

and so $[d(\tilde{x}_k{}^n, \tilde{x}_k{}^m)]^{p_k/M} < \varepsilon / L$ for all $n,m \geq n_0$ (2.8)

where $L = \inf\{d(\tilde{\lambda}_k, 0)^{p_k/M} \;\; k = 1,2,..\}$

This means that $\{\tilde{x}_k{}^n, \;\; n = 1,2,..\}$ is a modal fundamental sequence in gI. Since gI is a Banach space, $(\tilde{x}_k{}^n)$ is convergent. Now, let $\lim_n \tilde{x}_k{}^n = \tilde{x}_k$ for each $k \in N$ and $\tilde{x} = (\tilde{x}_k)$

Using the inequalities (1.4), (2.7), (2.8) and the fact that $(\tilde{x}_k{}^{n_0}) \in \lambda_0^p(gI)$ for each fixed n_0. It can be shown that $(\tilde{x}_k) \in \lambda_0^p(gI)$ and $\tilde{x}^n \to \tilde{x}$ *as* $n \to \infty$ *in* $\tau_\lambda(gI)$. Hence the condition is sufficient for $(\lambda_0^p(gI), \tau_{\tilde{\lambda}}(gI))$ to be complete.

Conversely, Suppose that $(\lambda_0^p(gI), \tau_{\tilde{\lambda}}(gI))$ is a complete metric space. If (2.6) is not true, then $\{d(\tilde{\lambda}_k, 0)^{p_k/M}\}$ contains a subsequence $\{d(\tilde{\lambda}_{k(i)}, 0)^{p_{k(i)}/M}\}$ which steadily decreases and tends to zero.

Consider the sequence of modals $\{\tilde{x}_k{}^n \;\; , n = 1,2,..\}$

where $\tilde{x}_k{}^n = \begin{cases} [1,1] & \text{if } k = k(1), k(2), \ldots k(n) \\ [0,0] & \text{otherwise} \end{cases}$

then $\tilde{x}_k{}^n \in c_0^p(gI)$ for all n=1,2,..

For $n > m$, we have $\tilde{d}_{\tilde{\lambda}}(\tilde{x}_k{}^m, \tilde{x}_k{}^n) = [d(\tilde{\lambda}_{k(n+1)}, \tilde{0})]^{p_{k(n+1)}/M} \to 0$ *as* $n, m \to \infty$

So that $\tilde{x}_k{}^n$ is a modal fundamental sequence in $(\lambda_0^p(gI), \tilde{d}_{\tilde{\lambda}})$.

If $\lim_n \tilde{x}_k{}^n$ exists, then $\lim_n \tilde{x}_k{}^n = \{[1,1],[1,1],\ldots\}$ which is not in $\lambda_0^p(gI)$. Thus $\lambda_0^p(gI)$ would cease to be complete, a contradiction. Hence (2.6) must hold whenever $\lambda_0^p(gI)$ is complete.

Corollary . The sequence space $(\lambda_0(gI), \tilde{d}_{\tilde{\lambda}})$ is a complete metric space if and only if

$$\lim_{k\to\infty}\{d(\tilde{\lambda}_k, \tilde{0})\} > 0 \text{ Where } \tilde{d}_{\tilde{\lambda}}(\tilde{x}, \tilde{y}) = \sup\{d(\tilde{\lambda}_k, \tilde{0}) d(\tilde{x}_k, \tilde{y}_k)\}.$$

Remark. $\chi(gI)$ is a complete metric space with respect to the metric $\tilde{\rho}(\bar{x}, \bar{y}) = \sup\{k![d(\tilde{x}_k, \tilde{y}_k)]^{1/k}\}$ for $\tilde{x} = (\tilde{x}_k)$, $\tilde{y} = (\tilde{y}_k)$ *in* $\chi(gI)$.

Theorem 2.4. $\tau(gI)$ is finer than $\tau_{\tilde{\lambda}}(gI)$ if and only if

$$\lim_{k\to\infty} \sup\{d(\tilde{\lambda}_k, \tilde{0})^{p_k/M}\} < \infty \qquad (2.9)$$

Proof. Suppose that (2.9) holds. Then

$$\sup\{d(\tilde{\lambda}_k,0)^{p_k/M}\} = D < \infty \tag{2.10}$$

For some positive real number D.

Let $\varepsilon > 0$ be any real number. Let $(\tilde{x}^n)$ be a any sequence of modal interval numbers converging to zero in $\lambda_0^p(gI)$ with respect $\tau(gI)$.Then there exists some n_0 such that $[d(\tilde{x}^n,\tilde{0})]^{p_k/M} < \varepsilon/D$ *for all* $n \geq n_0$.

Consequently, $\sup_k [d(\tilde{x}^n,\tilde{0})]^{p_k/M} < \varepsilon/D$ *for all* $n \geq n_0$ (2.11)

Now by using (2.10) and (2.11), $\tilde{d}_{\tilde{\lambda}}(\tilde{x}_k{}^n,\tilde{0}) = \sup_k\{d(\tilde{\lambda}_k,\tilde{0})[d(\tilde{x}_k{}^n,\tilde{0})]\}^{p_k/M}$

$$= \sup_k\{[d(\tilde{\lambda}_k,\tilde{0})]^{p_k/M}[d(\tilde{x}_k{}^n,\tilde{0})]^{p_k/M}\}$$

$$< \frac{\varepsilon}{D}D = \varepsilon \quad \text{for all } n \geq n_0$$

Therefore, $(\tilde{x}^n)$ converging to zero in $\lambda_0^p(gI)$ with respect $\tau(gI)$. In other words, the identity map on $(\lambda_0^p(gI),\tau(gI))$ onto $(\lambda_0^p(gI),\tau_{\tilde{\lambda}}(gI))$ is continuous. Hence $\tau(gI) \supset \tau_{\tilde{\lambda}}(gI)$

Conversely, Suppose that $\tau(gI)$ is finer than $\tau_{\tilde{\lambda}}(gI)$. If (2.9) were not true, then there exists a subsequence $\left\{[d(\tilde{\lambda}_{k(n)},\tilde{0})]^{p_{k(n)}/M}, n = 1,2,....\right\}$ which is strictly increasing and tends to infinity.

This implies that $\dfrac{1}{[d(\tilde{\lambda}_{k(n)},\tilde{0})]^{p_{k(n)}/M}} \to \tilde{0}$ *as* $n \to \infty$ (2.12)

Take $(\tilde{x}^i) = \{\theta,\theta,....\theta,\dfrac{[1,1]}{d(\tilde{\lambda}_{k(i)},\tilde{0})},\theta,.....\}$ where $d(\tilde{\lambda}_{k(i)},\tilde{0})$ occurs in the k(i)th place, for i=1,2,.. and zeros elsewhere. Then $\tilde{x}^1$, $\tilde{x}^2$,... $\tilde{x}^n$,..all belong to $\lambda_0^p(gI)$.

Also $\tilde{d}_{\tilde{\lambda}}(\tilde{x}^n,\tilde{0}) = 1$, so that $\tilde{x}^n$ does not tends to zero with respect to $\tau_{\tilde{\lambda}}(gI)$.But $[d(\tilde{x}^n,\tilde{0})]^{p_k/M} \to 0$ *as* $n \to \infty$ using (2.12)

Hence $\tau_{\tilde{\lambda}}(gI) \not\subset \tau(gI)$. Thus (2.9) is necessary.

Theorem 2.5. Let $\lambda_0^p(gI)$ be a complete metric space and let $q = (q_k)$ a bounded sequence of strictly positive real numbers. Then the following are equivalent.

(i) $\lambda_0^p(gI) \subset \lambda_0^q(gI)$

(ii) $\liminf_{k\to\infty}\left\{\dfrac{q_k}{p_k}\right\} > 0$

Proof. Proof of (i)$\Rightarrow$(ii):

Assume that (ii) is not true. Then we can determine an increasing sequence of positive integers $k(1) < k(2) < ....$ such that $q_{k(i)} < \dfrac{1}{i}p_{k(i)}$.

Define $\tilde{x}_k^{\ i} = \begin{cases} \left[\dfrac{1}{[id(\tilde{\lambda}_{k(i)},\tilde{0})]^{p_{k(i)}}}\right]^{1/p_{k(i)}} [1,1] & for\ k = k(i) \\ [0,0] & otherwise \end{cases}$

Then $[d(\tilde{\lambda}_{k(i)}\tilde{x}_{k(i)},\tilde{0})]^{p_{k(i)}} < \frac{1}{i} \to 0\ as\ i \to \infty$

Also $[d(\tilde{x}_{k(i)},\tilde{0})]^{p_{k(i)}} < \dfrac{1}{i[d(\tilde{\lambda}_{k(i)},\tilde{0})]^{p_{k(i)}}} < \dfrac{1}{iL} \to 0\ as\ i \to \infty$

Since $\lambda_0^p(gI)$ is complete , but $[d(\tilde{\lambda}_{k(i)}\tilde{x}_{k(i)},\tilde{0})]^{p_{k(i)}} > \exp[-\log i/i] > \exp(-1/2)$

This shows that $(\tilde{x})$ does not belong to $\lambda_0^q(gI)$ which contradicts (i) and so (ii) must hold.

Proof of (ii) $\Rightarrow$ (i):

Suppose (ii) holds. Then there exists $\alpha > 0$ such that $q_k > \alpha p_k$ for all sufficiently large k.

Let $\tilde{x} \in \lambda_0^p(gI)$ then for all sufficiently large k, $[d(\tilde{\lambda}_k\tilde{x}_k,\tilde{0})]^{q_k} \le \left\{[d(\tilde{\lambda}_k\tilde{x}_k,\tilde{0})]^{p_k}\right\}^{\alpha}$

Since $[d(\tilde{\lambda}_k\tilde{x}_k,\tilde{0})] \le 1$ for such k.

Hence $\{\tilde{\lambda}_k\tilde{x}_k\} \in c_0^q(gI))$ Also $\{\tilde{x}_k\} \in c_0^q(gI)$. Therefore $\tilde{x} \in \lambda_0^q(gI))$

Consequently, $\lambda_0^p(gI) \subset \lambda_0^q(gI)$

Hence (ii) $\Rightarrow$ (i).

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