ISSN 2347-1921

THE PERIOD OF 2-STEP AND 3-STEP SEQUENCES IN DIRECT PRODUCT OF MONOIDS

Melis Minisker
Mustafa Kemal University, Faculty of Education, Antakya, Hatay, TURKEY
minisker38@gmail.com

ABSTRACT

Let M and N be two monoids consisting of idempotent elements. By the help of the presentation which defines $M \times N$, the period of 2-step sequences and 3-step sequences in $M \times N$ is given.

Indexing Terms / Keywords

Monoid; period; sequence; direct product

Academic Discipline And Sub-Disciplines

Mathematics, Algebra and Number Theory, Semigroup Theory

SUBJECT CLASSIFICATION

20M05

TYPE (METHOD/APPROACH)

Combinatorial methods

ISSN 2347-1921

INTRODUCTION

The study of Fibonacci sequences in groups began with the earlier work of Wall [see 14]. He investigated the ordinary Fibonacci sequences in cyclic groups. The problem was extended to abelian groups in the mid eighties by Wilcox [see 15]. Campbell, Doostie and Robertson expanded the theory to some finite simple groups in [4]. Aydin and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. In [5, 6, 11] the theory has been generalized to the 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2,3,n. Then it is shown in [1] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent and nilpotency class . There has been much interest in applications of Fibonacci numbers and sequences for several years. 2-step Fibonacci sequences in finite nilpotent groups of nilpotency class 4 has been obtained by Karaduman and Aydin in [8]. Karaduman and Yavuz proved that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 are the periods of ordinary 2-step Fibonacci sequences [see 9].

A *k*-nacci sequence in a finite group is a sequence of group elements $x_0,x_1,x_2,...,x_n,...$ for which, given an initial (seed) set $x_0,x_1,...,x_{j-1}$, each element is defined by

$$x_0x_1x_2...x_{n-1} \text{ for } j\leq n < k$$

$$x_{n-k}\, x_{n-k+1}...\, x_{n-1} \text{ for } n\geq k$$

The initial elements of the sequence, $x_0,x_1,x_2,...,x_{j-1}$ generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The *k*-nacci sequence of a group generated by $x_0,x_1,x_2,...,x_{j-1}$ is denoted by $F_k(G;x_0,x_1,x_2,...,x_{j-1})$. 2-step Fibonacci sequence in the integers modulo can be written as $F_2(Z_m, 0,1)$. We call a 2-step Fibonacci sequence of a group elements a Fibonacci sequence of a finite group. A finite group is *k*-nacci sequenceable if there exists a *k*-nacci sequence of such that every element of the group appears in the sequence.

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence *a, b ,c, d, e, b, c, d, e, b, c, d, e,...* is periodic after the initial element *a* and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence *a, b ,c ,d, e, f, a, b, c, d, e, f, a, b, c, d, e, f,...* is simply periodic with period 6.

Semigroup presentations have been studied over a long period, usually as a means of providing examples of semigroups. In [10], B.H. Neumann introduced an enumeration method for finitely presented semigruops analogous to the Todd-Coxeter coset enumeration process for group[13]. For about semigroup presentations see[12]. To find a minimal presentation for an arbitrary semigroup is another branch of study in semigroup theory. In [3] a minimal presentation for CL_n is given. Thus it is shown that CL_n is an efficient semigroup. It is also shown that $CL_m\times CL_n$ is inefficient for arbitary $m,n\in N$.

Let *M* and *N* be two monoids consisting of idempotent elements. The direct product of monoids is $M\times N$. Assume that the number of generators of *M* is *m* and the number of generators of *N* is *n*. Let *p* denote the period of sequences in $M\times N$. In this paper we prove that the period of 2-step sequences in $M\times N$ is $p=(m+n-2)(m+n)+2$ and the period of 3-step sequences in $M\times N$ is $p=\frac{(m+n-3)}{2}(m+n)+2$. (if $m+n$ is odd) and $p=\frac{(m+n-2)}{2}(m+n)+2$ (if $m+n$ is even).

Let *A* be an alphabet. We denote by A^+ the free semigroup on *A* consisting of all non-empty words over *A*. A semigroup presentation is an ordered pair of $<A/R>$, where $R\subseteq A^+\times A^+$. A semigroup is said to be defined by the semigroup presentation $<A/R>$ if *S* is isomorphic to A^+/ρ where ρ is the congruence on A^+ generated by *R*. Let *u* and *v* be two words in A^+. We write $u\equiv v$ if *u* and *v* are identical words, and write $u=v$ if $(u,v)\in\rho$, that is *v* is obtained from *u* by applying relations from *R* or equivalently there is a finite sequence

$$u\equiv\alpha_1,\alpha_2,\alpha_3,...,\alpha_n\equiv v$$

of words from A^+ in which every α_i is obtained from α_{i-1} by applying a relation from *R*.(see [7, Proposition 1.5.9]). If both *A* and *R* are finite sets then $<A/R>$ is said to be a finite presentation. If a semigroup *S* can be defined by a finite presentation then *S* is said to be finitely presented.

DIRECT PRODUCT OF MONOIDS AND 2-STEP AND 3-STEP SEQUENCES

Let *M* be a monoid with generating set $A=\{a_1,a_2,...,a_m\}$ and *N* be a monoid with generating set $B=\{b_1,b_2,...,b_n\}$. Assume that *M* and *N* consist of idempotent elements. Also assume that M is defined by the presentation $<A/R>$ and *N* is defined by the presentation $<B/Q>$. Then $M\times N$ is defined by the presentation $<A,B/R,Q,C>$ where *C* is the group of relations $\{a_ib_j=b_ja_i\ (1\leq i\leq m, 1\leq j\leq n)\}$.

Now we define *2*-step sequences in $M\times N$ as $x_i=x_{i-n}x_{i-n-1}$ and 3-step sequences in $M\times N$ as

$x_i=x_{i-n}x_{i-n-1}x_{i-n-2}$ for $i>n$.

Theorem 2.1. Let M and N be two monoids. Assume that M is defined by the presentation $<A/R>$ and N is defined by the presentation $<B/Q>$. Also assume that M and N consists of only idempotent elements. Then 2-step sequences in $M\times N$ is periodic and the period of the sequence of is equal to $p=(m+n-2)(m+n)+2$.

Proof. The first $m+n$ terms of sequence are $a_1, a_2, a_3, \ldots, a_m, b_1, b_2, \ldots, b_n$. For simplicity, we use indices instead of generating elements of $M\times N$ in our process. Since $x_i=x_{i-n}x_{i-n-1}$, for $i>n$ we have

$$x_{m+n+1}=x_1x_2=a_1a_2$$

$$x_{m+n+2}=x_2x_3=a_2a_3$$

$$x_{m+n+3}=x_3x_4=a_3a_4$$

$$x_{m+n+4}=x_4x_5=a_4a_5$$

.

.

.

$$x_{2(m+n)}=x_{m+n}\, x_{m+n+1}=b_na_1a_2=a_1a_2b_n$$

$$x_{2(m+n)+1}=x_{m+n+1}\, x_{m+n+2}=a_1a_2a_3$$

$$x_{2(m+n)+2}=x_{m+n+2}\, x_{m+n+3}=a_2a_3a_4$$

$$x_{2(m+n)+3}=x_{m+n+3}\, x_{m+n+4}=a_3a_4a_5$$

.

.

.

$$x_{3(m+n)}=x_{2(m+n)}\, x_{2(m+n)+1}=a_1a_2a_3b_n$$

$$x_{3(m+n)+1}=x_{2(m+n)+1}\, x_{2(m+n)+2}=a_1a_2a_3a_4$$

$$x_{3(m+n)+2}=x_{2(m+n)+2}\, x_{2(m+n)+3}=a_2a_3a_4a_5$$

.

.

.

$$x_{(m+n-2)(m+n)}=x_{(m+n-3)(m+n)}x_{(m+n-3)(m+n)+1}=x_1x_2\ldots x_{(m+n-2)}x_{(m+n)}=$$

$$a_1a_2\ldots a_mb_1b_2\ldots b_{n-2}b_n$$

$$x_{(m+n-2)(m+n)+1}=a_1a_2\ldots a_mb_1b_2\ldots b_{n-1}$$

$$x_{(m+n-2)(m+n)+2}=a_2a_3\ldots a_mb_1b_2\ldots b_n$$

$$x_{(m+n-2)(m+n)+3}=a_1a_2a_3\ldots a_mb_1b_2\ldots b_n$$

Thus we have the period of 2-step sequences of $M\times N$ is $p=(m+n-2)(m+n)+2$. Now we will examine the period of 3-step sequences of $M\times N$.

Theorem 2.2. Let M and N be two monoids. Assume that M is defined by the presentation $<A/R>$ and N is defined by the presentation $<B/Q>$. Also assume that M and N consists of only idempotent elements. Then 3-step sequences in $M\times N$ is periodic and the period of the sequence of is equal to $p=((m+n-3)/2)(m+n)+2$ (if $m+n$ is odd) and $p=((m+n-2)/2))(m+n)+2$ (if $m+n$ is even).

Proof. The first $m+n$ terms of sequence are $a_1, a_2, a_3, \ldots, a_m, b_1, b_2, \ldots, b_n$. First of all we consider the case when $m+n$ is odd. Since we define 3-step sequences as $x_i=x_{i-n}x_{i-n-1}x_{i-n-2}$ for $i>n$ we have

$$x_{m+n+1}=x_1x_2x_3=a_1a_2a_3$$

$$x_{m+n+2}=x_2x_3x_4=a_2a_3a_4$$

$$x_{m+n+3}=x_3x_4x_5=a_3a_4a_5$$

.

.

.

$$x_{2(m+n)}=x_{(m+n)}x_{(m+n+1)}x_{(m+n+2)}=b_n(a_1a_2a_3)(a_2a_3a_4)=a_1a_2a_3a_4b_n$$

.

.

.

$$x_{(2(m+n)+1)}=x_{(m+n+1)}x_{(m+n+2)}x_{(m+n+3)}=(a_1a_2a_3)(a_2a_3a_4)(a_3a_4a_5)=a_1a_2a_3a_4a_5$$

$$x_{(2(m+n)+2)}=x_{(m+n+2)}x_{(m+n+3)}x_{(m+n+4)}=(a_2a_3a_4)(a_3a_4a_5)(a_4a_5a_6)=a_2a_3a_4a_5a_6$$

$$x_{(2(m+n)+3)}=x_{(m+n+3)}x_{(m+n+4)}x_{(m+n+5)}=(a_3a_4a_5)(a_4a_5a_6)(a_5a_6a_7)=a_3a_4a_5a_6a_7$$

.

.

$$x_{(3(m+n))}=x_{(2(m+n))}x_{(2(m+n)+1)}x_{(2(m+n)+2)}=$$

$$(a_1a_2a_3a_4b_n)(a_1a_2a_3a_4a_5)$$

$$(a_2a_3a_4a_5a_6)=a_1a_2a_3a_4a_5a_6b_n$$

$$x_{(3(m+n)+1)}=x_{(2(m+n)+1)}x_{(2(m+n)+2)}x_{(2(m+n)+3)}=(a_1a_2a_3a_4a_5)(a_2a_3a_4a_5a_6)$$

$$(a_3a_4a_5a_6a_7)=a_1a_2a_3a_4a_5a_6a_7$$

$$x_{(3(m+n)+2)}=x_{(2(m+n)+2)}x_{(2(m+n)+3)}x_{(2(m+n)+4)}=(a_2a_3a_4a_5a_6)(a_3a_4a_5a_6a_7)$$

$$(a_4a_5a_6a_7a_8)=a_2a_3a_4a_5a_6a_7a_8$$

.

.

.

$$x_{((m+n-3)/2)(m+n)}=x_{((m+n-5)/2)(m+n)}x_{((m+n-5)/2)(m+n)+1}x_{((m+n-5)/2)(m+n)+2}=$$

$$a_1a_2a_3\ldots a_mb_1b_2\ldots b_{n-3}b_n$$

$$x_{((m+n-3)/2)(m+n)+1}=a_1a_2\ldots a_mb_1b_2\ldots b_{(n-2)}$$

$$x_{((m+n-3)/2)(m+n)+2}=a_2a_3\ldots a_mb_1b_2\ldots b_n$$

$$x_{((m+n-3)/2)(m+n)+3}=a_1a_2\ldots a_mb_1b_2\ldots b_n$$

Thus we obtain the period of 3-step sequences of $M\times N$ is $p=((m+n-3)/2)(m+n)+2$ if $m+n$ is odd.

Now we consider the case when $m+n$ is even. As given above the first $m+n$ terms of sequence are a_1, a_2, a_3,..., a_m,b_1, b_2,..., b_n. Since we define 3-step sequences as $x_i=x_{i-n}x_{i-n-1}x_{i-n-2}$ for $i>n$ we have

$$x_{m+n+1}=x_1x_2x_3=a_1a_2a_3$$

$$x_{m+n+2}=x_2x_3x_4=a_2a_3a_4$$

$$x_{m+n+3}=x_3x_4x_5=a_3a_4a_5$$

.

.

.

$$x_{2(m+n)}=x_{(m+n)}x_{(m+n+1)}x_{(m+n+2)}=b_n(a_1a_2a_3)(a_2a_3a_4)=a_1a_2a_3a_4b_n$$

$$x_{(2(m+n)+1)}=x_{(m+n+1)}x_{(m+n+2)}x_{(m+n+3)}=(a_1a_2a_3)(a_2a_3a_4)(a_3a_4a_5)=a_1a_2a_3a_4a_5$$

$$x_{(2(m+n)+2)}=x_{(m+n+2)}x_{(m+n+3)}x_{(m+n+4)}=(a_2a_3a_4)(a_3a_4a_5)(a_4a_5a_6)=a_2a_3a_4a_5a_6$$

$$x_{(2(m+n)+3)}=x_{(m+n+3)}x_{(m+n+4)}x_{(m+n+5)}=(a_3a_4a_5)(a_4a_5a_6)(a_5a_6a_7)=a_3a_4a_5a_6a_7$$

.

.

$$x_{(3(m+n))}=x_{(2(m+n))}x_{(2(m+n)+1)}x_{(2(m+n)+2)}=$$

$$(a_1a_2a_3a_4b_n)(a_1a_2a_3a_4a_5)$$

$$(a_2a_3a_4a_5a_6)=a_1a_2a_3a_4a_5a_6b_n$$

$$x_{(3(m+n)+1)}=x_{(2(m+n)+1)}x_{(2(m+n)+2)}x_{(2(m+n)+3)}=(a_1a_2a_3a_4a_5)(a_2a_3a_4a_5a_6)$$

$$(a_3a_4a_5a_6a_7)=a_1a_2a_3a_4a_5a_6a_7$$

$$x_{(3(m+n)+2)}=x_{(2(m+n)+2)}x_{(2(m+n)+3)}x_{(2(m+n)+4)}=(a_2a_3a_4a_5a_6)(a_3a_4a_5a_6a_7)$$

$$(a_4a_5a_6a_7a_8)=a_2a_3a_4a_5a_6a_7a_8$$

.

.

.

$$x_{((m+n-2)/2)(m+n)}=a_1a_2a_3\ldots a_mb_1b_2\ldots b_{n-1}$$

$$x_{((m+n-2)/2)(m+n)+1}=a_1a_2\ldots a_mb_1b_2\ldots b_{(n-2)}$$

$$x_{((m+n-3)/2)(m+n)+2}=a_2a_3\ldots a_mb_1b_2\ldots b_{n-1}b_n$$

$$x_{((m+n-3)/2)(m+n)+3}=a_1a_2\ldots a_mb_1b_2\ldots b_n$$

Thus we have the period of 3-step sequences of $M\times N$ is $p=((m+n-2)/2)(m+n)+2$ when $m+n$ is even.

CONCLUSION

In this paper we determine the period of 2-step and 3-step sequences for the direct product of two monoids M and N ($M \times N$) which contain idempotent elements. In future studies it may be possible to examine the period of 2-step, 3-step and n-step sequences for different kinds of semigroups.

REFERENCES

[1] Aydin, H. and Dikici, R. 1998. General Fibonacci sequences in finite groups. Fibonacci Quarterly 36.3 (1998), 216-221.

[2] Aydin, H. and Smith, G.C. 1994. Finite-quotients of some cyclicaly presented groups. J. London Math. Soc. 49.2 (1994), 83-92.

[3] Ayik, H., Minisker,M. and Vatansever B. 2005. Minimal presentations and embedding into inefficient semigroups. Algebra Colloquium, 12 (2005) 59-65.

[4] Campbell, C.M., Doostie,H. and Robertson, E.F. 1990. Fibonacci length of generating pairs in groups. Applications of Fibonacci Numbers 3 d. G. E. Bergum et al.Kluwer Academic Publishers ,(1990) 27-35.

[5] Dikici, R. and Smith, G.C. 1995. Recurrences in finite groups, Turkish J. Math. 19 (1995), 321-329.

[6] Dikici, R. and Smith,G.C.1997. Fibonacci sequences in finite nilpotent Groups, Turkish J. Math. 21 (1997),133-142.

[7] Howie, J.M. 1995. Foundations of Semigroup Theory, Clarendon Press.

[8] Karaduman, E. and Aydin, H. 2003. General 2-step Fibonacci sequences in nilpotent groups of exponent p and nilpotency class 4. Applied Mathematics and Computation 141 (2003), 491-497.

[9] Karaduman, E. and Yavuz, U. 2003. On the period of Fibonacci sequences in nilpotent groups. Applied Mathematics and Computation 142 (2003) 321-332.Spector, A. Z. 1989.

[10] Neumann, B.H. 1967. Some remarks on semigroup presentations. Canad. J. Math. 19 (1967) 1018-1026.

[11] Ozkan,E. 2003. 3-step Fibonacci sequences in nilpotent groups. Applied Mathematics and Computation 144 (2003) 517-527.

[12] Robertson, E.F. and Unlu, Y. 1992. On semigroup presentations. Proceedings of Edingburg Mathematical Society 36 (1992) 55-68.

[13] Todd Coxeter, J.A. and H. S. M. 1936. A partical method for enumeratings cosets of a finite Abstract group. Proceedings of Edingburg Mathematical Society 5 (1936) 26-34.

[14] Wall, D.D 1969. Fibonacci series module. Amer. Math. Monthly 67 (1969) 525-532.

[15] Wilcox, H.J. 1986. Fibonacci sequences of period in groups. Fibonacci Quarterly 24 (1986) 356-361

Author' biography with Photo

Associate Prof. Dr. Melis Minisker was born in Adana, Turkey in 1974. She has graduated from Cukurova University, Mathematics Department in Adana in 1996. She has finished master's degree and doctoral degree in Cukurova University Mathematics Department. She has worked as a research asistant in Cukurova University Mathematics Department for 8 years. In 2005 she has begun to work as Asistant Prof. Dr. in Mustafa Kemal University Faculty of Education, Antakya, Hatay, TURKEY. She has taken Associate Prof. Dr. degree in 2010. She has studied on new research papers since then and has supervised 10 master's students. 5 of them have finished their master's degree on mathematics education. She has research papers and projects in algebra, semigroup theory. She has also some presentations in Mathematics Education symposiums in Turkey.