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ON GENERALIZATION OF INJECTIVE MODULES

Mehany, M.S.

EL-Arabey, A.E., Kamal, M.A., EL-Baroudy, M.H., and Ammar, A.Y.

ABSTRACT: Here we introduce the concept of CK-N-injectivity as a generalization of N-injectivity. We give a homomorphism diagram representation of such concept, as well as an equivalent condition in terms of module decompositions. The concept CK-N-jectivity is also dealt with, as a generalization of CK-N-injectivity. We introduce a generalization of N-injectivity, namely C-N-injectivity. Its generalization CI-N-injectivity (given in [8] as C-N-injectivity). In our study of C-N-injectivity, we discovered some mistake results (given in [1] as IC-Pseudo-injecyivity), and we dealt with their corrections. Finally we turn our attention to a more generalization of injective modules, namely the generalized extending modules (or module with (C_1^*)) and obtained some important results.

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1 INTRODUCTION

Throughtout this paper, R is an associative ring with identity and all modules are unitary right R-modules. A submodule N of an R-module M is called an essential submodule in M (denoted by $N \leq^e M$) if N \ K 6= 0 for any non-zero submodule K of M. For a submodule C of an R-module M is called closed in M (denoted by $N \leq^c M$) if C has no proper essential extensions in M. Clearly, every direct summand of M is closed in M. Moreover, if A is any submodule of M, then there exists, by zorn's Lemma, a submodule B of M maximal with respect to the property that A is an essential submodule of B, and in this case B is a closed submodule of M. A module M is an extending module (or a CS-module, or a module with (C_1)) if every closed submodule is a direct summand (or equivalently, if L ≤ M, then there is a decomposition $M = M_1$ © M_2, such that L ≤ M_1 and L © $M_2 \leq^e M$): For the properties of closed submodules and extending modules (see [2] , [9]): In[6] ; a module M has the condition (C_1^*) (given in [11] as (C_{11})) if every submodule of M has a complement which is adirect summand of M (equivalently, every closed submodule has a complement which is a direct summand, or if L ≤ M, then there is a decomposition $M = M_1$ © M_2, such that L \ M_2 = 0; and L © $M_2 \leq^e M$). It is well known that the condition (C_1) is inherited by direct summands, while the inheritance of modules having the condition (C_1^*) is not so (given by an example in [12]). In Lemma 22, we prove that if a module $M = M_1$ © M_2, then M_i (i = 1,2) has (C_1^*) if and only if for every submodule of M with zero intersection with M_j (j≠ i) has a complement summand submodule of M. As an immediate result of Lemma 22, we obtained Corollary 23, namely, if $M = Z_2(M)$ © F, then both have (C_1^*) if and only if every submodule C of M, with zero intersection with $Z_2(M)$ (or with F) has a complement summand containing F (or $Z_2(M)$). An extenging module M which satisfies the condition (C_2): (every submodule of M which is isomorphic to a direct summand of M, is itself direct summand), is called continuous. We introduce the concept of CK-injectivity as the following: Let M and N be an R-modules, M is said to be CK-N-injective if for every submodule X of N and every homomorphism f : X ! M, with ker f $\leq^c$ N can be extended to a homomorphism $\bar{f}$: N ! M. An R-module is CK-injective, if it is CK-N-injective for all R-modules N. Here we shows that a module M is CK-N-injective if and only if for every closed submodule L of M © N, with L \ M = 0, and L \ N $\leq^c$ N, there exists a submodule M′ of M © N, such that M ©N = M © M′; and L ≤ M′. We shows that the concept of CK-N-injectivity is inherited by direct summands on both ways, we also study the properties of such concept. It is clear that if M is N-injective, then M is CK-N-injective. Example 5, shows that there are CK-injective modules, which is not injective. An R-module M is said to be C-N-injective, if for every closed submodule N′ of N, and every monomorphism → : N′ ! N, and every homomorphism f : N′ ! M, there exists a homomorphism ℘ : N ! M, such that ℘→ = f. We prove that a module N is continuous if and only if K is C-N-injective for every closed submodule K of N. Example 35, tells us that there exists an R-modules that are C-injective modules, which are not injective. An R-module M is said to be CI-N-injective,if for every closed submodule N′ of N, and for every homomorphism f from N′ to M, there exists a homomorphism $\bar{f}$: N ! M, such that $\bar{f}$ j $_{N'}$ = f:

2 CK– INJECTIVE MODULES

Definition 1: A module M is said to be CK-N-injective, if for every submodule X of N and every homomorphism $f : X \to M$, with $\ker f \leq^c N$ can be extended to a homomorphism $\bar{f} : N \to M$.

Theorem 2: *Let M and N be R-modules. Then the following are equivalent:*

1. M is CK-N-injective.

2. For every submodule L of $M \oplus N$, with $L \cap M = 0$, and $L \cap N \leq^c N$, there exists a submodule

M′ of M ⊕ N, such that M ⊕ N = M ⊕ M′, and L ≤ M′.
3. For every closed submodule L of M ⊕ N, with L ∩ M = 0, and L ∩ N $\leq^c$ N, there exists a submodule M′ of M ⊕ N, such that M ⊕ N = M ⊕ M′, and that L ≤ M′.

Proof. (1) ⇒ (2): Let L ≤ M ⊕ N, L ∩ M = 0, and that L ∩ N $\leq^c$ N. Write K = N ∩ (L ⊕ M), and let π : L ⊕ M → M be the projection. Then ker ($\pi|_K$) = L ∩ K = L ∩ N $\leq^c$ N. Since M is CK-N-injective, we have that there exists a homomorphism f : N → M, such that $f|_K = \pi|_K$. Put M′ = { n – f (n) | n ∈ N }, then for all m ∈ M and n ∈ N, we have m + n = (m + f (n)) + (n – f (n)) ∈ M ⊕ M′, and hence M ⊕ N = M ⊕ M′. Now let l ∈ L, as l = m + n (m ∈ M, n ∈ N), we have l – m = n ∈ N ∩ (L ⊕ M), then $\pi|_K$ (l – m) =π|K (n) = f(n), then l = n + m = n – f(n) ∈M′, and hence L ≤ M′.

(2) ⇒ (1): Let X be a submodule of N, and f : X → M be a homomorphism, with ker f $\leq^c$ N. Choose W = { x - f(x) | x ∈ X }, it follows that W ∩ M = 0, and W ∩ N = ker f $\leq^c$ N. By assumption, there exists M′≤ M ⊕ N, such that M ⊕ N = M ⊕ M′, W ≤ M′. Let π denote the projection of M ⊕ M′ onto M, then for every x ∈ X, we have that π (x) = π (f(x) + (x - f(x)) = π (f(x)) = f(x). Therefore M is CK-N-injective.

(2) ⇒ (3): It is clear.

(3) ⇒ (2): Let L ≤ M ⊕ N, such that L ∩ M = 0, L ∩ N $\leq^c$ N, and let K be a maximal essential extension of L in M ⊕ N, then K $\leq^c$ M ⊕ N, and K ∩ M = 0. Since L $\leq^e$ K, we have that L ∩ N $\leq^e$ K ∩ N, and hence K ∩ N $\leq^c$ N. By assumption, there exists M′≤ M ⊕ N, such that M ⊕ N = M ⊕ M′, K ≤ M′, and hence there exists M′≤ M ⊕ N, such that M ⊕ N = M ⊕ M′, and that L ≤ M′.

Proposition 3: Let M be CK-N-injective, then M ⊕ N = M ⊕ C holds for every complement C of M in M ⊕ N, with C∩N $\leq^c$ N.

Proof. Let C be a complement of M in M ⊕ N, with C∩N $\leq^c$ N, then C $\leq^c$ M ⊕ N. By Theorem 2, there exists M′≤ M ⊕ N, such that M ⊕ N = M ⊕ M′ , C ≤ M′ . But C is maximal zero intersection of M in M ⊕ N, then M′ = C.

Lemma 4: Let M be N-injective, then M is CK-N-injective.

Proof. It is clear.

The following example shows that CK-N-injective need not be N-injective.

Example 5: $\mathbb{Z}_2$ is CK-$\mathbb{Z}$-injective, which is not injective.

Proposition 6: Let M = A ⊕ B, where B is CK-A-injective. Let A = A_1⊕ A_2, and B = B_1 ⊕ B_2. Then the following are satisfies (for i,j = 1,2) :

1. B_i is CK-A-injective.
2. B is CK-A_j-injective.
3. B_i is CK-A_j-injective.

Proof. For 1. Write M = A ⊕ B_1 ⊕ B_2. Let L ≤ A ⊕ B_1, such that L ∩ B_1 = 0, and that L ∩ A $\leq^c$ A, then L ≤ M, L ∩ B = 0. Since B is CK-A-injective, we have that there exists M′ ≤ M, such that M = M′⊕ B_1 ⊕ B_2, and that L ≤ M′. Then A ⊕ B_1 = [(A ⊕ B_1) ∩ (B2 ⊕ M′)] ⊕ B_1, L ≤ (A ⊕ B_1) ∩ (B2 ⊕ M′). Then B_1 is CK-A-injective.

For 2. Write M = A_1 ⊕ A_2 ⊕ B. Let L ≤ A_1 ⊕ B, such that L ∩ B = 0, and that L ∩ A $\leq^c$ A_1. It is clear that L ≤ M, and L ∩ A = L ∩ A_1. Since B is CK-A-injective, then there exists M′ ≤ M, such that M = M′⊕ B, and that L ≤ M′. Then A_1 ⊕ B = [(A_1⊕ B) ∩ M′] ⊕ B, and L ≤ (A_1 ⊕ B) ∩ M′. Hence B is CK-A_1-injective.

For 3. Follows from (1) and (2).

Proposition 7: *Let M be CK-N-injective, and N′ be a closed submodule of N. Then M is CK-N′-injective.*

Proof. Let X be a submodule of N′, and f be a homomorphism from X into M with ker f $\leq^c$ N′. Hence ker f $\leq^c$ N. Since M is CK-N-injective, then there exists a homomorphism $\bar{f}$ from N into M, such that $f^{-}|X = f$.

Proposition 8: *If M is CK-N-injective, and N isomorphic to W, then M is CK-W-injective.*

Proof. Let X be a submodule of W, and $f : X \to M$ be a homomorphism, with $\ker f \leq^c W$, and let ψ be an isomorphism from W into N, then $f\,\psi^{-1}$ be a homomorphism from $\psi(X)$ into M. Claim that $\ker(f\,\psi^{-1}) \leq^c N$. So let $\psi(\ker f) \leq^e K \leq N$, we have that $\ker f \leq^e \psi^{-1}(K) \leq W$, and that $\ker f = \psi^{-1}(K)$, then $\psi(\ker f) = K$, and consequently $(f\,\psi^{-1}) \leq^c N$. By assumption, there exists a homomorphism θ from N into M with $\theta|_{\psi(X)} = f$. Define $\theta\psi: W \to M$, then $\theta\psi(w) = f(w)$, for any $w \in W$. Hence M is CK-W-injective.

Proposition 9: *If M is CK-N-injective, and N′ is a direct summand submodule of N, then M is CK-N/N′-injective.*

Proof. Write $M = N' \oplus K$. Proposition 7, tells us that M is CK-K-injective. Since K is isomorphic to M/N′, then M is CK-N/N′-injective.

The following example shows that CK-N-injective need not be N/N′-injective, for every submodule N′ of N.

Example 10: *$\mathbb{Z}_2$ is CK-$\mathbb{Z}$-injective (by example 2.5) and $\mathbb{Z}_2$ is not CK-$\mathbb{Z}/p^n\mathbb{Z}$ (p- prime, n=2,3,4.....)-injective.*

Proposition 11: *Let M be CK-N-injective and N′ be a closed submodule of N. Then every monomorphism from a submodule X/N′ of N/N′ into M can be extended to a homomorphism from N/N′ into M.*

Proof. Let X be a submodule of N which contains N′, and let $\phi : X/N' \to M$ be a monomorphism. Let π denote the natural epimorphism of N onto N/N′and $\pi' = \pi|_X$, then $\ker(\phi\pi') = \pi'^{-1}(\ker\phi) = \pi'^{-1}(0) = \ker\pi' = N' \leq^c N$. Since M is CK-N-injective, we have that there exists a homomorphism θ from N to M, such that $\theta|_X = \phi\pi'$. Since $\theta(N') = \phi\pi'(N') = \phi(0) = 0$, we have that $\ker\pi \leq \ker\theta$, and consequently there exists a homomorphism ψ from N/N′ to M, such that $\psi\pi = \theta$. It follows that for every $x + N' \in X/N'$, $\psi(x + N') = \psi\,\pi'(x) = \theta(x) = \phi\pi'(x) = \phi(x + N')$. Thus ψ extends ϕ.

Lemma 12: ([6], Lemma 2.3.) It was shown that if $M = N \oplus K$, and C is a complement in N of a submodule A of N. Then

(1) $C \oplus K$ is a complement of A in M.

(2) C is a complement of $A \oplus K$ in M.

(In [6]) A module M is said to be N-jective if, for every complement C of M in $M \oplus N$ is a direct summand.

Definition 13: *A module M is said to be CK-N-jective if, for every complement C of M in $M \oplus N$ with $C \cap N \leq^c N$ is a direct summand.*

Proposition 14: *Let $M = A \oplus B$, where B is CK-A-jective. Let $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$. Then the following are satisfied (for i,j = 1,2):*

1. B_i is CK-A-jective.

2. B is CK-A_j-jective.

3. B_i is CK-A_j-jective.

Proof. For 1. Write $M = A \oplus B_1 \oplus B_2$. Let C be a complement of B_1 in $A \oplus B_1$, with $C \cap A \leq^c A$. Then by Lemma 12(2), we have that C is a complement of B in M. Since B is CK-A-jective, then $C \leq^{\oplus} M$, we have that $C \leq^{\oplus} A \oplus B_1$. Then B_1 is CK-A-jective.

For 2. Write $M = A_1 \oplus A_2 \oplus B$. Let C be a complement of B in $A_1 \oplus B$, with $C \cap A_1 \leq^c A_1$. Then by Lemma 12(1), we have that $C \oplus A_2$ is a complement of B in M. Since $C \cap A_1 \leq^c A_1$, we have

that $C \cap A_1$ is a complement of K in A_1, for some submodule K of A. Hence by Lemma 12(1), $(C \cap A_1) \oplus A_2$ is a complement of K in A. It is clear that $C \cap A_1 = C \cap A$, then $(C \oplus A_2) \cap A = (C \cap A) \oplus A_2 = (C \cap A_1) \oplus A_2 \leq^c A$. Since B is CK-A-jective, we have that $C \oplus A_2 \leq^{\oplus} M$, then $C \leq^{\oplus} A \oplus B_1$. Therefore B is CK-$A_1$-jective.
For 3. Follows from (1) and (2).

Lemma 15: *([10], Lemma 1) Let A and B be submodules of a module M, with $A \cap B = 0$. Then A is a complement of B in M if and only if A is a closed submodule of M, and $A \oplus B$ is essential in M.*

Proposition 16: *Let $M = A \oplus B$, B is CK-A-jective. If A is an extending module. then every closed submodule C of M, with $C \cap B = 0$ and $C \cap A \leq^c A$ is a direct summand of M.*
Proof. Since A is an extending module, we have $(C \oplus B) \cap A \leq^e A_1 \leq^{\oplus} A$, and hence $((C \oplus B) \cap A) \oplus B \leq^e A_1 \oplus B$. Since $C \oplus B = ((C \oplus B) \cap A) \oplus B$, we have that $C \oplus B \leq^e A_1 \oplus B$. By Lemma 15, C is a complement of B in $A_1 \oplus B$. It follows that $C \cap A = C \cap A_1$, and hence $C \cap A_1 \leq^c A_1$, Proposition 14, tell us that B is CK-A_1-jective. Therefore $C \leq^{\oplus} A_1 \oplus B \leq^{\oplus} M$.

3 GENERALIZED EXTENDING MODULES

(In[6]) A module M is said to have (C_1^*) if, for every submodule X of M, there exists a direct summand submodule K of M, which is a complement of X in M.

Proposition 17: *([6], Proposition 3.11.) Let M be an R-module, which has (C_1^*). Then the second singular submodule $Z_2(M)$ of M splits.*

Lemma 18: *([6], Lemma 3.14.) Let $A \leq B \leq M$. If C is a complement of A in M, then $C \cap B$ is a complement of A in B.*

Theorem 19: *([6], Theorem 3.2.) If $M = M_1 \oplus M_2$, where M_1 and M_2 are both have the condition (C_1^*), then M has (C_1^*).*

Remark 20 :

(1) *Let R be a commutative integral domain, and Let M be an R- module, which is not torsion. If M has (C_1), then its torsion submodule t(M) is injective (given in [5], Corollary 2.)*

(2) *Let R be a commutative integral domain , and let M be an R- module, which is not torsion. If M has (C_1^*), then its torsion submodule t(M) is not necessary to be injective.*

Example 21: *Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}$, it is clear that M is not torsion, and by Theorem 19, we have M has (C_1^*). But $\mathbb{Z}_2$ is not injective.*

Lemma 22: *If $M = M_1 \oplus M_2$, then M_i ($i = 1,2$) has (C_1^*) if and only if for every submodule L of M, with $L \cap M_j = 0$ ($j \neq i$), then there exists a submodule H of M, such that $H + M_j$ is a direct summand in M and is a complement of L in M.*
Proof. Suppose first that M_1 has (C_1^*). Let $L \leq M$, with $L \cap M_2 = 0$, then there exists $H \leq^{\oplus} M_1$, such that H is a complement of $(L \oplus M_2) \cap M_1$ in M_1. As $((L \oplus M_2) \cap M_1) \oplus H \leq^e M_1$, we have that $((L \oplus M_2) \cap M_1) \oplus H \oplus M_2 \leq^e M$. Since $L \oplus M_2 = ((L \oplus M_2) \cap M_1) \oplus M_2$, it follows that $L \oplus M_2 \oplus H \leq^e M$. Thus, by Lemma 15, $H \oplus M_2$ is a complement of L in M.
Conversely, suppose that for every submodule L of M, with $L \cap M_2 = 0$, there exists a submodule H of M, such that $H + M_2 \leq^{\oplus} M$, and that is a complement of L in M. let $C \leq M_1$, then there exists a submodule H of M, such that $H + M_2 \leq^{\oplus} M$, and that is a complement of C in M. By Lemma 18, we

have that $M_1 \cap (H + M_2)$ is a complement of C in M_1. It is clear that $M_1 \cap (H + M_2) \leq^{\oplus} M_1$.

Corollary 23: *If $M = Z_2(M) \oplus F$, then both have (C_1^*) if and only if every submodule C of M, with zero intersection with $Z_2(M)$ (or with F) has a complement summand containing F (or $Z_2(M)$).*

Proposition 24: *If $M = Z_2(M) \oplus F$, then M has the condition (C_1^*) if and only if $Z_2(M)$ and F both have (C_1^*).*

Proof. Suppose first that $Z_2(M)$ and F both have (C_1^*). By Theorem 19, we have M has (C_1^*). Conversely, write $M = Z_2(M) \oplus F$. Let N be nonsingular submodule of M, then $N \cap Z(M) = 0$. Since $Z(M) \leq^e Z_2(M)$, we have that $N \cap Z_2(M) = 0$. Since M has (C_1^*), we have that there exists $K \leq^{\oplus} M$, which is a complement of N in M. Write $M = K \oplus K'$, since $K \oplus N \leq^e M$, we have that $Z_2(K) \oplus Z_2(N) \leq^e Z_2(M)$, and that $Z_2(K) \leq^e Z_2(M)$, and consequently $Z_2(K) = Z_2(M)$. Hence $Z_2(M) \leq^{\oplus} K$. By Lemma 22, we have F has (C_1^*). Again, let L be a submodule of M, with $L \cap F = 0$. Since M has (C_1^*), we have that there exists $H \leq^{\oplus} M$, such that H is a complement of $(L \oplus F) \cap Z_2(M)$ in M. As $((L \oplus F) \cap Z_2(M)) \oplus H \leq^e M$, we have that $[(L \oplus F) \cap Z_2(M)] \oplus [Z_2(M) \cap H] \leq^e Z_2(M)$, then $[(L \oplus F) \cap Z_2(M)] \oplus F \oplus Z_2(H) \leq^e M$. Since $L \oplus F = [(L \oplus F) \cap Z_2(M)] \oplus F$, we have that $L \oplus F \oplus Z_2(H) \leq^e M$. It is clear that $F \oplus Z_2(H)$ is a direct summand submodule of M, and hence $F \oplus Z_2(H)$ is a complement of L in M. By Corollary 23, we have $Z_2(M)$ has (C_1^*).

Corollary 25: *Let M be an R-module has (C_1^*), and the second singular submodule $Z_2(M) \neq M$. Then for every submodule N of M, with $N \cap Z(M) = 0$, there exists a submodule H′ of F, such that $H' \oplus Z_2(M)$ is a direct summand of M, and is a complement of N in M.*

Proof. Write $M = Z_2(M) \oplus F$. Let N be a submodule of M, such that $N \cap Z(M) = 0$, then by Proposition 24, and Lemma 22, there exists $H + Z_2(M) \leq^{\oplus} M$, and it is a complement of N in M. Hence $H + Z_2(M) = Z_2(M) \oplus (F \cap (H + Z_2(M)))$. Choose $H' = F \cap (H + Z_2(M))$.

Corollary 26: *([11],Theorem 2.7.) A module M satisfies (C_1^*) if and only if $M = Z_2(M) \oplus K$, for some (nonsingular) submodule K of M, and $Z_2(M)$ and K both satisfy (C_1^*).*

Proof. Straightforward from Theorem 19, and Proposition 24.

Corollary 27: Let R be a commutative integral domain, and Let M be an R- module which is not torsion. If M has *(C_1^*)*, then the following are holds:

1. t(M) is contained in a complement of every torsion free submodule of M.
2. M= $t(M) \oplus F$, where t(M) and F both have *(C_1^*)*.

Definition 28: *An R-module M has the condition (*) "if every submodule of M has a unique complement in M "*

Proposition 29: *Let M be a right R-module has (*), then the following are equivalent :*

1. M has (C_1^*).
2. M is an extending module.
3. M is quasi - continuous.

Proof. (1) ⇒ (2): Let A be a closed submodule of M. By (C_1^*), there exists a decomposition $M = B \oplus C$, where B is a complement of A in M. Since $A \leq^c M$, then A is a complement of B in M. By (*), we have that A = C. Therefore M is an extending module.

(2) ⇒ (3): Let A and B are both direct summand submodules of M, and $A \cap B = 0$. Then by (C_1), there exists a decomposition $M = M_1 \oplus M_2$, where $A \oplus B \leq^e M_1$. Then M_2 is a complement of $A \oplus B$. Since M has the condition (*), we have that $A \oplus B = M_1$.

(3) ⇒ (1): It is clear from the fact that every extending module has (C_1^*).

Proposition 30: *Let $M = A \oplus B$, where B is A-jective. If $A = A_1 \oplus A_2$, where A_1 is an extending module, then for every closed submodule C of M, with $C \cap B = 0$, and $A_2 \leq C \oplus B$, is a summand of M.*

Proof. Since A_1 is an extending module, we have that $(C \oplus B) \cap A_1 \leq^e A_1' \leq^{\oplus} A_1$, and hence $[(C \oplus B) \cap A_1] \oplus A_2 \oplus B \leq^e A_1' \oplus A_2 \oplus B$. Since $C \oplus B = [(C \oplus B) \cap A_1] \oplus A_2 \oplus B$, we have that $C \oplus B \leq^e A_1' \oplus A_2 \oplus B$, and that C is a complement of B in $A_1' \oplus A_2 \oplus B$, and hence B is $A_1' \oplus A_2$ -jective. Therefore $C \leq^{\oplus} A_1' \oplus A_2 \oplus B \leq^{\oplus} M$.

4 C-INJECTIVE AND CI-INJECTIVE

Definition 31: *An R-module M is said to be C-N-injective if, for every closed submodule N′ of N, every monomorphism α from N′ to N, and every homomorphism f from N′ into M, then there exists a homomorphism ψ from N into M, such that ψα= f .*

Definition 32: *An R-module M is said to be CI-N-injective, if for every closed submodule N′ of N, and any homomorphism f from N′ into M, Can be extended to a homomorphism f from N′ into M.*

Lemma 33: *(In [9]) Let M be an R-module. Then M is continuous if and only if for every closed submodule C of M, and every monomorphism α from C into M, then α is split.*

Remark 34: *If M and N are right R-modules, then we have the following implications :*
M is N-injective ⇒ M is C-N-injective ⇒ M is CI- N-injective.
Generally neither of the converse implications is true, and we shows that by the following examples.

Example 35: *Let R be a von neumman regular ring. Suppose that a right R-module R is extending, then by ([7], exercises 6G, (38)), we have R as a right R-module is continuous. Let M be a right R-module which is not injective, and C be a closed submodule of R_R . Let α be a monomorphism from C to R_R, and f be a homomorphism from C to M. Then, by Lemma 33, we have $R_R = \alpha(C) \oplus K$, for some $K_R \leq R_R$. Let π be a projection homomorphism from $\alpha(C) \oplus K$ to α(C). Then for every $c \in C$, we have $f\alpha^{-1}\pi\alpha(c) = \alpha(c)$. Therefore M is C-R-injective.*

Example 36: *It is clear that $\mathbb{Z}$ is CI-$\mathbb{Z}$-injective. Let α be a monomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, where α(1)= n, n=2,3,4,.... . suppose that there exists a homomorphism ψ from $\mathbb{Z}$ to $\mathbb{Z}$, such that $\psi\alpha=I_{\mathbb{Z}}$ if and only if $n\mathbb{Z}$ is a direct summand of $\mathbb{Z}$. Then there is not exists a homomorphism ψ from $\mathbb{Z}$ to $\mathbb{Z}$, such that $\psi\alpha= I_{\mathbb{Z}}$. Then $\mathbb{Z}$ is not C- $\mathbb{Z}$-injective.*

Proposition 37: *Let M be an R-module. Then M is continuous if and only if N is C-M-injective for every closed submodule N of M.*

Proof. Suppose first that N is C-M-injective for every closed submodule N of M. Let $L \leq^{\oplus} M$, and let α be a monomorphism from L into M, and I_L denote the identity mapping on L. By assumption, there exists a homomorphism ψ from M to L, such that $\psi\alpha = I_L$, then $M = \alpha(L) \oplus \ker \psi$, by Lemma 33, we have M is continuous.
Conversely, suppose that M is continuous. Let M_1 and M_2 be closed submodules of M, and let α be a monomorphism from M_1 into M, and f be a homomorphism from M_1 to M_2. By Lemma 33, we have $M = \alpha(M_1) \oplus W$, for some submodule W of M. Let π be a projection homomorphism from $\alpha(M_1) \oplus W$ to $\alpha(M_1)$. Then for every $m_1 \in M_1$, we have $f\alpha^{-1}\pi\alpha(m_1) = f(m_1)$. Therefore N is-C-M-injective for every closed submodule N of M.

Proposition 38: *Let M be CI-N-injective and N′ be a closed submodule of N, then we have the following :*
1. M is CI-N′-injective .
2. M is CI-N/N′-injective .

Proof. For 1. Let $N'' \leq^c N'$, and let f be a homomorphism from N″ into M, then $N'' \leq^c N$. Since M be CI-N-injective, we have that there exists a homomorphism $\bar{f}$ from N into M, such that $\bar{f}|_{N''} = f$. Therefore M is CI-N′-injective.

For 2. Let X be a submodule of N, which contained N′, $X/N' \leq^c N/N'$, and let ϕ be a homomorphism from X/N′ into M. Let π denote the natural homomorphism of N onto N/N′, and $\pi' = \pi|_X$. Claim that $X \leq^c N$. Suppose that $X \leq^e L$, for some submodule L of N, since $N' \leq^c N$, we have that $X/N' \leq^e L/N' \leq N/N'$, and that $X \leq^c N$. Since M is CI-N-injective, we have that there exists a homomorphism θ from N to M, such that $\theta|_X = \phi\pi'$. Since $\theta(N') = \phi\pi'(N') = \phi(0) = 0$, we have that $\ker \pi \leq \ker \theta$, and consequently there exists a homomorphism ψ from N/N′ into M, such that $\psi\pi = \theta$. For every $x + N' \in X/N'$, $\psi(x + N') = \psi\pi'(x) = \theta(x) = \phi\pi'(x) = \phi(x + N')$. Thus ψ extends ϕ.

(In [7]) An abelian group D is divisible if, given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$, such that $nx = y$.

Lemma 39: *([7],chapter IV, Lemma 3.9.) An abelian group D is divisible if and only if D is injective.*

Lemma 40: *Let M be an abelian group, then the following are equivalent :*
1. M is an injective.
2. M is C-injective.
3. M is divisiblle.

Proof. (1) ⇒ (2): It is clear.
(2) ⇒ (3): Let $m \in M$ and $0 \neq n \in \mathbb{Z}$. Let α be a monomorphism from $\mathbb{Z}$ into $\mathbb{Z}$, such that $\alpha(1) = n$, and let ϕ be a homomorphism from $\mathbb{Z}$ into M, such that $\varphi(1) = m$. Since M is C-injective, we have that there exists a homomorphism ψ from $\mathbb{Z}$ into M, such that $\alpha\psi = \phi$. Put $\psi(1) = m'$, then $m = \phi(1) = \psi\alpha(1) = \psi(n) = n\,\psi(1) = nm'$. Hence M is divisiblle.
(3) ⇒ (1): Clear from Lemma 39.

Proposition 41: *Let M and N be an R-modules. If M is C-N-injective, and L is a direct summand submodule of M and K is a closed submodule of N, then we have the following :*
1. L is C-N-injective;
2. M is C-K-injective;
3. L is C-K-injective.

Proof. For 1. Let N′ be a closed submodule of N, let α be a monomorphism from N′ into N, and f be a homomorphism from N′ into L. Consider ι_L be the inclusion monomorphism from L into M. Since M is C-N-injective, we have that there exists a homomorphism ψ from N to M, such that $\psi\alpha = \iota_L f$. Let π be a projection homomorphism from M into L. Define $\pi\psi$ from N to L, then for every $n' \in N'$, we have that $\pi\psi\alpha(n') = \pi\iota_L f(n') = f(n')$. Therefore L is C-N-injective.
For 2. Let K′ be a closed submodule of K, let α be a monomorphism from K′ into K, and f be a homomorphism from K′ into M. Consider ι_K be inclusion monomorphism from K into N. Since $K' \leq^c N$, and M is C-N-injective, we have that there exists a homomorphism ψ from N into M, such that $\psi\,\iota_K\alpha = f$. Then for every $k' \in K'$, we have $\psi\,\iota_K\alpha(k') = \psi\alpha(k') = f(k')$.Therefore M is C-K-injective.
For 3. Follows from (1) and (2).

Corollary 42: *Let M and N be right R-modules. Then M is C-N-injective if and only if M is C-X-injective, for every closed submodule X of N.*

Corollary 43: *A direct summand of quasi-C-injective is a quasi-C-injective.*

Recall that the ring R is said to be principal right ideal ring (for short right PI-ring) if every right ideal of R is principal. This concept (given in [4] and [1] as R is pri-ring) and generalizing these concept to modules, an R-module M is called epi-retractable if every submodule of M is a homomorphic image of M .

In [7], a ring R is said to be right hereditary if every right ideal of R is projective as a right R-module, that is equivalent to submodules of projective right R-modules are projective.

In [3], a module M is called an hereditary module if every submodule of M is projective.

Proposition 44: *[[4],Proposition 2.5.] Let R be a right hereditary ring, then R is PI-ring if and only if every free right R-module is epi-retractable.*

Lemma 45: *Let R be a ring, such that the right ideal x^0 is a direct summand of R, for every $x \in R$. Then every right ideal of R is projective.*

Proof. Let $x \in R$, by assumption, write $R = x^0 \oplus D$. Since R is projective, then D is projective, and consequently R/x^0. Then xR is projective.

Corollary 46: *Let R be a ring, such that the right ideal x^0 is a direct summand of R, for every $x \in R$. Then R is right hereditary ring.*

Corollary 47: *Let R be a ring, such that the right ideal x^0 is a direct summand of R, for every $x \in R$. Then R is PI-ring if and only if every free right R-module is epi-retractable.*

Remark 48: *In [1], Proposition 2.3., tell us if R is right hereditary PI-ring .Then every free R-module is continuous. But this is not true, for example $\mathbb{Z}$ as a $\mathbb{Z}$-module, and we will correct them in the following Proposition.*

Proposition 49: *Let R be a right PI-ring with the right ideal x^0 is a direct summand of R, for every $x \in R$. Then every submodule of $R^{(I)}$ (for some index set I) is isomorphic to a summand.*

Proof. Let X ba a submodule of $R^{(I)}$. Since $R^{(I)}$ is free, then by Proposition 44, we have $R^{(I)}$ is epi-retractable, and consequently there exists an epimorphism α from $R^{(I)}$ to X. Let I_X be the identity mapping on X, by Corollary 46, we have that X is projective, and consequently there exists a monomorphism β from X to $R^{(I)}$, such that $\alpha\beta = I_X$. Then $R^{(I)} = \beta(X) \oplus \ker \alpha$. Hence X is isomorphic to a summand of $R^{(I)}$.

Remark 50:

1. *If R be a ring, which satisfies all conditions in Proposition 49, then every free right R-module need not to be continuous for example $\mathbb{Z}_{\mathbb{Z}}$.*
2. *If R be a ring which satisfies all conditions in Proposition 49. Then every free right R-module, which has the condition (C_2) is semisimple R-module.*

Proposition 51: *([3], Proposition 9) Let R be any ring,and M an hereditary continuous right R-module. Then M is a direct sum of Neotherian uniform submodules, each with a division endomorphism ring .*

Remark 52: *Proposition 2.4 in [1], tell us if R is a right hereditary PI-ring, then every projective*

R-module is a direct sum of Neotherian uniform submodules each with a division endomorphism ring. But this is not true, for example $\mathbb{Z}$ as a $\mathbb{Z}$-module and we will reformulate them in the following Proposition.

Proposition 53: *Let R be a right PI-ring with the right ideal x^0 is a direct summand of R, for every $x \in R$ and let M be a projective right R-module whose closed submodules are C-M-injective. Then M is a direct sum of Neotherian uniform submodules each with a division endomorphism ring.*

Proof. Straightforward from Proposition 37, and Proposition 51.

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