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Some convergence theorems for order-Mcshane equi-integral in Riesz space

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ABSTRACT

In this paper we prove some convergence theorems of order-Macshane equi -integrals on Banach lattice and arrive same result in L-space as on Mcshane norm-integrals.

Keywords

Riesz space; Order-Mcshane integration; Order-Henstock-integration; order- Mcshane equi-integration.

SUBJECT CLASSIFICATION

Mathematics Subject Classification; 28B15, 28B05, 28B10.

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INTRODUCTION

Preliminaries.

Recently, there are many papers paying attention to the integration in Riesz space. There are introduced and studied the notions of order-type integrals, for functions taking their values in ordered vector spaces, and in Banach lattices. In particular we can see [5], [7], [3], [10], [9], [6], [4], [8], [11]. We are affected from the works of Candeloro and Sambucini [5] as well as Boccuto et al.[1-2] about order –type integrals.

From now on, *T* will denote a compact metric space, and $\mu: \mathfrak{B} \to \mathbb{R}_0^+$ any regular, nonatomic σ-additive measure on the σ-algebra $\mathfrak{B}$ of Borel subsets of *T*.

A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \le p_n, \forall n \in \mathbb{N}$.

(see also [9], [11]), and we will write (o) lim $_n r_n = r$.

A *gage* is any map $\gamma: T \to \mathbb{R}^+$. A *partition* Π of T is a finite family $\Pi = \{(E_i, t_i): i = 1, \dots, k\}$ of pairs such that the sets E_i are pairwise disjoint sets whose union is *T* and the points t_i are called *tags*. If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock partition*. Otherwise, if t_i is not necessary to be in E_i, we say that it is a *free* or *McShane* partition.

Given a gage γ, we say that Π is γ-*fine* if $d(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and *i* = 1, . . . ,*k*. Clearly, a gage γ can also be defined as a mapping associating with each point $t \in T$ an open ball centered at *t* and cover E_i.

Let us assume now that *X* is any Banach lattice with an order-continuous norm. For the sake of completeness we recall the main notions of integral we are interested in.

Definition 1.1.

A function $f: T \to X$ is called (oM)-integrable ((oH)-integrable) and $J \in X$ is its (oM)-McShane integral ((oH)-integral) if for every for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition (H-partition) $\{(I_i, t_i), i = 1, \dots, p\}$ of T holds the inequality

$$|\sigma(f, \Pi) - J| \le b_n.$$

Where $\sigma(f, \Pi) = \sum_{i=1}^{p} f(t_i)\mu(E_i)$. We denote

$$J = (oM)\int_T f.$$

respectively

$$J = (oH)\int_T f.$$

Theorem 1.2 [5].

Let $f: T \to X$ be any mapping. Then f is (oH)–integrable ((oM)-integrable) if and only if there exist an (o) –sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that for every n, as soon as Π'', Π' are two γ_n-fine Henstock (Mcshane) partitions, the following holds true:

$$|\sigma(f, \Pi'') - \sigma(f, \Pi')| \le b_n$$

Proposition1.3 [5].

Let *f* : *T* →*X* be any (oH)-integrable function.Then, there exist an (*o*)-sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that, for every *n* and every γ_n-fine Henstock partition Π it holds

$$\sum_{E \in \Pi} Ob_n(f, E) \le b_n$$

Where $Ob_n(E) = \sup_{\Pi} \left\{ \left| \sum_{F'' \in \Pi''} f(\tau_{F''})\mu(F'') - \sum_{F' \in \Pi'} f(\tau_{F'})\mu(F') \right| \right\}$

Lemma1.5 (Saks-Henstock).

Assume that $f: T \to X$ is (oM)–integrable. Given (o)- sequence $(b_n)_n$ assume that a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0,+\infty[$ on T such that for every n

$$\left|\sum_{i=1}^{k} f(t_i)\,\mu(J_i) - (oM)\int_T f\right| \leq b_n$$

for every γ_n-fine M- partition Π= $\{(I_i, t_i): i = 1, \ldots, k\}$, of T:

Then if $\{(K_j, r_j): j = 1, \ldots, m\}$ is an arbitrary γ_n-fine M-system we have

$$\left|\sum_{j=1}^{m} (f(r_j)\,\mu(K_j) - (oM)\int_{K_j} f)\right| \leq b_n$$

Proof.

$$\left|\sum_{i=1}^{k_l} f(s_i^l)\,\mu(J_i^l) - (OM)\int_{M_l} f\right| \leq \frac{a_n}{p+1}$$

Provided $\{(J_i^l, s_i^l): i = 1, \ldots, k_l\}$ is $\gamma_n -$ M-partition γ_n of the interval M_l. The sum

$$\sum_{j=1}^{m} f(r_j)\,\mu(K_j) + \sum_{l=1}^{p}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l)$$

represents an integral sum corresponds one M-partition γ_n $-$ fine of the interval T and consequently by the assumption we have

$$\left|\sum_{j=1}^{m} f(r_i)\,\mu(K_i) + \sum_{l=1}^{p}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - (oM)\int_I f\right| < b_n.$$

Hence

$$\left|\sum_{j=1}^{m} f(r_i)\,\mu(K_i) - (oM)\int_{K_j} f\right| \leq$$

$$\leq \left|\sum_{j=1}^{m} f(r_j)\,\mu(K_i) + \sum_{l=1}^{p}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - (oM)\int_I f\right| +$$

$$+ \sum_{l=1}^{p}\left|\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - (oM)\int_{M_l} f\right| \leq b_n + p\,\frac{a_n}{p+1} < b_n + a_n$$

We obtain the proof of the theorem.

2. Convergence theorems of order-Mcshane equi-integrals

Definition 2.1.

A collection $\mathcal{F}$ of functions $f: T \to X$ is called (oM)-equi-integrable ((oH)-equi-integrable) if every $f \in \mathcal{F}$ is $(\mathrm{o})-$McShane integrable ($(\mathrm{o})-$Henstock-Kurzweil integrable) and for any o-sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$of gauges such that for every n and any $f \in \mathcal{F}$ the inequality

$$\left|\sum_{i=1}^{p} f(t_i)\,\mu(I_i) - (oM)\int_T f\right| \leq b_n$$

$$(\left|\sum_{i=1}^{p} f(t_i)\,\mu(I_i) - (oH)\int_T f\right| \leq b_n)$$

holds provided $\{(I_i, t_i), i = 1, \ldots, p\}$ is γ_n –fine M-partition (K-partition) of T.

Theorem 2.2.

Assume that $\mathcal{F} = \{f_k: T \to X; k \in \mathbb{N}\}$ is (oM)-equi-integrable sequence such that.

$$(o)-\lim_{k\to\infty} f_k(t) = f(t), \qquad t \in T$$

Then the function $f: T \to X$ is (o)-McShane integrable and holds the equation

$$(o)-\lim_{k\to\infty}(oM)\int_T f_k = (oM)\int_T f.$$

Proof. If $(\gamma_n)_n$ is the sequence of gauges from the definition of equi-integrability of the sequence (f_k) corresponding to (o)- sequence $(b_n)_n$ then for any $k \in \mathbb{N}$ we have

$$\left|\sum_{i=1}^{p} f_k(t_i)\mu(I_i) - (oM)\int_T f_k\right| \le b_n \qquad (1)$$

for every n and for every (γ_n) -fine M-partition $\{(I_i, t_i), i = 1, \dots, p\}$ of T.

If the partition $\{(I_i, t_i), i = 1, \dots, p\}$ is fixed then the pointwise convergence $f_k \to f$ yields

$$(o)-\lim_{k\to\infty}\sum_{i=1}^{p} f_k(t_i)\mu(I_i) = \sum_{i=1}^{p} f(t_i)\mu(I_i).$$

Choose $k_0 \in \mathbb{N}$ such that for $k > k_0$ the inequality

$$\left|\sum_{i=1}^{p} f_k(t_i)\mu(I_i) - \sum_{i=1}^{p} f(t_i)\mu(I_i)\right| \le b_n$$

holds.Then we have

$$\left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - (oM)\int_T f_k\right|$$
$$\le \left|\sum_{i=1}^{p}[f(t_i)\mu(I_i) - f_k(t_i)\mu(I_i)]\right|$$
$$+\left|\sum_{i=1}^{p}[f_k(t_i)\mu(I_i) - (oM)\int_T f_k]\right| < 2b_n$$

for $k > k_0$.This gives for $k, l > k_0$ the inequality

$$\left|(oM)\int_T f_k - (oM)\int_T f_l]\right| \le 4b_n$$

which shows that the sequence $(oM)\int_T f_k$,$k \in \mathbb{N}$ of elements of X is Cauchy and therefore

$$(o)-\lim_{k\to\infty}(oM)\int_T f_k = A \in X \ (2)$$

exists.

Let(o)- sequence $(b_n)_n$. By hypothesis there is a corresponding sequence $(\gamma_n)_n$of gauges such that (1) holds for all $k \in \mathbb{N}$ whenever $\{(I_i, t_i), i = 1, \dots, p\}$ is a γ_n-fine M-partition of T.By (2) choose $N \in \mathbb{N}$ such that

$$\left|(oM)\int_T f_k - A\right| < b_n$$

for all $k \ge N$. Suppose that $\{(I_i, t_i), i = 1, \dots, p\}$ is a γ_n-fine M-partition of T for every n. Since f_k converges pointwise to f there is a $k_1 \ge$ N such that

$$\left|\sum_{i=1}^{p} f_{k_1}(t_i)\mu(I_i) - \sum_{i=1}^{p} f(t_i)\mu(I_i)\right| \le b_n$$

Therefore

$$\left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - A\right| \le \left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - \sum_{i=1}^{p} f_{k_1}(t_i)\mu(I_i)\right| +$$

$$+ \left|\sum_{i=1}^{p} f_{k_1}(t_i)\,\mu(I_i) - (oM)\int_T f_{k_1}\right| + \left|(oM)\int_T f_{k_1} - A\right| \le 3b_n$$

and it follows that f is (o)- Mcshane integrable on T and

$$(o) - \lim_{k\to\infty}(oM)\int_T f_k = A = (oM)\int_T f$$

The (o) –Henstock variant of the theorem can be proved analogously.

Proposition 2.3.

A function $f: T \to X$ is (o)-Mcshane integrable if and only if the set $\{x^*(f); x^* \in B(X^*)\}$ is (oM)-equi-integrable.

Proof. If f is (o)- Mcshane integrable then for every for any $(o)-$ sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0,+\infty[$ for every nsuch that

$$\left|\sum_{i=1} f(t_i)\,\mu(I_i) - (oM)\int_T f\right| \le b_n$$

for every γ_n-fine M-partition $\{(I_i,t_i), i = 1,\dots,p\}$onT. For an arbitrary $x^* \in X^*$ we have

$$|\sum_i x^*(f(t_i))\mu(I_i) - x^*((oM)\int_T f)|$$

$$\le |x^*|\cdot\left|\sum_i f(t_i)\mu(I_i) - (oM)\int_T f\right| \le b_n \cdot sup|x^*|$$

and therefore $\{x^*(f); x^* \in B(X^*)\}$ is (o)- Mcshane-equi-integrable. If on the other hand $\{x^*(f); x^* \in B(X^*)\}$ is (o)- Mcshane-equi-integrable then for every $(o)-$ sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0,+\infty[$ for every n on T such that

$$|\sum_i x^*(f(t_i))\mu(I_i) - (oM)\int_T x^*(f)| \le b_n$$

for every γ_n-fine M-partition $\{(I_i,t_i), i = 1,\dots,p\}$ onT and $x^* \in B(X^*)$.

Hence if $\{(I_i,t_i)\},\{(J_j,s_j)\}$ are γ_n-fine M-partition of T we get

$$|x^*(\sum_i f(t_i)\mu(I_i) - \sum_i f(s_j)\mu(J_j))|$$

$$= |\sum_i x^*(f(t_i))\mu(I_i) - \sum_j x^*\left(f(s_j)\right)\mu(J_j)| \le 2b_n$$

for every $x^* \in B(X^*)$. Hence

$$|\sum_i f(t_i)\mu(I_i) - \sum_j f(s_j)\mu(J_j)| \quad \le 2b_n$$

and by Theorem (2) the function f is (o)- McShane integrable.

Concerning the concept of an equi-integrable collection given by Proposition (1.3) let us note that we have the following result which represents a certain Bolzano-Cauchy condition for equi-integrability of an equi-integrable collection $\mathcal{F}$ of functions $f: T \to X$.

Theorem 2.4.

A collection $\mathcal{F}$ of functions $f: T \to X$ is(oM)- equi-integrable if and only if for every (o)- sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0,+\infty[$ onT such that

$$\left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - \sum_{j=1}^{r} f(s_j)\mu(J_j)\right| < b_n$$

for every n and γ_n –fine M-partition $\{(I_i, t_i), i = 1, \dots, p\}$ and $\{(J_i, s_i), i = 1, \dots, r\}$ of T and every $f \in \mathcal{F}$.

Proof. If $\mathcal{F}$ is equi-integrable then the condition evidently holds for a sequence of gauges $(\gamma_n)_n$ which corresponds to $\frac{1}{2} b_n$ in Definition (2.1) of equi-integrability.

If the condition of the theorem is satisfied then every individual function $f \in \mathcal{F}$ is (oM)-integrable with the same corresponding sequence $(\gamma_n)_n$ of gauges for a given (o)- sequence $(b_n)_n$ independently of the choice of $f \in \mathcal{F}$ and this proves the theorem.

Let us close by an analogue of the Saks-Henstock Lemma (1.5) which holds for equi-integrable collections of.

Lemma 2.5.

Assume that an (oM) –equi-integrable collection $\mathcal{F}$ of $f: T \to X$,sequence (o)- sequence $(b_n)_n$ is given assume that the corresponding sequence $(\gamma_n)_n$ of gauge on T is such that

$$\left|\sum_{i=1}^{k} f(t_i)\,\mu(J_i) - (oM)\int_T f\right| \le b_n$$

For every γ_n-fine M- partition $\Pi = \{(I_i, t_i): i = 1, \dots, k\}$, of T:

Then if $\{(K_j, r_j): j = 1, \dots, m\}$ is an arbitrary γ_n-fine M-system we have

$$\left|\sum_{j=1}^{m}(f(r_j)\mu(K_j) - (oM)\int_{K_j} f)\right| \le b_n \text{ for any } f \in \mathcal{F}$$

For the proof of this statement the proof of the Saks-Henstock Lemma can be repeated word for word.

Notation. To simplify writing from now we will use the notation $\{(U_l, u_l)\}$ for M-systems instead of $\{(U_l, u_l); 1 = 1, \dots, r\}$ which specifies the number *r* of elements of the M-system. For a function $f: T \to X$ and an M-system $\{(U_l, u_l)\}$ we write $\sum_l f(u_l)\,\mu(U_l)$ instead of

$$\sum_{l=1}^{r} f(u_l)\mu(U_l).$$

Proposition 2.6.

Assume that X is L-space and $f_k: T \to X,\ k \in N$ are (o)- McShane integrable functions such that

1. $f_k(t) \to f(t)$ for t$\in T$,
2. the set $\{f_k: k \in N\}$ forms an (oM)-integrable sequence.

Then for every $(o) -$ sequence $(b_n)_n$ there exists an $\eta > 0$ such thatif F is closed, G open, $F \subset G \subset T$ and $\mu(G \setminus F) < \eta$ then there is a sequence $(\gamma_n)_n$ corresponding gauge $\gamma_n: T \to]0, \infty[$ such that

$$B(t, \gamma_n(t)) \subset G \quad \text{for } t \in G,$$

$$B(t, \gamma_n(t)) \cap T \subset T \setminus F \quad \text{for } t \in T \setminus F.$$

Proof. Denote $\Phi_k(J) = (oM)\int_J f_k$ for an interval $J \subset T$ (the indefinite integral or primitive of f_k) and put $\overline{b_n} = \frac{b_n}{10}$. Since f_k are (oM)-equi-integrable, the Saks-Henstock lemma implies that there is a sequence Δ_n gauge on T such that

$$\left|\sum_j [f_k(r_j)\mu(K_j) - \Phi_k(K_j)]\right| \le \overline{b_n}$$

for every Δ_n-fine M-system $\{(K_j, r_j)\}$ and $k \in \mathbb{N}$. Assume that $\{(W_p, w_p)\}$ is a fixed Δ_n-fine M-partition of T. Let $k_0 \in \mathbb{N}$ for all p and thanks to the fact X is L-space be such that

$|f_k(w_p) - f(w_p)| < a$

for $k > k_0$ and $a \in X$.

Put

$$s = \sup_{p,k \le k_0} \{a + |f(w_p|, |f_k(w_p)|\}$$

Then $$|f_k(w_p)| < s$$

for all $k_0 \in \mathbb{N}$ and p. Assume that $\eta > 0$ satisfies

$$\eta . s \le b_n$$

And take $0 < \gamma_n(t) \le \Delta_n(t), t \in T$

Since the sets G and T \ Fare open, the gauge γ_n can be chosen such that $B(t, \gamma_n(t)) \subset G$

For $t \in G$, and $B(t, \gamma_n(t)) \cap T \subset T \setminus F$ for $t \in T \setminus F$.

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