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FIXED POINT RESULTS IN QUASI CONE METRIC SPACE FOR GENERALIZED $\alpha-\psi$ CONTRACTIVE MAPPINGS USING DIAMETER OF ORBITS

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ABSTRACT

E. Karapinar and B. Samet (2012) [1] have given some fixed point theorems related $\alpha-\psi$ Contractive Mappings in metric space. N. Bilgili, E. Karapinar and B. Samet (2014) [2] have proved these results in quasi metric space. In this paper we prove some new results for the existence and uniqueness of fixed point in quasi cone metric space using generalized $\alpha-\psi$ Contractive Mappings related with diameter of orbits.

Indexing terms/Keywords

Fixed point; orbit; $\alpha-\psi$ contractive mappings; quasi cone metric space.

Academic Discipline And Sub-Disciplines

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INTRODUCTION

Huang and Zhang [12] have reviewed the concept of cone metric spaces replacing the real axis in the definition of distance by an ordered Banach space. They have proved some theorems about fixed points in cone metric spaces for contractive mappings and their results generalize some fixed point theorems in metric space. Abdeljaward and Karapinar [4] have given the definition of quasi cone metric space and there are many authors who have worked in this direction. E. Karapinar and B. Samet (2012) [1] have given some fixed point theorems related $\alpha - \psi$ Contractive Mappings in metric space. N. Bilgili, E. Karapinar and B. Samet (2014) [2] have proved these results in quasi metric space. As the quasi cone metric spaces form a bigger category than the one usual metric spaces and quasi metric space, it is the main aim of this paper to extend the result of N. Bilgili, E. Karapinar and B. Samet (2014) and many others and giving some new results.

PRELIMINARIES

Definition 1. [14] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if

(i) P is closed, $P \neq \Phi$, $P \neq \{0\}$;

(ii) for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in R^+$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in$ int P, where int P denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that $0 \leq x \leq y \Rightarrow ||x|| \leq K||y||$, for all $x, y \in P$. The least positive K satisfying this, is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent; that is if x_n is a sequence such that $x_1 \leq x_2 \leq \ldots \leq x_n \leq y$, for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every sequence which is bounded below is convergent.

Definition 2. Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies following conditions:

(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 3. [14] Let X be a nonempty set. Suppose the mapping $q: X \times X \to E$ satisfies following conditions:

(i) $0 \leq q(x, y)$ for all $x, y \in X$;
(ii) $q(x, y) = 0$ if and only if $x = y$;
(iii) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

then q is called a *quasi-cone metric* on X, and (X, q) is called a *quasi-cone metric space*.

Remark 1. Note that any cone metric space is a quasi-cone metric space.

Now we introduce the appropriate generalization in quasi-cone metric spaces by considering the established notions in quasi-metric spaces.

Definition 5. [7] Let (X, q) be a quasi–cone metric space.

A sequence $\{x_n\}$ in X is called right (left) Cauchy if for each $c \in$ int P, there is $n_0\ x \in N$ such that $q(x_n, x_m) \ll c$ ($q(x_m, x_n) \ll c$ resp.) for all $n \geq m \geq n_0$.

The sequence $\{x_n\}_{n \in N}$ in X is called Cauchy if and only if it is both left and right Cauchy.

Definition 6. Let (X, q) be a quasi-cone metric space. Let $\{x_n\}_{n \in N}$ be a sequence in X. We say that the sequence $\{x_n\}_{n \in N}$ is right convergent to $x \in X$ if $q(x, x_n) \to 0$. We say that the sequence $\{x_n\}_{n \in N}$ is convergent to $x \in X$ if the sequence is right and left convergent to x. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Definition 7. A quasi-cone metric space (X, q) is called complete if every Cauchy sequence in X converges.

Definition 9. Let $O(x) = \{x, Tx, T^2x, \ldots\}$ where $x \in X$. The set $O(x)$ is called orbit of x.

Definition 10. Let $M \subseteq X$ where X is a quasi-cone metric space.

$\delta(M) = \sup\{q(x, y), q(y, x), x, y \in M\}$ is called diameter of M.

Define $\delta(O(x) \cup O(y)) = \max\{q(T^i x, T^j x), q(T^k y, T^m y), q(T^i x, T^k y)\}$ for $i, j, k, m \in N_0$.

The orbit $O(x)$ is called bounded if there exist a $c \in P$, $\delta(O(x)) \leq c$.

Definition 12. [4] The function ψ: $P \to P$ which satisfies the following conditions

1. $\forall\, t \in P$, $\psi(t) < t$;
2. $\forall\, t_1, t_2 \in P$, $t_1 < t_2 \Rightarrow \psi(t_1) < \psi(t_2)$;
3. $\lim_{n\to\infty} \|\psi^n(t)\| = 0, t \in P$

is called a *comparison function*.

Definition 13. [3] Let (X, q) be a quasi-cone metric space and T: $X \to X$ be a given function.

We say that T is $\alpha - \psi$ contractive mapping if there exist two functions α: $X \times X \to [0, \infty)$ and ψ a comparison function that satisfy the nonlinear contraction condition:

$$\alpha(x, y)q(T(x), T(y)) \leq \psi(q(x, y)).$$

Definition 14. Let T: $X \to X$ and α: $X \times X \to [0, \infty)$. We say that T is α-admissible if for all x, y $\in X$ and we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

MAIN RESULTS

The following results Theorem 1 and Theorem 2 are generalization of N. Bilgili and E. Karapinar in quasi-cone metric space because cone metric space is bigger category than metric space.

Theorem 1. Let (X, q) be a complete Hausdorff quasi-cone metric space and let T: $X \to X$ be a continuous function that satisfies the nonlinear contraction condition:

$$\alpha(x, y)q(T(x), T(y)) \leq \psi(M(x, y)).$$

where $M(x, y) = \max\{q(x, y), q(Tx, x), q(Ty, y), \frac{1}{2}[\, q(Tx, y) + q(x, Ty)]\}$ for all $x, y \in X$ and ψ: $P \to P$ satisfies the conditions $\psi(0) = 0$, $\forall\, t \in P$, $\psi(t) < t$ and ψ is semi-lower continuous. Suppose that

(i) α: $X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ for every $n, m \in N$.
Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0\}_{n\in N}$. Now we have to prove that the sequence $\{T^n x_0\}_{n\in N}$ is right Cauchy.

Firstly, we see that the sequence $\{q(T^{n+1}x_0, T^n x_0)\}_{n\in N}$ is monoton decreasing.

$$q(T^{n+1}x_0, T^n x_0) \leq \alpha(T^n x_0, T^{n-1}x_0)\, q(T^{n+1}x_0, T^n x_0) \leq \psi(M(T^n x_0, T^{n-1}x_0))$$

where

$$M(T^n x_0, T^{n-1}x_0) = \max\{q(T^n x_0, T^{n-1}x_0), q(T^{n+1}x_0, T^n x_0), q(T^n x_0, T^{n-1}x_0), ½\,[q(T^{n+1}x_0, T^{n-1}x_0), q(T^n x_0, T^n x_0)]\}$$
$$= \max\{q(T^n x_0, T^{n-1}x_0), q(T^{n+1}x_0, T^n x_0)\}$$

We shall examine two cases.

Case 1. $M(T^n x_0, T^{n-1}x_0) = q(T^{n+1}x_0, T^n x_0)$, so $q(T^{n+1}x_0, T^n x_0) \leq \psi(q(T^{n+1}x_0, T^n x_0))$ which is a contradiction.

Case 2. $M(T^n x_0, T^{n-1}x_0) = q(T^n x_0, T^{n-1}x_0)$, so $q(T^{n+1}x_0, T^n x_0) \leq \psi(q(T^n x_0, T^{n-1}x_0)) \leq q(T^n x_0, T^{n-1}x_0)$ for all $n \geq 1$.

Let denote $C_k = \sup\{q(T^i x_0, T^j x_0), I > j > k\}$. The sequence $\{C_k\}$ is monoton decreasing and bounded below, so it converges to $C_0 \in P$. So we have

$$\forall\, p \in N, \exists\, i_p > j_p > p, C_p - \frac{C_0}{p} \leq q(T^{i_p}x_0, T^{j_p}x_0) \leq C_p \Rightarrow q(T^{i_p}x_0, T^{j_p}x_0) \to C_0 \text{ as } p \to \infty.$$

We prove now that $C_0 = 0$.

$$q(T^{i_p+1}x_0, T^{j_p+1}x_0) \leq \alpha(T^{i_p}x_0, T^{j_p}x_0)\, q(T^{i_p+1}x_0, T^{j_p+1}x_0) \leq \psi(M(T^{i_p}x_0, T^{j_p}x_0))$$

Taking the limit when $p \to \infty$, we have that $C_0 \le \psi(C_0)$, so $C_0 = 0$.

In the same manner we prove that the sequence $\{T^n x_0\}$ is left Cauchy. So it is a Cauchy sequence and since the space is complete, it is convergent to x^*.

$$\lim_{n\to\infty} q(T^n x_0, x^*) = \lim_{n\to\infty} q(x^*, T^n x_0) = 0.$$

Since T is continuous

$$\lim_{n\to\infty} q(T^n x_0, Tx^*) = \lim_{n\to\infty} q(T(T^{n-1}x_0), Tx^*)) = 0 \text{ and } \lim_{n\to\infty} q(Tx^*, T^n x_0) = \lim_{n\to\infty} q(Tx^*, T(T^{n-1}x_0)) = 0.$$

Thus, we have $\lim_{n\to\infty} q(T^n x_0, Tx^*) = \lim_{n\to\infty} q(Tx^*, T^n x_0) = 0$.

As X is Hausdorff, we have that $x^* = Tx^*$ and x^* is a fixed point of T.

In the following theorem we take a non continuous function T, but we take the cone normal.

Theorem 2. Let (X, q) be a complete, Hausdorff quasi-cone metric space with cone normal and let

$T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$\alpha(x, y)q(Tx, Ty)) \le \psi(N(x, y))$$

where $N(x, y)) = \max\{q(x, y), \frac{1}{a}[q(Tx, x) + q(Ty, y)], \frac{1}{a}[q(Tx, y) + q(x, Ty)]\}$ for all $x, y \in X$, $a \ge K$, $\psi : P \to P$ satisfies the conditions $\psi(0) = 0$, $\forall\, t \in P$, $\psi(t) < t$ and ψ is semi-lower continuous. Suppose that

(i) $\alpha : X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \ge 1$ for every $n, m \in N$.

(ii) If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \ge 1$ for all n and $T^n x \to x^*$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}$ such that $\alpha(T^{n_k} x, x^*) \ge 1$, $\alpha(x^*, T^{n_k} x) \ge 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in N}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0\}_{n\in N}$. Continuing the same procedure as Theorem 1, we prove that the sequence $\{T^n x_0\}_{n\in N}$ is Cauchy and so it converges to x^*.

From (iii) there exist a subsequence $\{T^{n_k} x_0\}$ of $\{T^n x_0\}$ such that $\alpha(T^{n_k} x, x^*) \ge 1$, $\alpha(x^*, T^{n_k} x) \ge 1$ for all k. We see

$$q(Tx^*, T^{n_k+1}x_0) \le \alpha(x^*, T^{n_k}x_0)q(Tx^*, T^{n_k+1}x_0) \le \psi(N(x^*, T^{n_k}x_0)).$$

$$N(x^*, T^{n_k}x_0) = \max\{q(x^*, T^{n_k}x_0), \tfrac{1}{a}[q(Tx^*, x^*) + q(T^{n_k+1}x_0, T^{n_k}x_0)], \tfrac{1}{a}[q(Tx, T^{n_k}x_0) + q(T^{n_k+1}x_0, x^*)]\}$$

Taking the limit, we have $\lim_{k\to\infty} N(x^*, T^{n_k}x_0) = \frac{1}{a} q(Tx^*, x^*)$.

From $q(Tx^*, T^{n_k+1}x_0) \le \psi(N(x^*, T^{n_k}x_0)) < N(x^*, T^{n_k}x_0)$ and taking the limit, we get

$$\|q(Tx^*, T^{n_k+1}x_0)\| < \frac{K}{a} \|\, q(Tx^*, x^*)\, \| < \|q(Tx^*, x^*)\|,$$

which is a contradiction. So, we have that $q(Tx^*, x^*) = 0$ and $Tx^* = x^*$ and x^* is a fixed point of T.

Theorem 3. Let (X, q) be a complete,Hausdorff quasi-cone metric space and let $T: X \to X$ be a continuous function that satisfies the nonlinear contraction condition:

$$\alpha(x, y)q(T(x), T(y)) \le \psi(\delta(O(x) \cup O(y))$$

for all $x, y \in X$, where $\psi: P \to P$ is a comparison function. Suppose that

(ii) $\alpha : X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \ge 1$ for every $n, m \in N$.

Moreover for $x_0 \in X$, the orbit $O(x_0)$ is bounded. Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.

Proof. We have that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \ge 1$. Define the sequence $\{T^n x_0\}_{n\in N}$. Now we have to prove that the sequence $\{T^n x_0\}_{n\in N}$ is right Cauchy. For this, we take $x = T^{n+i}x_0$, $y = T^{n+j}x_0$, where $i, j, n \in N$ and $I > j$.

$$q(Tx, Ty) = q(T^{n+i+1}x_0, T^{n+j+1}x_0) \le \alpha(q(T^{n+i}x_0, T^{n+j}x_0))\, q(T^{n+i+1}x_0, T^{n+j+1}x_0) \le \psi(\delta(O(T^{n+i}x_0) \cup O(T^{n+j}x_0))) \le \psi(\delta(O(T^n x_0))$$

So it is true that $q(T^{n+i+1}x_0, T^{n+j+1}x_0) \le \psi(\delta(O(T^n x_0)))$ for every i, j, n and $I > j$.

Also $q(T^{n+i+1}x_0, T^{n+j+1}x_0) \le \delta(O(T^{n+1}x_0)) = \max\{ q(T^{n+i+1}x_0, T^{n+j+1}x_0), i, j \in N\} \le \psi(\delta(O(T^n x_0)))$.

From this we have that

$$q(T^{n+i+1}x_0, T^{n+j+1}x_0) \le \psi(\delta(O(T^n x_0))) \le \psi^2(\delta(O(T^{n-1}x_0))) \le \cdots \le \psi^n(\delta(O(x_0))) \le \psi^n(c).$$

Due to $\lim_{n\to\infty} ||\psi^n(c)|| = 0 \Leftrightarrow (\forall \frac{\varepsilon}{K} > 0)(\exists n_0 \in N)(\forall n > n_0 \Rightarrow ||\psi^n(c)|| < \frac{\varepsilon}{K}$, where K is the constant of normality of cone, we have

$$||q(T^{n+i+1}x_0, T^{n+j+1}x_0)|| \le K||\psi^n(c)|| < K\frac{\varepsilon}{K} = \varepsilon \text{ for } n > n_0, \text{ and } I > j.$$

So we have that the sequence $\{T^n x_0\}_{n\in N}$ is right Cauchy. In the same manner we prove that the sequence $\{T^n x_0\}_{n\in N}$ is left Cauchy. Since (X, q) is complete, exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n\in N}$ is convergent to $x^* \in X$ so $q(T^n x_0, x^*) \to 0$ as $n \to \infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to \infty$.

Now we have to prove that x^* is a fixed point of T, $Tx^* = x^*$. We have that $\lim_{n\to\infty} q(T^n x_0, x^*) = 0$ and $\lim_{n\to\infty} q(x^*, T^n x_0) = 0$. By using the continuity of T, we have $\lim_{n\to\infty} q(T(T^n x_0), Tx^*) = 0$ and $\lim_{n\to\infty} q(Tx^*, T(T^n x_0)) = 0$.

By uniqueness of the limit, we conclude that $Tx^* = x^*$.

Example 1. Let $X = [0, 1]$, $E = R^2$, and P a normal cone, $P = \{(x, y) \in R^2, x, y \ge 0\}$. Determine $q: X \times X \to P$,

$$q(x, y) = \begin{cases} (\frac{y}{2}, y), & x < y \\ (0,0), & x = y \\ (x, 2x), & x > y \end{cases}$$ is a quasi-cone metric and (X, q) is quasi-cone metric space.

Let $T: X \to X$, $T(x) = x^2$ be a continuous function, $\psi: P \to P$, $\psi(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ be a comparison function and

$\alpha: X \times X \to [0, \infty)$, $\alpha(x, y) = \begin{cases} \frac{1}{2}\max(x, y), & x \ne y \\ a, & x = y \end{cases}$, where $a \ge 1$. We prove that the function satisfies the condition of Theorem 1. Taking $x_0 = 0$ we have that $\alpha(T^n x_0, T^m x_0) = a \ge 1$.

Now let see that T satisfy the contraction condition of Theorem 3.

Case 1. $x = y$. This case is trivial because $q(Tx, Ty) = 0$.

Case2. $x < y$. In this case

$$q(Tx, Ty) = q(x^2, y^2) = \left(\frac{y^2}{2}, y\right), \alpha(x, y) = \frac{y}{2},$$

$$\alpha(x, y)\, q(Tx, Ty) = \left(\frac{y^3}{2}, y^3\right)$$

$$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (y, 2y), \ \psi(\delta(O(x) \cup O(y)) = \left(\frac{y}{2}, y\right) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

Case 3. $x > y$.

$$q(Tx, Ty) = q(x^2, y^2) = (x^2, 2x^2), \alpha(x, y) = \frac{x}{2},$$

$$\alpha(x, y)\, q(Tx, Ty) = \left(\frac{x^3}{2}, x^3\right)$$

$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$

$\delta(O(x) \cup O(y)) = (x, 2x)$, $\psi(\delta(O(x) \cup O(y)) = \left(\frac{x}{2}, x\right) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \leq \psi(\delta(O(x) \cup O(y))$.

So the function T has fixed points. We see that $x = 0$ and $x = 1$ are the fixed points of T.

Theorem 4. Let (X, q) be a complete Hausdorff quasi-cone metric space with constant of normality K and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$\alpha(x, y) q(T(x), T(y)) \leq \psi\left(\frac{1}{a}\, \delta(O(x) \cup O(y))\right)$$

for all $x, y \in X$ and $a \geq K$, where $\psi: P \to P$ is a comparison function. Suppose that

(i) $\alpha : X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ for every $n, m \in N$.

(ii) If $\{T^n x\}_{n \in N}$ is a sequence such that for $T^n x \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}_{n \in N}$ such that: $\alpha(T^{n_k} x, T^q x^*) \geq 1$, $\alpha(T^{n_k} x^*, x^*) \geq 1$, $\alpha(x^*, T^{n_k} x) \geq 1$,

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in N}$ is convergent to x^*..

Moreover for every $z \in X$ the orbit $O(z)$ is bounded. Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in N}$ is convergent to x^*.

Proof. We have that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$. Define the sequence $\{T^n x_0\}_{n \in N}$.

We prove that the sequence $\{T^n x_0\}_{n \in N}$ is Cauchy using the same method in Theorem 3. Since (X, q) is complete, exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n \in N}$ is convergent to $x^* \in X$ so $q(T^n x_0, x^*) \to 0$ as $n \to \infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to \infty$.

Now we have to prove that x^* is a fixed point of T, $Tx^* = x^*$. For this we need to prove that the sequence $\{T^m x^*\}_{m \in N}$ converges to x^*. Suppose that this sequence converges to $l \in X$.

$q(T^{n_k+1} x^*, T^{n_k+1} x_0) \leq \alpha(T^{n_k} x^*, T^{n_k} x_0)\, q(T^{n_k+1} x^*, T^{n_k+1} x_0) \leq \psi(\frac{1}{a}\, \delta(O(T^{n_k} x^*) \cup O(T^{n_k} x_0))) < \frac{1}{a}\, \delta(O(T^{n_k} x^*) \cup O(T^{n_k} x_0))$

$q(T^{n_k+1} x^*, T^{n_k+1} x_0) < \frac{1}{a}\, \delta(O(T^{n_k} x^*) \cup O(T^{n_k} x_0)) = \frac{1}{a} \max\{q(T^{n_k+i} x^*, T^{n_k+j} x^*), q(T^{n_k+p} x_0, T^{n_k+r} x_0), q(T^{n_k+i} x^*, T^{n_k+p} x_0)\}$ for $I, j, p, r \in N_0$.

Taking the limit of both sides when $k \to \infty$, we have $||q(l, x^*)|| < \frac{K}{a} ||q(l, x^*)|| \leq ||q(l, x^*)||$. So this is a contradiction, so $q(l, x^*) = 0 \Rightarrow l = x^*$.

Now we see

$q(T^{n_k+1} x^*, Tx^*) \leq \alpha(q(T^{n_k} x^*, x^*))\, q(T^{n_k+1} x^*, Tx^*) \leq \psi(\frac{1}{a}\, \delta(O(T^{n_k} x_0) \cup O(x^*))) < \frac{1}{a}\, \delta(O(T^{n_k} x_0) \cup O(x^*))$

$= \frac{1}{a} \max\{q(T^{n_k+s} x^*, T^{n_k+r} x^*), q(x^*, Tx^*), q(x^*, T^i x^*), q(T^{n_k} x^*, Tx^*)\} = \frac{1}{a} \max\{q(x^*, Tx^*), q(T^{n_k} x^*, Tx^*)\}$.

Taking the limit of both sides when $k \to \infty$, we have $||q(x^*, Tx^*)|| < \frac{K}{a} ||q(x^*, Tx^*)|| \leq ||q(x^*, Tx^*)||$. So this is a contradiction, then $q(x^*, Tx^*) = 0 \Rightarrow Tx^* = x^*$. So $q(Tx^*, x^*) = 0 \Rightarrow Tx^* = x^*$.

Example 2. Let $X = [0, 1]$, $E = R^2$, and P a normal cone, $P = \{(x, y) \in R^2, x, y \geq 0\}$. The constant of normality of cone $K = 1$. Determine $q: X \times X \to P$, such that

$$q(x, y) = \begin{cases} (\frac{y}{2}, y), & x < y \\ (0, 0), & x = y \\ (x, 2x), & x > y \end{cases}$$

is a quasi-cone metric and (X, q) is quasi-cone metric space.

Let T: $X \to X$, $T(x) = \begin{cases} x^3, & 0 \le x < \frac{1}{16} \\ \frac{1}{16}, & \frac{1}{16} \le x \le 1 \end{cases}$ be a non-continuous function, ψ: $P \to P$, $\psi(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ be a comparison function and α: $X \times X \to [0, \infty)$, $\alpha(x, y) = \begin{cases} \frac{1}{4}\min\{x, y\}, & (x, y) \in [0, \frac{1}{16}) \times [\frac{1}{16}, 1] \cup [\frac{1}{16}, 1] \times [0, \frac{1}{16}) \\ b, & x = y \\ \max\{x,y\}, & \text{otherwise} \end{cases}$, where $b \ge 1$. We prove that the function satisfies the condition of Theorem 4.

Taking $x_0 = 0$ we have that $\alpha(T^n x_0, T^m x_0) \ge 1$.

Also $T^n 0 \to 0$ when $n \to \infty$, so $\alpha(T^n 0, 0) \ge 1$.

Now let see that T satisfy the contraction condition of Theorem 4.

Case 1. $x = y$. This case is trivial because $q(Tx, Ty) = 0$.

Case 2.1. $x, y \in [0, \frac{1}{16})$, $x < y$. In this case

$$q(Tx, Ty) = q(x^3, y^3) = \left(\frac{y^3}{2}, y^3\right),\ \alpha(x, y) = \frac{y}{2},$$

$$\alpha(x, y)\, q(Tx, Ty) = \left(\frac{y^4}{4}, \frac{y^4}{2}\right)$$

$$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (y, 2y),\ \psi(\delta(O(x) \cup O(y)) = \left(\frac{y}{2}, y\right) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

Case 2.2. $x, y \in [0, \frac{1}{16})$, $x > y$. In this case

$$q(Tx, Ty) = q(x^3, y^3) = (x^3, 2x^3),\ \alpha(x, y) = \frac{x}{2},$$

$$\alpha(x, y)\, q(Tx, Ty) = \left(\frac{x^4}{2}, x^4\right)$$

$$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (x, 2x),\ \psi(\delta(O(x) \cup O(y)) = \left(\frac{x}{2}, x\right) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

Case 3.1. $x, y \in [\frac{1}{16}, 1]$, $x < y$ or $x > y$. In this case

$$q(Tx, Ty) = q(\tfrac{1}{16}, \tfrac{1}{16}) = 0 \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

Case 4.1. $x \in [0, \frac{1}{16})$, $y \in [\frac{1}{16}, 1]$. In this case

$$q(Tx, Ty) = q(x^3, \tfrac{1}{16}) = (\tfrac{1}{32}, \tfrac{1}{16}),\ \alpha(x, y) = \frac{x}{4}$$

$$\alpha(x, y)\, q(Tx, Ty) = (\tfrac{x}{128}, \tfrac{x}{64}),$$

$$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (\tfrac{1}{16}, \tfrac{1}{8}),\ \psi(\delta(O(x) \cup O(y)) = (\tfrac{1}{32}, \tfrac{1}{16}) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

Case 4.2 $y \in [0, \frac{1}{16})$, $x \in [\frac{1}{16}, 1]$. In this case

$$q(Tx, Ty) = q(\tfrac{1}{16}, y^3) = (\tfrac{1}{16}, \tfrac{1}{8}),\ \alpha(x, y) = \frac{y}{4}$$

$$\alpha(x, y)\, q(Tx, Ty) = (\tfrac{y}{64}, \tfrac{y}{32}),$$

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$$\delta(O(x) \cup O(y)) = \max\{q(x, y), q(x, Tx), q(y, Ty), q(x, T^i x), q(y, T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (\tfrac{1}{16}, \tfrac{1}{8}),\ \psi(\delta(O(x) \cup O(y)) = (\tfrac{1}{32}, \tfrac{1}{16}) \Rightarrow \alpha(x, y)\, q(Tx, Ty) \le \psi(\delta(O(x) \cup O(y)).$$

So the function T has fixed points. We see that $x = 0$ and $x = \frac{1}{16}$ are the fixed points of T.

The conditions of above theorems give us the existence of fixed points, but they don't guaranty the uniqueness of them.

If we add the following condition to theorems 1, 2, 3 and 4, we prove that T has a unique fixed point.

Note, we prove the theorem 3. Similarly can be proved the others theorems.

Theorem 5. Suppose we are in condition of Theorem 3, T is α admissibleand for all $x, y \in X$ and there exist $z \in X$ such that

$$\alpha(x, z) \ge 1, \alpha(z, x) \ge 1, \text{ and } \alpha(y, z) \ge 1,, \alpha(z, y) \ge 1,$$

then the function T has a unique fixed point.

Proof. Suppose there exist another fixed point $x^{**} \in X$, $T(x^{**}) = x^{**}$. So there exist $z \in X$ such that $\alpha(x^*, z) \ge 1$, $\alpha(z, x^*) \ge 1$, $\alpha(x^{**}, z) \ge 1$, $\alpha(z, x^{**}) \ge 1$. We have

$$\alpha(x^*, T^n z) \ge 1, \alpha(T^n z, x^*) \ge 1, \alpha(x^{**}, T^n z) \ge 1, \alpha(T^n z, x^{**}) \ge 1, \forall\, n.$$

Define the sequence $\{T^n z\}_{n\in N_0}$ and see

$$q(T^{n+1}z, x^*) \le \alpha(T^n z, x^*)\, q(T^{n+1}z, x^*) \le \le \psi(\delta(O(T^n z) \cup O(x^*)).$$

$$q(T^{n+1}z, x^*) \le \delta((O(T^n z) \cup O(x^*)) = \max\{q(q(T^{n+i+1}x, T^{n+j+1}y), q(T^s x^*, T^r x^*), q(T^{n+i+1}z, T^s x)\}$$

So $\quad \delta((O(T^n z) \cup O(x^*)) \le \psi(\delta(O(T^n z) \cup O(x^*)).$

By iterating process we have

$$q(T^{n+1}z, x^*) \le \psi(\delta(O(T^n z) \cup O(x^*)) \le \psi^2(\delta(O(T^{n-1}z) \cup O(x^*)) \le \cdots \le \psi^n(\delta(O(z) \cup O(x^*)) \le \psi^n(c)$$

$$\|q(T^{n+1}z, x^*)\| \le K\|\psi^n(c)\|$$

Taking limit of both sides, we have $q(T^{n+1}z, x^*) \to 0$ as $n \to \infty$.

So the sequence $\{T^n z\}_{n\in N_0}$ is right convergent to x^*. In the same manner we prove that sequence $\{T^n z\}_{n\in N_0}$ is left convergent to x^*. So we have $\lim_{n\to\infty} T^n z = x^*$.

Similarly, we prove that $\lim_{n\to\infty} T^n z = x^{**}$. As X is Hausdorff, $x^* = x^{**}$ and x^* is unique fixed point of T.

The following results are true for $\alpha(x, y) = 1$.

Corollaries

Corollary 1. (see Jachymski [5]). Let (X, q) be a complete, Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le \psi(\delta(O(x) \cup O(y)).$$

for all $x, y \in X$, where $\psi: P \to P$ is a comparison function.

Moreover for every $u \in X$ the orbit $O(u)$ is bounded. Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. If we take$\alpha(x, y) = 1$, we are in condition of Theorem 4.

Corollary 2 (see Babu [4], Berinde [6]). Let (X, q) be a complete, Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le \psi(q(x, y))$$

for all $x, y \in X$, where $\psi: P \to P$ is a comparison function.

Moreover for every $u \in X$ the orbit $O(u)$ is bounded. Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. We see that

$$q(x, y) \le \delta(O(x) \cup O(y)) \Rightarrow \psi(q(x, y)) \le \psi(\delta(O(x) \cup O(y)) \Rightarrow q(T(x), T(y)) \le \psi(q(x, y)) \le \psi(\delta(O(x) \cup O(y))$$

So we are in condition of theorem and T has a fixed point, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Corollary 3. Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le \psi(\max\{q(x, y), q(Tx, x), q(Ty, y), q(Tx, y), q(x, Ty)\}$$

for all $x, y \in X$, where $\psi: P \to P$ is a comparison function.

Moreover for every $u \in X$ the orbit $O(u)$ is bounded. Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. We see that

$$\max\{q(x, y), q(Tx, x), q(Ty, y), q(Tx, y), q(x, Ty)\} \le \delta(O(x) \cup O(y)) \Rightarrow$$

$$\psi(\max\{q(x, y), q(Tx, x), q(Ty, y), q(Tx, y), q(x, Ty)\}) \le \psi(\delta(O(x) \cup O(y))) \Rightarrow$$

$$q(T(x), T(y)) \le \psi(\max\{q(x, y), q(Tx, x), q(Ty, y), q(Tx, y), q(x, Ty)\}) \le \psi(\delta(O(x) \cup O(y))).$$

So we are in condition of theorem and T has a fixed point, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Corollary 4. Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le \psi(\max\{q(x, y), q(Tx, x), q(Ty, y), \tfrac{1}{2}[q(Tx, y) + q(x, Ty)]\})$$

for all $x, y \in X$, where $\psi: P \to P$ is a comparison function.

Moreover for every $u \in X$ the orbit $O(u)$ is bounded. Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. We see that

$$\max\{q(x, y), q(Tx, x), q(Ty, y), \tfrac{1}{2}[q(Tx, y) + q(x, Ty)]\} \le \delta(O(x) \cup O(y)) \Rightarrow$$

$$\psi(\max\{q(x, y), q(Tx, x), q(Ty, y), \tfrac{1}{2}[q(Tx, y) + q(x, Ty)]\}) \le \psi(\delta(O(x) \cup O(y))) \Rightarrow$$

$$q(T(x), T(y)) \le \psi(\max\{q(x, y), q(Tx, x), q(Ty, y), \tfrac{1}{2}[q(Tx, y) + q(x, Ty)]\}) \le \psi(\delta(O(x) \cup O(y)))$$

So we are in condition of theorem and T has a fixed point, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Corollary 5. (Ciric [9]). Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le h \max\{q(x, y), \tfrac{1}{2}[q(x, Tx) + q(y, Ty)], \tfrac{1}{2}[q(x, Ty) + q(y, Tx)]\}$$

for all $x, y \in X$, where $h \in (0, 1)$.

Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. Taking $\psi(t) = ht$, $h \in (0, 1)$, we have

$$q(T(x), T(y)) \le h \max\{q(x, y), \tfrac{1}{2}[q(x, Tx) + q(y, Ty)], \tfrac{1}{2}[q(x, Ty) + q(y, Tx)]\}$$

Corollary 5 follows from theorem 2.

Corollary 6. (Banach Contraction Principle [8]) Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the contraction condition:

$$q(T(x), T(y)) \le h\, q(x, y),$$

for all $x, y \in X$, where $h \in (0, 1)$.

Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Proof. We use the same manner as in Corollary 5.

Corollary 7. (Kannan [10]). Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the contraction condition:

$$q(T(x), T(y)) \le h\, [q(x, Tx) + q(y, Ty)]$$

for all $x, y \in X$, where $h \in (0, ½)$.

Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Corollary 8. (Chatterja [11]). Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X \to X$ be a function that satisfies the contraction condition:

$$q(T(x), T(y)) \le h\, [q(x, Ty) + q(y, Tx)]$$

for all $x, y \in X$, where $h \in (0, ½)$.

Then T has a unique fixed point $x^* \in X$ and for every $u \in X$, the sequence $\{T^n u\}_{n\in N}$ is convergent to x^*.

Corollary 9. (Sila [15]) Let (X, q) be a complete Hausdorff quasi-cone metric space and let $T: X\to X$ be a function that satisfies the nonlinear contraction condition:

$$q(T(x), T(y)) \le \psi(\max\{q(x, y), q(Tx, x), q(Ty, y), q(Tx, y), q(x, Ty)]\}) \qquad (1)$$

for all $x, y \in X$, where $\varphi : P \to P$ is a comparison function. Let $x_0 \in X$ such that $O(x_0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.

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