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The global attractors and their Hausdorff and fractal dimensions estimation for the higher-order nonlinear Kirchhoff-type equation*

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Abstract

We investigate the global well-posedness and the longtime dynamics of solutions for the higher-order Kirchhoff-type equation with nonlinear strongly dissipation: $u_{tt}+(-\Delta)^m u_t+\phi(\|D^m u\|^2)(-\Delta)^m u+g(u)=f(x)$. Under of the proper assume, the main results are that existence and uniqueness of the solution is proved by using priori estimate and Galerkin method, the existence of the global attractor with finite-dimension, and estimation Hausdorff and fractal dimensions of the global attractor.

Key words: Higher order; Attractor; Kirchhoff; Hausdorff dimension;Fractal dimension

1 Introduction

We consider the problem

$$u_{tt}+(-\Delta)^m u_t+\phi(\|(-\Delta)^{\frac{m}{2}}u\|^2)(-\Delta)^m u+g(u)=f(x),\ x\in\Omega,\ t>0, m>1 \tag{1.1}$$

$$u(x,t)=0,\ \frac{\partial^i u}{\partial v^i}=0,\ x\in\partial\Omega,\ i=1,2,\dots,m-1 \tag{1.2}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) \tag{1.3}$$

Where Ω is a bounded domain of R^n, with a smooth Dirichlet boundary $\partial\Omega$ and initial value, the damping coefficient is function of the L_2-norm of the gradient m power, g(u) is a nonlinear forcing, $(-\Delta)^m u_t$ is a strongly dissipation.

There have been many researches on the global attractors existence of the Kirchhoff equation with strong dissipation, we can see [1,2,3]. There are lots of recent results on the global attractor of Kirchhoff equation, we can refer [4,5,6,7].

Zhijian Yang and Pengyan Ding [8] studied the longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on R^n:

$$u_{tt}-\Delta u_t-\mathrm{M}(\|\nabla u\|^2)\Delta u+u_t+g(x,u)=f(x). \tag{1.4}$$

They obtain the well-posedness, the existence of the global and exponential attractors in $H=H^1(R^n)\times L^2(R^n)$ in critical nonlinearity case. Their novelty is that it overcomes the essential difficulties that is the Sobolev embedding on R^n and the critical growth of g cause the lack of compactness.

Recently, Zhijian Yang, Pengyan Ding and Lei Li [9] also studied longtime dynamics of the Kirchhoff equation with fractional damping and supercritical nonlinearity:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + (-\Delta)^{\alpha} u_t + f(u) = g(x),\ x \in \Omega,\ t>0, \tag{1.5}$$

$$u|_{\partial\Omega} = 0,\ \ u(x,0)=u_0(x),\ \ u_t(x,0) = u_1(x), \tag{1.6}$$

Where $\alpha \in (\frac{1}{2},1)$, Ω is a bounded domain in R^n with the smooth boundary, they show that (i) even if p (the growth exponent p of the nonlinearity $f(u)$), $1 \le p \le \frac{N+4\alpha}{(N-4\alpha)^+}$, the well-posedness and longtime behavior of the solutions of the equation are of the characters of the parabolic equation; (ii) when $\frac{N+4\alpha}{(N-4\alpha)^+} \le p < \frac{N+4}{(N-4)^+}$, the limit solutions exist and possesses a weak global attractor.

Chueshov [10] first studied the well-posedness and the global attractor for the IBVP of Kirchhoff wave models with strong nonlinear damping:

$$u_{tt} - \sigma(\|\nabla u\|^2)\Delta u_t - \phi(\|\nabla u\|^2)\Delta u + g(u) = h(x) \tag{1.7}$$

He established a finite-dimensional global attractor in the sense of partially strong topology. In particular, in nonsupercritical case: (i) the partially strong topology becomes strong; (ii) an exponential attractor is obtained in natural energy space $\mathrm{H}(\Omega) = H^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega)$.

Guigui Xu and Guoguang Lin [11] studied the global attractor and their dimensions estimation for the generalized Boussinesq equation:

$$u_{tt} - \Delta u - \Delta u_{tt} + \alpha\Delta^2 u + \beta\Delta^2 u_{tt} - \Delta u_t - \Delta |u|^p = f(x). \tag{1.8}$$

Under the existence of the global solution, it is discussed that the global attractor and infinte Hausdorff dimension and fractional dimension.

The main details of this paper are arranged as follow:

In section 2, under the assume of Lemma 2.1 and Lemma 2.2, we get the existence and uniqueness of solution; in section 3, we obtain the global attractor of the problems (1.1)-(1.3); in section 4, we consider the finite Hausdorff dimension and fractal dimension of the global attractor.

2 Main results of the paper

For convenience, we denote the simple symbol, $\|.\|$ represents norm, $(.,.)$ represents inner product,

and $f = f(x)$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $H^{2m}(\Omega) = H^{2m}$, $(-\Delta)^{\frac{m}{2}} = D^m$, $\|.\| = \|.\|_{L^2}$, C_i $(i=0,1\cdots 7)$ is a constant, m_i $(i=0,1\cdots 4)$ is also a constant.

Lemma 2. 1. Assume

(1) $\phi(\|D^m u\|^2)\ : R^+ \to R^+$ is a differentiable function;

(2) $\varepsilon\phi(\|D^m u\|^2)\|D^m u\|^2 \geq \varepsilon\Phi(\|D^m u\|^2) + \frac{1}{4}\varepsilon^2\|D^m u\|^2 \quad where \ \ \Phi' = \phi$;

(3) $\Phi(\|D^m u\|^2) \geq \varepsilon\|D^m u\|^2 + C_0$;

(4) $g(u)u \geq \varepsilon G(u) + \xi u^2, where G'(u) = g(u)u_t$;

(5) $J(u) = \int G(u)dx$;

(6) $f(x) \in L^2(\Omega)$.

Then the solution (u, v) of the problems（1.1）-（1.3）satisfies $(u,v) \in H^m(\Omega)\times L^2(\Omega)$，and satisfies：

$$\|(u,v)\|^2_{H^m\times L^2} = \|D^m u\|^2 + \|v\|^2 \leq W(0)e^{-\alpha t} + \frac{C}{\alpha}(1-e^{-\alpha t}) . \tag{2.1}$$

Where $v = u_t + \varepsilon u$,$W(0) = \|v_0\|^2 + \varepsilon^2\|u_0\|^2 + \varepsilon\|D^m u_0\|^2 + 2J(u_0)$, there exist $t = t_1 > 0$ and R_0 ,such that

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2 \leq \frac{C}{\alpha} = R_0 . \tag{2.2}$$

Proof : Let $v = u_t + \varepsilon u$ we use v multiply both sides of equation (1.1) and obtain

$$(u_{tt} + (-\Delta)^m u_t + \phi(\|D^m u\|^2)(-\Delta)^m u + g(u), v) = (f(x), v) . \tag{2.3}$$

$$\begin{aligned}(u_{tt}, v) &= (v_t - \varepsilon u_t, v) = (v_t, v) - \varepsilon(v - \varepsilon u, v)\\ &= \frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{\varepsilon^2}{2}\frac{d}{dt}\|u\|^2 - \|v\|^2 + \varepsilon^3\|u\|^2 .\end{aligned} \tag{2.4}$$

$$\begin{aligned}&((-\Delta)^m u_t, v)\\ &= (D^m v - \varepsilon D^m u, D^m v)\\ &= \|D^m v\|^2 - \varepsilon(D^m u, D^m u_t + \varepsilon D^m u)\\ &= \|D^m v\|^2 - \frac{\varepsilon}{2}\frac{d}{dt}\|D^m u\|^2 - \varepsilon^2\|D^m u\|^2\\ &\geq m_0\lambda_1\|v\|^2 - \frac{\varepsilon}{2}\frac{d}{dt}\|D^m u\|^2 - \varepsilon^2\|D^m u\|^2 .\end{aligned} \tag{2.5}$$

where $\lambda_1(>0)$ is the first eigenvalue of the operator Δ^m .

$$\begin{aligned}
&(\phi(\|D^m u\|^2)(-\Delta)^m u, v) \\
&= (\phi(\|D^m u\|^2)(-\Delta)^m u, u_t + \varepsilon u) \\
&= \phi(\|D^m u\|^2)\frac{1}{2}\frac{d}{dt}\|D^m u\|^2 + \varepsilon\phi(\|D^m u\|^2)\|D^m u\|^2 \\
&= \frac{1}{2}\frac{d}{dt}\Phi(\|D^m u\|^2) + \varepsilon\phi(\|D^m u\|^2)\|D^m u\|^2 \\
&\ge \frac{1}{2}\frac{d}{dt}\Phi(\|D^m u\|^2) + \varepsilon\Phi(\|D^m u\|^2) + \frac{1}{4}\varepsilon^2\|D^m u\|^2 .
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&(g(u), v) \\
&= (g(u), u_t) + \varepsilon(g(u), u) \\
&= \frac{d}{dt}\int G(u)dx + \varepsilon(g(u), u) \\
&\ge \frac{d}{dt}\int G(u)dx + \varepsilon^2\int G(u)dx \\
&\ge \frac{d}{dt}J(u) + J(u).
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&(f(x), v) \\
&\le \frac{1}{2\varepsilon^2}\|f\|^2 + \frac{\varepsilon^2}{2}\|v\|^2 .
\end{aligned} \tag{2.8}$$

From the above ,we have

$$\begin{aligned}
&\frac{d}{dt}[\|v\|^2 + \varepsilon^2\|u\|^2 + \varepsilon\|D^m u\|^2 + 2J(u)] \\
&+ (2m_0\lambda_1 - \varepsilon^2 - 2\varepsilon)\|v\|^2 + 2\varepsilon^3\|u\|^2 + (2\varepsilon - \varepsilon^2)\|D^m u\|^2 + 2\varepsilon^2 J(u) \\
&\le \frac{1}{2\varepsilon^2}\|f\|^2 + C_0 := C .
\end{aligned} \tag{2.9}$$

Where we take proper constant m_0 and ε , such that:

$$\begin{cases} a_1 = 2m_0\lambda_1 - \varepsilon^2 - 2\varepsilon \ge 0 \\ a_2 = 2\varepsilon - \varepsilon^2 \ge 0 \end{cases} \tag{2.10}$$

Then we take $\alpha = \min\left\{a_1, 2\varepsilon, \frac{a_2}{\varepsilon}, \varepsilon^2\right\}$, we obtain:

$$\frac{d}{dt}W(t) + \alpha W(t) \le C , \tag{2.11}$$

where

$$W(t) = \|v\|^2 + \varepsilon^2\|u\|^2 + \varepsilon\|D^m u\|^2 + 2J(u) . \tag{2.12}$$

By using Gronwall inequality, we obtain:

$$W(t) \le W(0)e^{-\alpha t} + \frac{C}{\alpha}(1 - e^{-\alpha t}) , \tag{2.13}$$

where

$$W(0)=\|v_0\|^2+\varepsilon^2\|u_0\|^2+\varepsilon\|D^m u_0\|^2+2J(u_0). \tag{2.14}$$

So, we have:

$$\|(u,v)\|^2_{H^m\times L^2}=\|D^m u\|^2+\|v\|^2\le W(0)e^{-\alpha t}+\frac{C}{\alpha}(1-e^{-\alpha t}). \tag{2.15}$$

And

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H^m\times L^2}\le\frac{C}{\alpha}. \tag{2.16}$$

Thus there exist $t=t_1(\Omega)$ and R_0, such that

$$\|(u,v)\|^2_{H^m\times L^2}\le\frac{C}{\alpha}=R_0\ (t>t_1). \tag{2.17}$$

Lemma 2.2. Assume

(1) $g(u)\le C_1(1+|u|^p),\ \ p\le\frac{2n}{n-2m},\ \ n\ge 3$;

(2) $\varepsilon_1\le\mu_0\le\phi(s)\le\mu_1,\ \mu=\begin{cases}\mu_0\ , & \frac{d}{dt}\|D^m u\|^2\ge 0\\ \mu_1\ , & \frac{d}{dt}\|D^m u\|^2<0\end{cases}$;

(3) $0\le\frac{d\phi(s)}{ds}\le C_2$;

(4) $f(x)\in L^2(\Omega)$.

Then the solution (u,v) of the problems（1.1）-（1.3）satisfies $(u,v)\in H^{2m}(\Omega)\times H^m(\Omega)$, and satisfies

$$\|(u,v)\|^2_{H^{2m}\times H^m}=\|(-\Delta)^m u\|^2+\|D^m v\|^2\le M(0)e^{-\beta t}+\frac{C_3}{\beta}(1-e^{-\beta t}), \tag{2.18}$$

where $v=u_t+\varepsilon_1 u$, $M(0)=\|D^m v_0\|^2+\varepsilon_1^2\|D^m u_0\|^2+(\mu-\varepsilon_1)\|(-\Delta)^m u_0\|^2$, there exist $t=t_2(\Omega)$ and R_1,such that

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H^{2m}\times H^m}\le\frac{C_3}{\beta}=R_1. \tag{2.19}$$

Proof Let $(-\Delta)^m v=(-\Delta)^m u_t+\varepsilon_1(-\Delta)^m u$ we use $(-\Delta)^m v$ multiply both sides of equation (1.1) and obtain

$$(u_{tt}+(-\Delta)^m u_t+\phi(\|D^m u\|^2)(-\Delta)^m u+g(u),(-\Delta)^m v)=(f(x),(-\Delta)^m v). \tag{2.20}$$

$$\begin{aligned}
&(u_{tt},(-\Delta)^m v)\\
&=(v_t-\varepsilon_1 u_t,(-\Delta)^m v)\\
&=\frac{1}{2}\frac{d}{dt}\|D^m v\|^2-\varepsilon_1(v-\varepsilon_1 u,(-\Delta)^m v)\\
&=\frac{1}{2}\frac{d}{dt}\|D^m v\|^2-\varepsilon_1\|D^m v\|^2+\varepsilon_1^2(u,(-\Delta)^m u_t+\varepsilon_1(-\Delta)^m u)\\
&=\frac{1}{2}\frac{d}{dt}\|D^m v\|^2-\varepsilon_1\|D^m v\|^2+\varepsilon_1^2\frac{1}{2}\frac{d}{dt}\|D^m u\|^2+\varepsilon_1^3\|D^m u\|^2 .
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&((-\Delta)^m u_t,(-\Delta)^m v)\\
&=((-\Delta)^m v-\varepsilon_1(-\Delta)^m u,(-\Delta)^m v)\\
&=\|(-\Delta)^m v\|^2-\varepsilon_1((-\Delta)^m u,(-\Delta)^m v)\\
&=\|(-\Delta)^m v\|^2-\frac{1}{2}\varepsilon_1\frac{d}{dt}\|(-\Delta)^m u\|^2-\varepsilon_1^2\|(-\Delta)^m u\|^2\\
&\ge\frac{1}{8}m_1\lambda_2\|D^m v\|^2+\frac{1}{2}\|(-\Delta)^m v\|^2+\frac{1}{4}\|(-\Delta)^m v\|^2\\
&+\frac{1}{8}\|(-\Delta)^m v\|^2-\frac{1}{2}\varepsilon_1\frac{d}{dt}\|(-\Delta)^m u\|^2-\varepsilon_1^2\|(-\Delta)^m u\|^2 .
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&(\phi(\|D^m u\|^2)(-\Delta)^m u,(-\Delta)^m v)\\
&=\phi(\|D^m u\|^2)\frac{1}{2}\frac{d}{dt}\|(-\Delta)^m u\|^2+\varepsilon_1\phi(\|D^m u\|^2)\|(-\Delta)^m u\|^2\\
&\ge\mu\frac{1}{2}\frac{d}{dt}\|(-\Delta)^m u\|^2+\varepsilon_1\mu_0\|(-\Delta)^m u\|^2 .
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&(g(u),(-\Delta)^m v)\\
&\ge-\frac{1}{2}\|g(u)\|^2-\frac{1}{2}\|(-\Delta)^m v\|^2 .
\end{aligned} \tag{2.24}$$

According to assume (1), we can get $\|g(u)\|^2\le C_4\|u\|^{2p}+C_5$, and accord to Poincare inequality $\|g(u)\|^2\le C_4 m_2\lambda_3\|D^m u\|^{2p}+C_5$, then accord to Lemma2.1. $\|D^m u\|^{2p}<\infty$, so, we have $\|g(u)\|^2\le C_6$.

$$\begin{aligned}
&(g(u),(-\Delta)^m v)\\
&\ge-C_6-\frac{1}{2}\|(-\Delta)^m v\|^2 .
\end{aligned} \tag{2.25}$$

$$(f(x),(-\Delta)^m v)\le\|f\|^2+\frac{1}{4}\|(-\Delta)^m v\|^2 . \tag{2.26}$$

From the above , we have

$$\begin{aligned}
&\frac{d}{dt}[\|D^m v\|^2+\varepsilon_1^2\|D^m u\|^2+(\mu-\varepsilon_1)\|(-\Delta)^m u\|^2]\\
&+(\frac{1}{4}m_2\lambda_3-2\varepsilon_1)\|D^m v\|^2+2\varepsilon_1^3\|D^m u\|^2+(-2\varepsilon_1^2+2\varepsilon_1\mu_0)\|(-\Delta)^m u\|^2\\
&+\frac{1}{4}\|(-\Delta)^m v\|^2\le\|f\|^2+C_6:=C_3 .
\end{aligned} \tag{2.27}$$

Next, accord to assume (2), we see $\mu-\varepsilon_1\ge 0$, $-2\varepsilon_1^2+2\varepsilon_1\mu_0\ge 0$ and we take proper constant m_2 such that $\frac{1}{4}m_2\lambda_3-2\varepsilon_1\ge 0$.

Then, we take $\beta = \min\{\frac{1}{4}m_2\lambda_3 - 2\varepsilon_1, 2\varepsilon_1, \frac{-2\varepsilon_1^2+2\varepsilon_1\mu_0}{\mu-\varepsilon_1}\}$, we obtain

$$\frac{d}{dt}M(t) + \beta M(t) \le C_3, \tag{2.28}$$

where

$$M(t) = \| D^m v\|^2 + \varepsilon_1^2 \| D^m u\|^2 + (\mu - \varepsilon_1)\|(-\Delta)^m u\|^2. \tag{2.29}$$

By using Gronwall inequality, we obtain

$$\|(u,v)\|^2_{H^{2m}\times H^m} = \|(-\Delta)^m u\|^2 + \| D^m v\|^2 \le M(0)e^{-\beta t} + \frac{C_3}{\beta}(1-e^{-\beta t}). \tag{2.30}$$

And

$$\overline{\lim_{t\to\infty}} \|(u,v)\|^2_{H^{2m}\times H^m} \le \frac{C_3}{\beta} = R_1. \tag{2.31}$$

Thus there exist $t = t_2(\Omega)$ and R_1, such that

$$\|(u,v)\|^2_{H^{2m}\times H^m} \le \frac{C_3}{\beta} = R_1, (t > t_2). \tag{2.32}$$

Theorem 2.1. Lemma 2.1, Lemma 2.2 holds; the initial boundary value problem (1.1) with Dirichlet boundary exists unique smooth solution $(u,v) \in L^\infty([0,+\infty); H^{2m}\times H^m)$.

Proof. By Lemma 2.1-Lemma 2.2 and Glerkin method, we can easily obtain the existence of solution of equation $(u,v) \in L^\infty([0,+\infty); H^{2m}\times H^m)$, the procedure is omitted . Next, we prove the uniqueness of solution in detail.

Let u, v are two solutions of equation (1.1), we denote $w = u - v$, then two equations subtract and obtain

$$w_{tt} + (-\Delta)^m w_t + \phi(\| D^m u\|^2)(-\Delta)^m u - \phi(\| D^m v\|^2)(-\Delta)^m v + g(u) - g(v) = 0 \tag{2.33}$$

By using w_t to inner product of the equation (2.33), and we have

$$(w_{tt} + (-\Delta)^m w_t + \phi(\| D^m u\|^2)(-\Delta)^m u - \phi(\| D^m v\|^2)(-\Delta)^m v + g(u) - g(v), w_t) = 0 \tag{2.34}$$

$$((-\Delta)^m w_t, w_t) = \| D^m w_t\|^2 \ge m_3\lambda_4 \| w_t\|^2. \tag{2.35}$$

$$\begin{aligned}
&(\phi(\|D^m u\|^2)(-\Delta)^m u - \phi(\|D^m v\|^2)(-\Delta)^m v, w_t) \\
&= (\phi(\|D^m u\|^2)(-\Delta)^m u - \phi(\|D^m u\|^2)(-\Delta)^m v + \phi(\|D^m u\|^2)(-\Delta)^m v - \phi(\|D^m v\|^2)(-\Delta)^m v, w_t) \\
&= \phi(\|D^m u\|^2)((-\Delta)^m w, w_t) + \phi'(\xi)(\|D^m u\| + \|D^m v\|)(\|D^m u\| - \|D^m v\|)((-\Delta)^m v, w_t) \\
&= \frac{1}{2}\phi(\|D^m u\|^2)\frac{d}{dt}\|D^m w\|^2 + \phi'(\xi)(\|D^m u\| + \|D^m v\|)(\|D^m u\| - \|D^m v\|)((-\Delta)^m v, w_t),
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}
&\phi'(\xi)(\|D^m u\| + \|D^m v\|)(\|D^m u\| - \|D^m v\|)((-\Delta)^m v, w_t) \\
&\le \|\phi'(\xi)\|_\infty (\|D^m u\| + \|D^m v\|)\|D^m w\|\cdot\|(-\Delta)^m v\|\cdot\|w_t\|.
\end{aligned} \tag{2.37}$$

According to Lemma 2.1 , Lemma 2.2 and young inequality ; so , exist a constant C_7 such that

$$\begin{aligned}
&\phi'(\xi)(\|D^m u\| + \|D^m v\|)(\|D^m u\| - \|D^m v\|)((-\Delta)^m v, w_t) \\
&\le \|\phi'(\xi)\|_\infty (\|D^m u\| + \|D^m v\|)\|D^m w\|\cdot\|(-\Delta)^m v\|\cdot\|w_t\| \\
&\le C_7\|D^m w\|\cdot\|w_t\| \\
&\le \frac{C_7}{2}(\|D^m w\|^2 + \|w_t\|^2).
\end{aligned} \tag{2.38}$$

According to (2.37) – (2.39), we have

$$\begin{aligned}
&(\phi(\|D^m u\|^2)(-\Delta)^m u - \phi(\|D^m v\|^2)(-\Delta)^m v, w_t) \\
&\ge \frac{\mu}{2}\frac{d}{dt}\|(D^m w\|^2 - \frac{C_7}{2}(\|D^m w\|^2 + \|(w_t\|^2) \\
&\ge \frac{\mu}{2}\frac{d}{dt}\|D^m w\|^2 - \frac{C_7}{2}\|D^m w\|^2 - \frac{C_7}{2}\|w_t\|^2 .
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
&(g(u) - g(v), w_t) \\
&\ge -\frac{m_4\lambda_5}{2}(\|D^m w\|^2 + \|w_t\|^2).
\end{aligned} \tag{2.40}$$

From the above , we obtain

$$\frac{d}{dt}[\|w_t\|^2 + \mu\|D^m w\|^2] + (2m_3\lambda_4 - m_4\lambda_5 - C_7)\|w_t\|^2 - (C_7 + m_4\lambda_5)\|D^m w\|^2 \le 0 \tag{2.41}$$

Take $\gamma = \min\{\frac{-(C_7 + m_4\lambda_5)}{\mu}, 2m_3\lambda_4 - m_4\lambda_5 - C_7\}$, we have

$$\frac{d}{dt}N(t) + \gamma N(t) \le 0, \tag{2.42}$$

where

$$N(t) = \|w_t\|^2 + \mu\|D^m w\|^2 . \tag{2.43}$$

By using Gronwall inequality, we obtain

$$N(t) \le N(0)e^{\gamma t} = 0. \tag{2.44}$$

Therefore

$$u = v. \tag{2.45}$$

So we prove the uniqueness of the solution .

3 Global attractor

Theorem 3.1. [12] Let E_1 be a Banach space , and $\{S(t)\}(t \geq 0)$ are the semigroup operator on E_1 . $S(t): \mathrm{E}_1 \to \mathrm{E}_1$, $S(t+s) = S(t)S(s)\ (\forall t, s \geq 0)$, $S(0)=I$, where I is a unit operator .set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is uniformly bounded , namely $\forall R > 0$, $\| u \|_{\mathrm{E}_1} \leq R$, it exists a constant $C(R)$, so that $\| S(t)u \|_{\mathrm{E}_1} \leq C(R)\ (t \in [0, +\infty))$;

2) It exists a bounded absorbing set $B_0 \subset \mathrm{E}_1$, namely, $\forall B \subset \mathrm{E}_1$, it exists a constant t_0 , so that $S(t)B \subset B_0\ (t \geq t_0)$;

Where B_0 and B are bounded sets.

3) When $t > 0$, $S(t)$ is a completely continuous operator A .

Therefore , the semigroup operator $S(t)$ exists a compact global attractor.

Theorem 3.2. [12] Under the assume of Theorem 2.1, equations have global attractor

$$\mathrm{A} = \omega(B_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B_0}.$$

Where $B_0 = \{(u,v) \in H^{2m}(\Omega) \times H^m(\Omega) : \| (u,v) \|^2_{H^{2m} \times H^m} = \| u \|^2_{H^{2m}} + \| v \|^2_{H^m} \leq R_0 + R_1\}$, B_0 is the bounded absorbing set of $H^{2m}(\Omega) \times H^m(\Omega)$ and satisfies

(1) $S(t)\mathrm{A} = \mathrm{A}$, $t>0$;

(2) $\lim\limits_{t \to \infty} \mathrm{dist}(S(t)B, \mathrm{A}) = 0$, here $B \subset H^{2m} \times H^m$ and it is a bounded set,

$$\lim_{t \to \infty} \mathrm{dist}(S(t)B, \mathrm{A}) = \sup_{x \in B} \left(\inf_{y \in \mathrm{A}} \| S(t)x - y \|_{H^{2m} \times H^m} \right) \to 0,\ t \to \infty .$$

Proof .Under the conditions of Theorem 2.1, it exists the solution semigroup $S(t)$, $S(t): \mathrm{E}_1 \to \mathrm{E}_1$, here $E_1 = H^{2m} \times H^m$.

(1) from Lemma 2.1 to Lemma 2.2, we can get that $\forall B \subset H^{2m} \times H^m$ is a bounded set that includes in the ball $\{\| (u,v) \|_{H^{2m} \times H^m} \leq R\}$,

$$\| S(t)(u_0, v_0) \|^2_{H^{2m} \times H^m} = \| u \|^2_{H^{2m}} + \| v \|^2_{H^m} \le \| u_0 \|^2_{H^{2m}} + \| v_0 \|^2_{H^m} + C \le R^2 + C \ \ (t \ge 0, (u_0, v_0) \in B)$$

This shows that $S(t)\ (t \ge 0)$ is uniformly bounded $H^{2m} \times H^m$.

(2) Furthermore, for any $(u_0, v_0) \in H^{2m} \times H^m$,when $t \ge \max\{t_1, t_2\}$,we have,

$$\| S(t)(u_0, v_0) \|^2_{H^{2m} \times H^m} = \| u \|^2_{H^{2m}} + \| v \|^2_{H^m} \le R_0 + R_1$$

So we get B_0 is the bounded absorbing set.

(3) Since $H^{2m} \times H^m \to H^m \times L^2$ is compact embedded, which means that the bounded set in $H^{2m} \times H^m$ is the compact set in $H^m \times L^2$, so the semigroup operator $S(t)$ exists a compact global attractor A .

4 Hausdorff and fractal dimensions for the global attractor

Theorem 4.1. under the conditions of Theorem 3.2. , the global attractor A of problem (1.1)-(1.3) has infinte Hausdorff dimension and fractal dimension, and $d_H(A) < \frac{2}{5} n$, $d_F(A) < \frac{7}{5} n$.

Proof .problem (1.1) can be written

$$u_{tt} + A^m u_t + \phi(\| A^{\frac{m}{2}} u \|^2) A^m u + g(u) = f(x), \tag{4.1}$$

where $-\Delta = A$.

Let $\psi = R_\varepsilon \varphi = (u, v), \varphi = (u, u_t), v = u_t + \varepsilon u$, $R_\varepsilon : \{u, u_t\} \to \{u, u_t + \varepsilon u\}$,is an isomorphic mapping, so the equation of (4.1) is

$$\psi_t + \Lambda_\varepsilon \psi + \bar{g}(\psi) = \bar{f}. \tag{4.2}$$

Where $\psi = \{u, u_t + \varepsilon u\}^T$, $\bar{\mathrm{g}}(\psi) = \{0, \mathrm{g}(\mathrm{u})\}^T$, $\bar{f}(x) = \{0, f(x)\}^T$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ (\phi(\| A^{\frac{m}{2}} u \|^2) - \varepsilon) A^m + \varepsilon^2 I & A^m - \varepsilon I \end{pmatrix}.$$

$$\psi_t = \bar{f} - \Lambda_\varepsilon \psi - \bar{g}(\psi) . \tag{4.3}$$

Let $F : E_1 \to E_1$ is Frechet differentiable, the linearized equation of (4.3) is

$$P_t + \Lambda_\varepsilon P + \bar{g}'(\psi) P = 0 , \tag{4.4}$$

where $P = (U, U_t + \varepsilon U)$, $\bar{g}'(\psi)U = (0, g_t(u)U)$. U is solution of (4.2).

For a fixed $(u_0, v_0) \in E_1$, let $\gamma_1, \gamma_2 \dots \gamma_N$ are N elements of E_1, let $P_1(t), P_2(t) \dots P_N(t)$ are N solutions of linear equation (4.4) with initial value $P_1(0) = \gamma_1, P_2(0) = \gamma_2 \dots P_N(0) = \gamma_N$, so, we have

$$\| P_1(t) \Lambda P_2(t) \Lambda \dots \Lambda P_N(t) \|^2_{\Lambda E_1} = \| \gamma_1 \Lambda \gamma_1 \Lambda \dots \Lambda \gamma_N \|_{\Lambda E_1} \exp\left(\int_0^t trF'(\psi(\tau)) \cdot Q_N(\tau) d\tau\right). \tag{4.5}$$

Where Λ represents the outer product, Tr represents the trace, $Q_N(\tau)$ is an orthogonal projection from the space E_1 to the subspace spanned by $\{P_1(t), P_2(t), \dots, P_N(t)\}$.

For a give time τ, let $\theta_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^T$, $j = 1, 2, \dots N$,is the standard orthogonal basis of the space $span\{P_1(t), P_2(t), \dots, P_N(t)\}$.

We denote inner product in E_1, $((\xi, \eta), (\bar{\xi}, \bar{\eta})) = ((\xi, \bar{\xi}) + (\eta, \bar{\eta}))$.

From the above , we have

$$\begin{aligned} TrF_t(\psi(\tau)) \cdot Q_N(\tau) &= \sum_{j=1}^{N} (F_t(\psi(\tau)) \cdot Q_N(\tau)\theta_j(\tau), \theta_j(\tau))_{E_1} \\ &= \sum_{j=1}^{N} (F_t(\psi(\tau))\theta_j(\tau), \theta_j(\tau))_{E_1} \end{aligned} \tag{4.6}$$

Where $$(F_t(\psi(\tau))\theta_j(\tau), \theta_j(\tau))_{E_1} = -(\Lambda_\varepsilon \theta_j, \theta_j) - (g_t \theta_j, \theta_j). \tag{4.7}$$

$$\begin{aligned} &(\Lambda_\varepsilon \theta_j, \theta_j) \\ &= ((\varepsilon\xi_j - \eta_j, (\phi(\| A^{\frac{m}{2}} u \|^2) - \varepsilon) A^m \xi_j - \varepsilon^2 \xi_j + A^m \eta_j - \varepsilon \eta_j), (\xi_j, \eta_j)) \\ &= (\varepsilon\xi_j - \eta_j, \xi_j) + (\phi(\| A^{\frac{m}{2}} u \|^2) - \varepsilon) A^m \xi_j - \varepsilon^2 \xi_j + A^m \eta_j - \varepsilon \eta_j, \eta_j) \\ &= \varepsilon \| \xi_j \|^2 - (1 + \varepsilon^2)(\xi_j, \eta_j) + (\phi(\| A^{\frac{m}{2}} u \|^2) - \varepsilon)(A^m \xi_j, \eta_j) + \| D^m \eta_j \|^2 - \varepsilon \| \eta_j \|^2 . \end{aligned} \tag{4.8}$$

$$\begin{aligned} &-(1 + \varepsilon^2)(\xi_j, \eta_j) + (\phi(\| A^{\frac{m}{2}} u \|^2) - \varepsilon)(A^m \xi_j, \eta_j) \\ &\geq l_1(\mu_1 - \varepsilon)(\xi_j, \eta_j) - (1 + \varepsilon^2)(\xi_j, \eta_j) \\ &\geq (l_1(\mu_1 - \varepsilon) - (1 + \varepsilon^2))(\xi_j, \eta_j). \end{aligned} \tag{4.9}$$

There exists a constant l_1, such that

$$l_1(\mu_1 - \varepsilon) - (1 + \varepsilon^2) \geq 0. \tag{4.10}$$

So we have

$$\begin{aligned}
&(\Lambda_\varepsilon \theta_j, \theta_j) \\
&= \varepsilon \|\xi_j\|^2 + \|D^m \eta_j\|^2 - \varepsilon \|\eta_j\|^2 \\
&\geq \varepsilon \|\xi_j\|^2 + (l_2 - \varepsilon)\|\eta_j\|^2
\end{aligned} \tag{4.11}$$

There exists a constant l_2, such that

$$l_2 - \varepsilon \geq 0 \tag{4.12}$$

Take $\delta = \min(\varepsilon, l_2 - \varepsilon)$, so

$$\begin{aligned}
&(\Lambda_\varepsilon \theta_j, \theta_j) \\
&\geq \varepsilon \|\xi_j\|^2 + (l_2 - \varepsilon)\|\eta_j\|^2 \\
&\geq \delta(\|\xi_j\|^2 + \|\eta_j\|^2).
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&(g_t(\psi)\theta_j, \theta_j) \\
&= (0, g_t(u)\xi_j) \cdot (\xi_j, \eta_j) \\
&= (g_t \xi_j, \eta_j) \\
&\geq -\|g_t \xi_j\| \cdot \|\eta_j\|.
\end{aligned} \tag{4.14}$$

Now, suppose that $(u_0, u_1) \in A$, A is a bounded absorbing set in E_1; $\psi(t) = (u(t), u_t(t) + \varepsilon u(t)) \in E_1$,

$u(t) \in D(A)$. Then there exists a $s \in [0,1]$, we have mapping $g_t : D(A) \to \sigma(V_s, H^m)$, such that

$$\sup \|g_t\| \leq r < \infty. \tag{4.15}$$

According to (4.7),(4.13),(4.14), we have

$$\begin{aligned}
&(F_t(\psi(\tau))\theta_j(\tau), \theta_j(\tau))_{E_1} \\
&\leq -\delta(\|\xi_j\|^2 + \|\eta_j\|^2) + r\|\xi_j\| \cdot \|\eta_j\| \\
&\leq -\frac{\delta}{2}(\|\xi_j\|^2 + \|\eta_j\|^2) + \frac{r}{2}\|\xi_j\|^2.
\end{aligned} \tag{4.16}$$

Because of $\theta_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^T$, $j = 1, 2, \ldots N$, is the standard orthogonal basis, so

$$\|\xi_j\|^2 + \|\eta_j\|^2 = 1. \tag{4.17}$$

$$\sum_{j=1}^{N}(F_t(\psi(\tau))\theta_j(\tau),\theta_j(\tau))_{E_1}$$

$$\leq -\frac{N\delta}{2}+\frac{r}{2}\sum_{j=1}^{N}\|\xi_j\|^2 . \tag{4.18}$$

Almost to all t, making

$$\sum_{j=1}^{N}\|\xi_j\|^2 \leq \sum_{j=1}^{N}\lambda_j^{s-1}. \tag{4.19}$$

So we have

$$TrF_t(\psi(\tau))\cdot Q_N(\tau)$$

$$\leq -\frac{N\delta}{2}+\frac{r}{2}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{4.20}$$

Let

$$q_N(t)=\sup_{\psi_0\in A}\ \sup_{\substack{\eta_j\in E_1\\ \|\eta_j\|\leq 1}}\left(\frac{1}{t}\int_0^t trF'(S(\tau)\psi_0)\cdot Q_N(\tau)d\tau\right). \tag{4.21}$$

$$q_N=\lim_{t\to\infty}q_N(t). \tag{4.22}$$

According to (4.20), we have

$$q_m \leq -\frac{N\delta}{2}+\frac{r}{2}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{4.23}$$

Therefore, the Lyapunov exponent of A is uniformly bounded.

$$\kappa_1+\kappa_2+\ldots+\kappa_N \leq -\frac{N\delta}{2}+\frac{r}{2}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{4.24}$$

Then, exist a $s\in[0,1]$, such that

$$(q_j)_+ \leq -\frac{N\delta}{2}+\frac{r}{2}\sum_{j=1}^{N}\lambda_j^{s-1}\leq\frac{r}{2}\sum_{j=1}^{N}\lambda_j^{s-1}\leq\frac{N\delta}{7}, \tag{4.25}$$

where λ_j is eigenvalue of A^m, and $\lambda_1<\lambda_2<\cdots<\lambda_m$.

And

$$q_N \le -\frac{N\delta}{2}(1-\frac{r}{N\delta}\sum_{j=1}^{N}\lambda_j^{s-1}) \le -\frac{5}{14}N\delta. \tag{4.26}$$

So

$$\max_{1\le j\le N}\frac{(q_j)_+}{|q_m|} \le \frac{2}{5}. \tag{4.27}$$

So, we can acquire $d_H(A) < \frac{2}{5}n$, $d_F(A) < \frac{7}{5}n$.

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